9.1 Non-Sinusoidal Voltages & Currents (Harmonics):

In communication, elecronic & power generation there is a harmonic waves represent a sum of infinite number of sine and cosine functions. The solution can be determined by superposing the partial functions with sinusoidal steady-state analysis. The harmonic waves are generated from fundamental wave with frequency multiplied by an integer number.

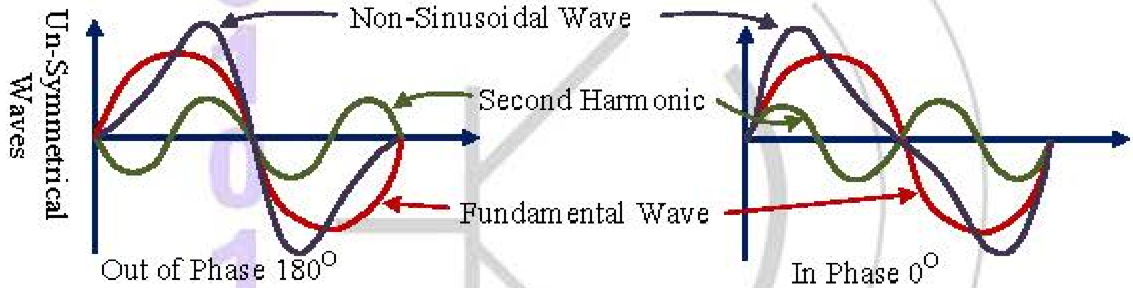
- The fundamental wave has fundamental frequency (f_1).
- The nth harmonic wave frequency (f_n) represented by: $f_n = nf_1$ where n represent harmonic wave number.

if n=2 for 2nd harmonic with ($2f_1$) frequency, n=3 for 3rd harmonic with ($3f_1$) frequency, ...etc.

$Non-sinusoidal\ wave = Fundamental\ wave + \sum harmonic\ waves$

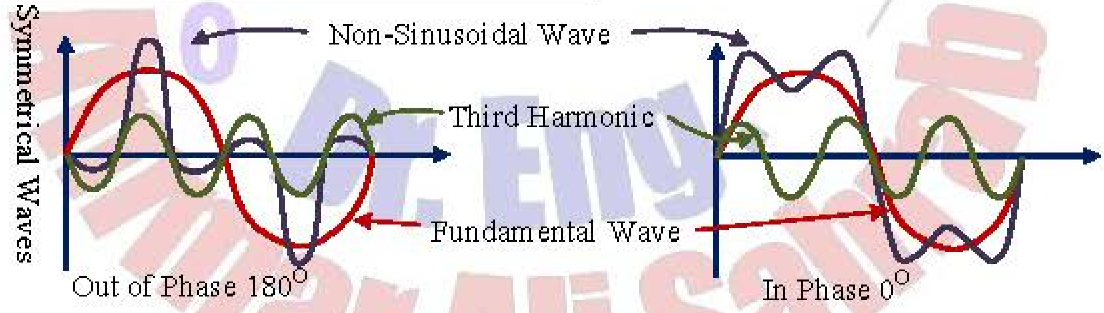
(vector summation) and this equation called instantanuous equation for voltage and current.

a. Fundamental Wave + Second Harmonic:



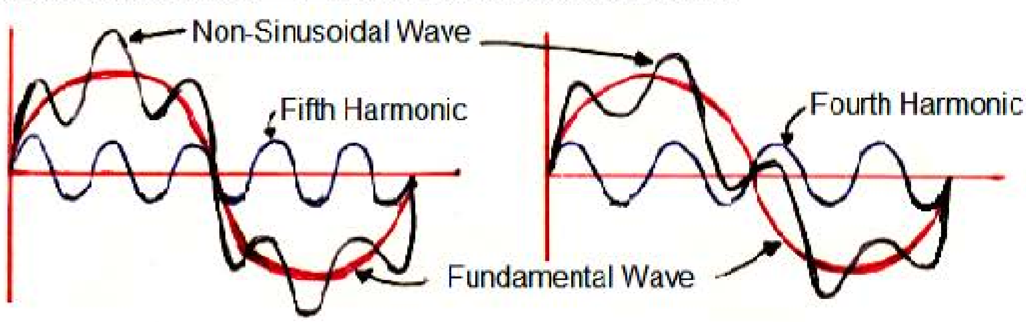
Un-symmetrical non-sinusoidal waves composite of fundamental wave + even harmonics (n=2, 4, ...etc).

