Digital Convolution

The Convolution Sum or Superposition Sum Representation of LTI Systems:

The **convolution** allows us to find the output signal from any LTI processor in response to any input signal. We can find the output signal $y(n)$ from an LTI processor by convolving its input signal $x(n)$ with a second function representing the impulse response $h(n)$ of the processor. The convolution sum or superposition sum of the sequences $x(n)$ and $h(n)$ can be represented by

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

$N= N_1+N_2-1$. Where **N_1** = number of samples of $x(n)$, **N_2** = number of samples of $h(n)$, and N= total number of samples.

This operation is represented symbolically as **$x(n)*h(n)$**

Properties of Convolution:

1- *Commutativity*

Convolution is a commutative operation, meaning signals can be convolved in any order.

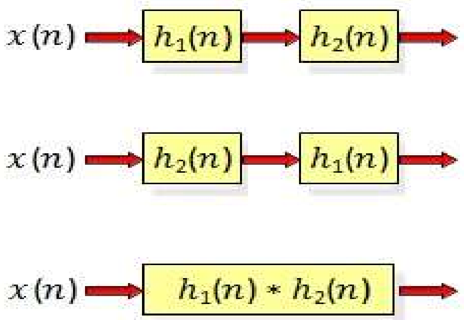
$$x(n)*h(n) = h(n)*x(n)$$

$$\sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

2- Associativity (Cascaded Connection)

Convolution is associative, meaning that convolution operations in series can be done in any order.

$$(x(n)*h(n))*g(n) = x(n)*(h(n)*g(n))$$



$y(n) = x(n) * [h_1(n) * h_2(n)]$

$y(n) = x(n) * [h_2(n) * h_1(n)]$

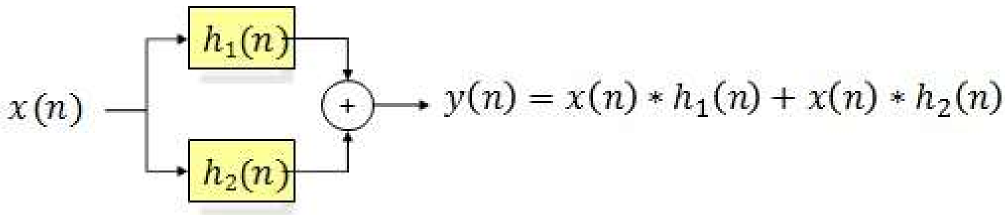
$y(n) = x(n) * h_1(n) * h_2(n)$

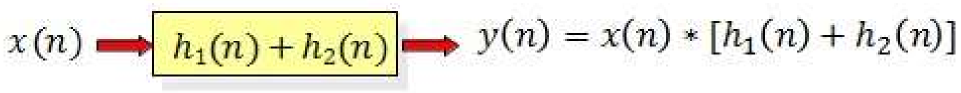
These systems are identical.

3- Distributivity (Parallel Connection)

Convolution is distributive over addition

$$x(n) * [h(n) + g(n)] = x(n) * h(n) + x(n) * g(n)$$





These two systems are identical.

The linear convolution can be performed by ***direct, graphical***, ***table lookup***, and ***matrix by vector*** methods.

Graphical Method:

The convolution sum of two sequences can be found by using the following steps:

Step 1. Obtain the reversed sequence h(- k).

Step 2. Shift h(- k) by n samples to get h(n - k). If $n \geq 0$, h(- k) will be shifted to the right by n samples; but if $n < 0$, h(- k) will be shifted to the left by n samples.

Step 3. Perform the convolution sum that is the sum of the products of two sequences $x(k)$ and h(n - k) to get y(n).

Step 4. Repeat steps 1 to 3 for the next convolution value $y(n)$.

Example: Find the convolution of the two sequences $x[n]$ and h[n] given by $x[n]$ = {**3**, 1, 2} and h[n] = {**3**, 2, 1}. The bold number shows where n=0. Using:

a. Direct method.

b. Graphical method.

sol.