b. Fundamental Wave + Third Harmonic:



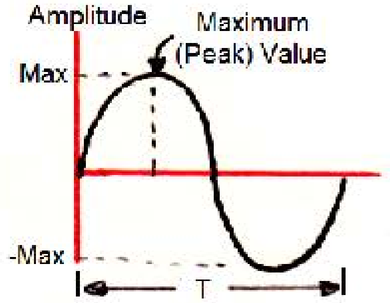
Symmetrical non-sinusoidal waves composite of fundamental wave + odd harmonics (n=3, 5, ...etc).

c. Fundamental Wave + Fourth & Fifth Harmonics:



9.2 Non-Sinusoidal Instantaneous Value Equation:

There are infinite number of harmonic peak values which are usually less than fundamental wave peak value (i.e. decreases from fundamental to nth harmonic) for both voltages and currents.



$\omega = 2\pi f = \frac{2\pi}{T}$ rad/sec

f = fundamental wave frequency

Fundamental wave:

$i_1 = I_{m1}\sin\omega t \qquad e_1 = V_{m1}\sin\omega t$

Second harmonic:

$i_2 = I_{m2}\sin 2\omega t \qquad e_2 = V_{m2}\sin 2\omega t$

Third harmonic:

$i_3 = I_{m3}\sin 3\omega t \qquad e_3 = V_{m3}\sin 3\omega t$

Nth harmonic:

$i_n = I_{mn}\sin n\omega t \qquad e_n = V_{mn}\sin n\omega t$

The instantaneous value for non-sinusoidal wave is algebric sum of instantaneous values for fundamental and harmonic waves.

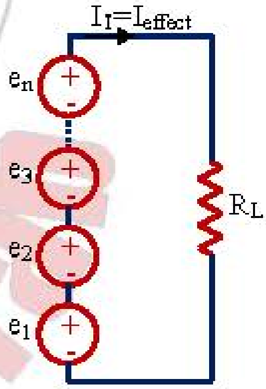
$$i = i_1 + i_2 + i_3 + \cdots + i_n = I_{m1}\sin\omega t + I_{m2}\sin 2\omega t + I_{m3}\sin 3\omega t + \cdots + I_{mn}\sin n\omega t$$

$$e = e_1 + e_2 + e_3 + \cdots + e_n = V_{m1}\sin\omega t + V_{m2}\sin 2\omega t + V_{m3}\sin 3\omega t + \cdots + V_{mn}\sin n\omega t$$

Notes:

1. Not necessary that all harmonics are presented in the instantaneous wave equation.
2. The fundamental and harmonics waves start at the same time, which is a special case.

9.3 Effective Values for Non-Sinusoidal Waves:



The non-sinusoidal waves could be grouped into different frequency sine-wave sources connected in series to the load. Each source supply the load by a power calculated using superposition theorem, and the total power is the sum of them.

$e_1 \rightarrow P_1, e_2 \rightarrow P_2, e_3 \rightarrow P_3, \ldots, e_n \rightarrow P_n$

The total power is: $P_T = P_1 + P_2 + P_3 + \cdots + P_n$

from power triangle:

$$P_1 = \frac{V_1^2}{R_L} = \frac{V_{m1}^2}{2R_L} = I_1^2 R_L = \frac{I_{m1}^2 R_L}{2}$$

$$P_2 = \frac{V_2^2}{R_L} = \frac{V_{m2}^2}{2R_L} = I_2^2 R_L = \frac{I_{m2}^2 R_L}{2}$$

$$P_3 = \frac{V_3^2}{R_L} = \frac{V_{m3}^2}{2R_L} = I_3^2 R_L = \frac{I_{m3}^2 R_L}{2}$$

$$P_n = \frac{V_n^2}{R_L} = \frac{V_{mn}^2}{2R_L} = I_n^2 R_L = \frac{I_{mn}^2 R_L}{2}$$

The power dissipated as heat in a resistance due to non-sinusoidal wave depends on constant current value. Assume this current as DC current give the same heat power at the same time in the same resistance, it is equal the effective current value (non-sinusoidal). $P_T = I_{eff}^2 R_L = \frac{V_{eff}^2}{R_L}$

$$I_{eff} = \sqrt{\frac{I_{m1}^2 + I_{m2}^2 + I_{m3}^2 + \cdots + I_{mn}^2}{2}} \quad \text{and} \quad V_{eff} = \sqrt{\frac{V_{m1}^2 + V_{m2}^2 + V_{m3}^2 + \cdots + V_{mn}^2}{2}}$$

9.4 General Case for Non-Sinusoidal Wave Equation:

DC: I_0 and V_0

Fundamental wave:

$i_1 = I_{m1}\sin(\omega t \mp \phi_1)$ and $e_1 = V_{m1}\sin(\omega t \mp \theta_1)$

Second harmonic:

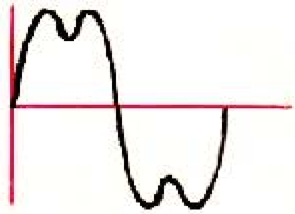
$i_2 = I_{m2}\sin(2\omega t \mp \phi_2)$ and $e_2 = V_{m2}\sin(2\omega t \mp \theta_2)$

Third harmonic:

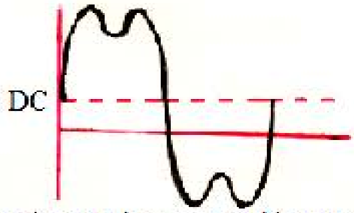
$i_3 = I_{m3}\sin(3\omega t \mp \phi_3)$ and $e_3 = V_{m3}\sin(3\omega t \mp \theta_3)$

Nth harmonic:

$i_n = I_{mn}\sin(n\omega t \mp \phi_n)$ and $e_n = V_{mn}\sin(n\omega t \mp \theta_n)$



Fundamental wave + 3rd harmonic



Fundamental wave + 3rd harmonic + DC voltage

$e = V_O + V_{m1}\sin\omega t + V_{m3}\sin 3\omega t$

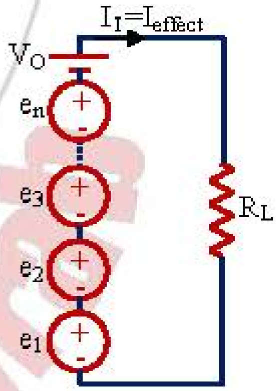
The instantaneous value for non-sinusoidal wave is algebric sum of instantaneous values for DC, fundamental and harmonics.

$$i = I_0 + i_1 + i_2 + i_3 + \cdots + i_n = I_{m1}\sin(\omega t \mp \phi_1) + I_{m2}\sin(2\omega t \mp \phi_2) + I_{m3}\sin(3\omega t \mp \phi_3) + \cdots + I_{mn}\sin(n\omega t \mp \phi_n)$$

$$e = V_0 + e_1 + e_2 + e_3 + \cdots + e_n = V_{m1}\sin(\omega t \mp \theta_1) + V_{m2}\sin(2\omega t \mp \theta_2) + V_{m3}\sin(3\omega t \mp \theta_3) + \cdots + V_{mn}\sin(n\omega t \mp \theta_n)$$

<u>Note:</u> In general case there is individual phase angle for fundamental and harmonic waves. Each phase angle related to its harmonic not to other harmonics or fundamental waves.

9.5 General Case for Effective Values:



The non-sinusoidal waves could be grouped into different sine-wave frequency sources connected in series to the load. Each source supply the load by a power calculated using superposition theorem, and the total power is the sum of them. The total power is: $P_T = P_0 + P_1 + P_2 + P_3 + \cdots + P_n$

from power triangle:

$$P_0 = \frac{V_0^2}{R_L} = I_0^2 R_L$$

$$P_1 = \frac{V_1^2}{R_L} = \frac{V_{m1}^2}{2R_L} = I_1^2 R_L = \frac{I_{m1}^2 R_L}{2}$$

$$P_2 = \frac{V_2^2}{R_L} = \frac{V_{m2}^2}{2R_L} = I_2^2 R_L = \frac{I_{m2}^2 R_L}{2}$$

$$P_3 = \frac{V_3^2}{R_L} = \frac{V_{m3}^2}{2R_L} = I_3^2 R_L = \frac{I_{m3}^2 R_L}{2}$$

$$P_n = \frac{V_n^2}{R_L} = \frac{V_{mn}^2}{2R_L} = I_n^2 R_L = \frac{I_{mn}^2 R_L}{2}$$

The power dissipated as heat in a resistance due to non-sinusoidal wave depends on constant current value. Assume this current as DC current give the same heat power at the same time in the same resistance; it is equal the effective current value (non-sinusoidal). $P_T = I_{eff}^2 R_L = \frac{V_{eff}^2}{R_L}$

$$I_{eff} = \sqrt{I_0^2 + \frac{I_{m1}^2 + I_{m2}^2 + I_{m3}^2 + \cdots + I_{mn}^2}{2}} \quad \text{and} \quad V_{eff} = \sqrt{V_0^2 + \frac{V_{m1}^2 + V_{m2}^2 + V_{m3}^2 + \cdots + V_{mn}^2}{2}}$$

Example: non-sinusoidal wave with fundamental frequency (200Hz) and maximum value of fundamental current (45mA). The wave contain third harmonic with maximum value of current (22.5mA). Also the wave contain sixth harmonic with maximum value of current (3mA). Write the non-sinusoidal current wave equation.

Solution:

$\omega_1 = 2\pi f = 2\pi \times 200 = 1256.6\, rad/sec$

$\omega_3 = 3\omega_1 = 3 \times 1256.6 = 3770\, rad/sec$

$\omega_6 = 6\omega_1 = 6 \times 1256.6 = 7540\, rad/sec$

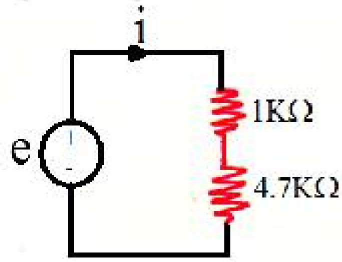
$i_1 = 45 \times 10^{-3} \sin(1256.6\, t)\, A$

$i_3 = 22.5 \times 10^{-3} \sin(3770\, t)\, A$

$i_6 = 3 \times 10^{-3} \sin(7540\, t)\, A$

$$\boxed{i = 45 \sin(1256.6\, t) + 22.5 sin(3770\, t) + 3 sin(7540\, t)\, mA}$$

Example: non-sinusoidal wave source has a voltage equation (e=0.45sinωt+0.18sin2ωt) volt, applied on a series circuit contain (R_1=1KΩ) and (R_2=4.7KΩ). Find the voltage effective value on (4.7KΩ) resistor and the current effective value in the circuit?



Solution:

$R_T = 1 \times 10^3 + 4.7 \times 10^3 = 5.7\, K\Omega$

$i = \frac{0.45}{5.7\times10^3} sin\omega t + \frac{0.18}{5.7\times10^3} sin2\omega t$

$= 80 sin\omega t + 60 sin2\omega t\, \mu A$

$I_{eff} = \sqrt{\frac{(80\times10^{-6})^2+(30\times10^{-6})^2}{2}} = 60\, \mu A$

$V_{eff4.7K} = I_{eff} R_{4.7K} = 60 \times 10^{-6} \times 4.7 \times 10^3 = 0.28\, V$

another way:

$V_{eff} = \sqrt{\frac{(0.45)^2+(0.18)^2}{2}} = 0.343\, V$ and $I_{eff} = \frac{V_{eff}}{R_T} = \frac{0.343}{5.7\times10^3} = 60\, \mu A$

using VDR: $V_{eff4.7K} = V_{eff} \times \frac{R_{4.7K}}{R_{4.7K}+R_{1K}} = 0.343 \times \frac{4.7\times10^3}{4.7\times10^3+1\times10^3} = 0.28\, V$

Example: Find the active value of voltage and current, and the average power for an electrical circuit if the voltage supplied is: e=200+100cos(500t+30°)+75cos(1500t+60°) V and the current is: i=3.53cos(500t+75°)+3.55cos(1500t+78.45°) A.

Solution:

$I_{eff} = \sqrt{\frac{(3.53)^2+(3.55)^2}{2}} = 3.54\, A$

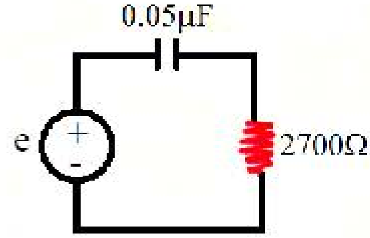
$V_{eff} = \sqrt{(200)^2 + \frac{(100)^2+(75)^2}{2}} = 218.66\, V$

$P_{DC} = 200 \times 0 = 0\, W$

$P_1 = \frac{100}{\sqrt{2}} \times \frac{3.53}{\sqrt{2}} \cos(75^O - 30^O) = 124.8\, W$

$P_3 = \frac{75}{\sqrt{2}} \times \frac{3.55}{\sqrt{2}} \cos(78.45^O - 60^O) = 126.3\, W$

$P_T = P_{DC} + P_1 + P_3 = 0 + 124.8 + 126.3 = 251.1\, W$



Example: The non-sinusoidal voltage (e=150sin6280t+45sin12560t+15sin31400t) V is applied to the electric circuit shown beside.

a. Derive the current equation.

b. Derive the voltage equation across resistance & capacitance.

c. Find the total power?

Solution:

a. fundamental: $X_{C1} = \frac{1}{6280 \times 0.05 \times 10^{-6}} = 3184.7\ \Omega$

$Z_{T1} = 2700 - j3184.7 = 4175\angle{-49.7^o}\ \Omega$

$i_1 = \frac{150\angle 0^o}{4175\angle{-49.7^o}} = 35.9\angle 49.7^o\ mA$

2nd harmonic: $X_{C2} = \frac{X_{C1}}{2} = 1592.35\ \Omega$

$Z_{T2} = 2700 - j1592.35 = 3134.6\angle{-30.5^o}\ \Omega$

$i_2 = \frac{45\angle 0^o}{3134.6\angle{-30.5^o}} = 14.36\angle 30.5^o\ mA$

5th harmonic: $X_{C5} = \frac{X_{C1}}{5} = 637\ \Omega$

$Z_{T5} = 2700 - j637 = 2774\angle{-13.3^o}\ \Omega$

$i_5 = \frac{15\angle 0^o}{2774\angle{-13.3^o}} = 5.41\angle 13.3^o\ mA$

$\therefore i = 35.9\sin(6280t + 49.7^o) + 14.36\sin(12560t + 30.5^o)$
$+5.41\sin(31400t + 13.3^o)\ mA$

b. $e_R = i \times R = 97.2\sin(6280t + 49.7^o) + 38.8\sin(12560t + 30.5^o)$
$+14.6\sin(31400t + 13.3^o)\ V$

$e_C = i \times (-jX_C) = 35.9 \times 10^{-3} \times 3184.7\sin(6280t + 49.7^o - 90^o)$
$+14.36 \times 10^{-3} \times 1592.35\sin(12560t + 30.5^o - 90^o)$
$+5.41 \times 10^{-3} \times 637\sin(31400t + 13.3^o - 90^o)\ V$

$e_C = 114.6\sin(6280t - 40.3^o) + 22.9\sin(12560t - 59.5^o)$
$+3.45\sin(31400t - 76.7^o)\ V$

c. $I_{eff} = \sqrt{\frac{(35.9\times10^{-3})^2+(14.36\times10^{-3})^2+(5.41\times10^{-3})^2}{2}} = 27.7\ mA$

$P_T = I_{eff}^2 R = (27.7 \times 10^{-3})^2 \times 2700 = 2.07\ W$

another method:

$V_{Reff} = \sqrt{\frac{(97.2)^2+(38.8)^2+(14.6)^2}{2}} = 74.72\ V$

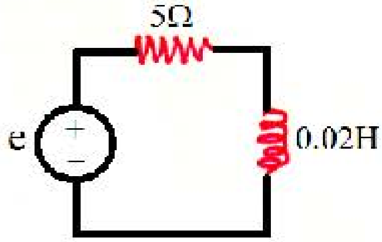
$P_T = \frac{V_{Reff}^2}{R} = \frac{(74.72)^2}{2700} = 2.07\ W$

another method:

$P_T = \frac{150}{\sqrt{2}} \times \frac{35.9\times10^{-3}}{\sqrt{2}}\cos(49.7^o) + \frac{45}{\sqrt{2}} \times \frac{14.36\times10^{-3}}{\sqrt{2}}\cos(30.5^o)$
$+\frac{15}{\sqrt{2}} \times \frac{5.41\times10^{-3}}{\sqrt{2}}\cos(13.3^o) = 2.07\ W$

Note: One can use any method to calculate the total power but should be careful about which voltage or current will be used depending on previous knowledge in basic circuit analysis.

Example: The non-sinusoidal voltage ($e=100+50\sin\omega t+25\sin 3\omega t$) V is applied to the electric circuit shown with $\omega=500$ rad/sec.



a. Derive the current equation.

b. Derive the voltage equation across resistance & inductance.

c. Find the total power?

Solution:

a. DC: $I_{DC} = \frac{100}{5} = 20\ A$ (inductance is short circuit)

fundamental: $X_{L1} = 500 \times 0.02 = 10\ \Omega$

$Z_{T1} = 5 + j10 = 11.18\angle 63.4^{o}\ \Omega$

$i_1 = \frac{50\angle 0^{o}}{11.18\angle 63.4^{o}} = 4.47\angle -63.4^{o}\ A$

3rd harmonic: $X_{L3} = 3X_{L1} = 30\ \Omega$

$Z_{T3} = 5 + j30 = 30.4\angle 80.54^{o}\ \Omega$

$i_3 = \frac{25\angle 0^{o}}{30.4\angle 80.54^{o}} = 0.822\angle -80.54^{o}\ A$

$\therefore i = 20 + 4.47\sin(500t - 63.4^{o}) + 0.822\sin(1500t - 80.54^{o})\ A$

b. $e_R = i \times R = 100 + 22.35\sin(500t - 63.4^{o}) + 4.11\sin(1500t - 80.54^{o})\ V$

$e_L = i \times (jX_L) = 20 \times 0 + 4.47 \times 10\sin(500t - 63.4^{o} + 90^{o})$
$+0.822 \times 30\sin(1500t - 80.54^{o} + 90^{o})$

$e_L = 44.7\sin(500t + 26.6^{o}) + 24.66\sin(1500t + 9.46^{o})\ V$

c. $I_{eff} = \sqrt{(20)^2 + \frac{(4.47)^2 + (0.822)^2}{2}} = 20.256\ A$

$P_T = I_{eff}^2 R = (20.256)^2 \times 5 = 2051.6\ W$

another method:

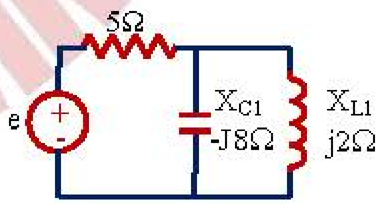
$V_{Reff} = \sqrt{(100)^2 + \frac{(22.35)^2 + (4.11)^2}{2}} = 101.3\ V$

$P_T = \frac{V_{Reff}^2}{R} = \frac{(101.3)^2}{5} = 2051.6\ W$

another method:

$P_T = 20 \times 100 + \frac{50}{\sqrt{2}} \times \frac{4.47}{\sqrt{2}}\cos(63.4^{o}) + \frac{25}{\sqrt{2}} \times \frac{0.822}{\sqrt{2}}\cos(80.54^{o}) = 2051.6\ W$

Example: For the circuit shown, find the total current equation if the supply voltage is: $e=50+20\sin 500t+10\sin 1000t$ V.



Solution:

DC: $I_{DC} = \frac{50}{5} = 10\ A$ (L is s/c & C is o/c)

fundamental:

$Z_{T1} = 5 + (j2//(-j8)) = 5 + j2.667 = 5.667\angle 28^{o}\ \Omega$

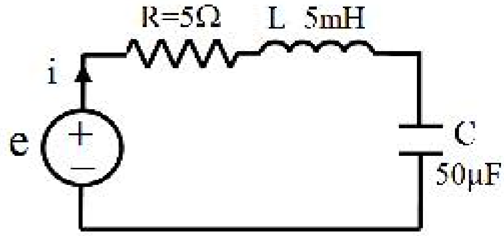
$i_1 = \frac{20\angle 0^{o}}{5.667\angle 28^{o}} = 3.53\angle -28^{o}\ A$

2nd harmonic: $X_{L2} = 2X_{L1} = 4\ \Omega$ & $X_{C2} = \frac{X_{C1}}{2} = 4\ \Omega$

$Z_{T2} = 5 + (j4//(-j4)) = \infty\ \Omega$ open circuit $i_2 = \frac{10\angle 0^{o}}{\infty} = 0\ A$

$\therefore i = 10 + 3.53\sin(500t - 28^{o})\ A$

Example: For the circuit shown, if the applied non-sinusoidal voltage is: e=150sin1000t+100sin2000t+75sin3000t V. Find the active value of the current passes through the circuit and the average consumed power in the circuit?



Solution:

fundamental: $X_{L1} = 1000 \times 5 \times 10^{-3} = 5\,\Omega$ & $X_{C1} = \frac{1}{1000\times 50\times 10^{-6}} = 20\,\Omega$

$Z_{T1} = 5 - j15 = 15.8\angle{-71.5^o}\,\Omega \Rightarrow i_1 = \frac{150\angle 0^o}{15.8\angle{-71.5^o}} = 9.4\angle 71.5^o\,A$

2nd harmonic: $X_{L2} = 2X_{L1} = 10\,\Omega$ & $X_{C2} = \frac{X_{C1}}{2} = 10\,\Omega$

$Z_{T2} = 5 = 5\angle 0^o\,\Omega \Rightarrow i_2 = \frac{100\angle 0^o}{5\angle 0^o} = 20\angle 0^o\,A = 20\,A$

3rd harmonic: $X_{L3} = 3X_{L1} = 15\,\Omega$ & $X_{C3} = \frac{X_{C1}}{3} = 6.667\,\Omega$

$Z_{T3} = 5 + j8.333 = 9.7\angle 59.2^o\,\Omega \Rightarrow i_3 = \frac{75\angle 0^o}{9.7\angle 59.2^o} = 7.7\angle{-59.2^o}\,A$

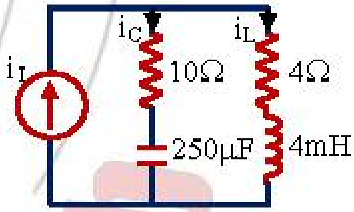
$\therefore i = 9.4\sin(1000t + 71.5^o) + 20\sin\,2000t + 7.7\sin\,(3000t - 59.2^o)\,A$

$I_{eff} = \sqrt{\frac{(9.4)^2+(20)^2+(7.7)^2}{2}} = 16.54\,A \Rightarrow P_T = I_{eff}^2 R = (16.54)^2 \times 5 = 1370\,W$

another method:

$P_T = \frac{150}{\sqrt{2}} \times \frac{9.4}{\sqrt{2}}\cos(-71.5^o) + \frac{100}{\sqrt{2}} \times \frac{20}{\sqrt{2}}\cos(0^o) + \frac{75}{\sqrt{2}} \times \frac{7.7}{\sqrt{2}}\cos(59.2^o) = 1370\,W$

Example: Derive the current equation in each branch and calculate the consumed power in the circuit if the supplied current is: i=10+4sin500t+2sin1500t A.



Solution:

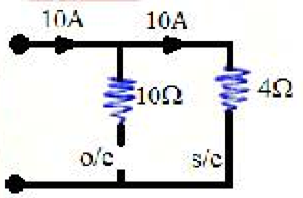
DC: $I_{CDC} = 0\,A$ & $I_{LDC} = 10\,A$

fundamental: $X_{L1} = 500 \times 4 \times 10^{-3} = 2\,\Omega$

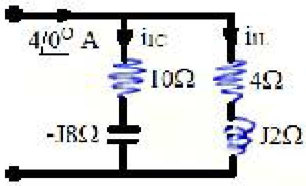
$X_{C1} = \frac{1}{500\times 250\times 10^{-6}} = 8\,\Omega$

$i_{1C} = 4\angle 0^o \times \frac{4+j2}{14-j6} = 1.2\angle 49.2^o\,A$

$i_{1L} = 4\angle 0^o \times \frac{10-j8}{14-j6} = 3.36\angle{-15^o}\,A$



3rd harmonic: $X_{L3} = 3X_{L1} = 6\,\Omega$ & $X_{C3} = \frac{X_{C1}}{3} = 2.67\,\Omega$

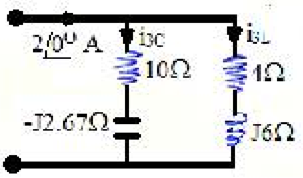


$i_{3C} = 2\angle 0^o \times \frac{4+j6}{14+j3.33} = 1\angle 42.9^o\,A$

$i_{3L} = 2\angle 0^o \times \frac{10-j2.67}{14+j3.33} = 1.4\angle{-28.2^o}\,A$

$i_C = 1.2\sin(500t + 49.2^o) + 1\sin\,(1500t + 42.9^o)\,A$

$i_L = 10 + 3.36\sin(500t - 15^o) + 1.4\sin\,(1500t - 28.2^o)\,A$



$I_{Ceff} = \sqrt{\frac{(1.2)^2+(1)^2}{2}} = 1.132\,A$

$I_{Leff} = \sqrt{(10)^2 + \frac{(3.36)^2+(1.4)^2}{2}} = 10.3\,A$

$P_T = P_C + P_L = I_{Ceff}^2 R_C + I_{Leff}^2 R_L = (1.132)^2 \times 10 + (10.3)^2 \times 4 = 436.1\,W$