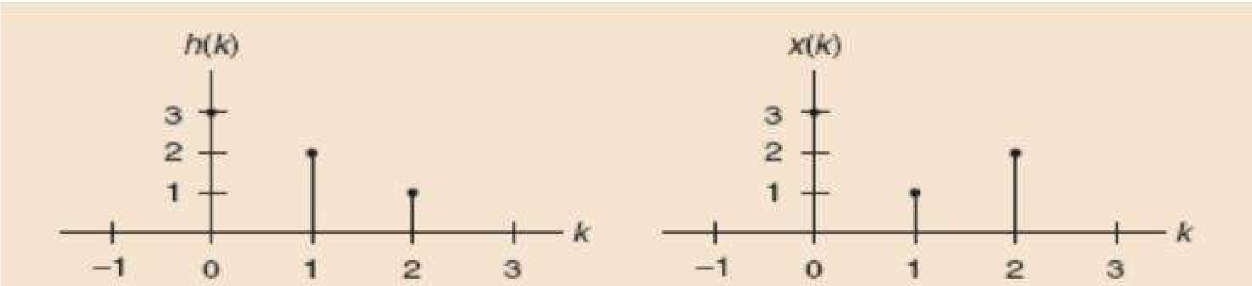
a. Using $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

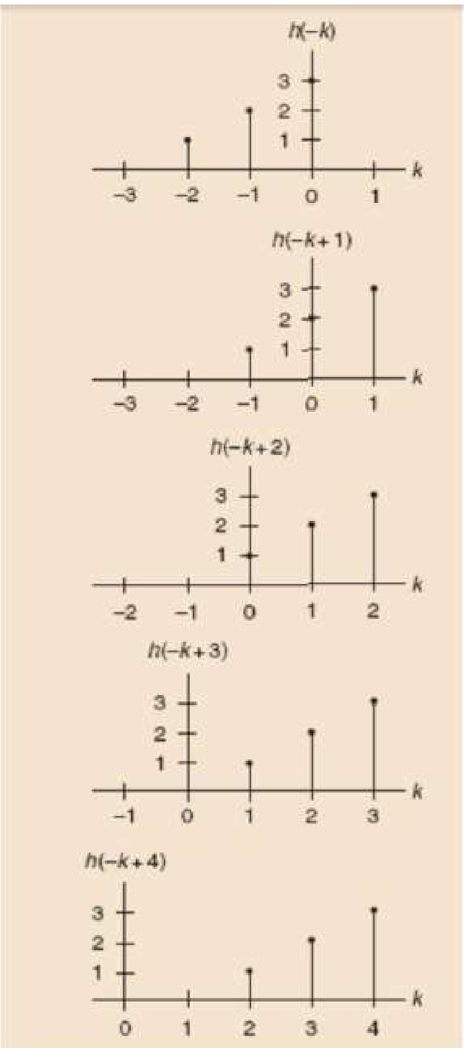
$x[n]$ = {**3**, 1, 2} and h[n] = {**3**, 2, 1}

Total number of samples N=N_1+N_2-1=3+3-1=5 samples.

$$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9,$$
$$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9,$$
$$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11,$$
$$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5.$$
$$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2,$$
$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.$$

b.





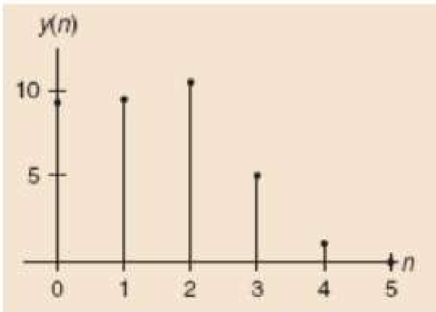
3x3=9

2x3+3x1=9

1x3+2x1+3x2=11

0x3+1x1+2x2+3x0=5

0x3+0x1+1x2+2x0=2



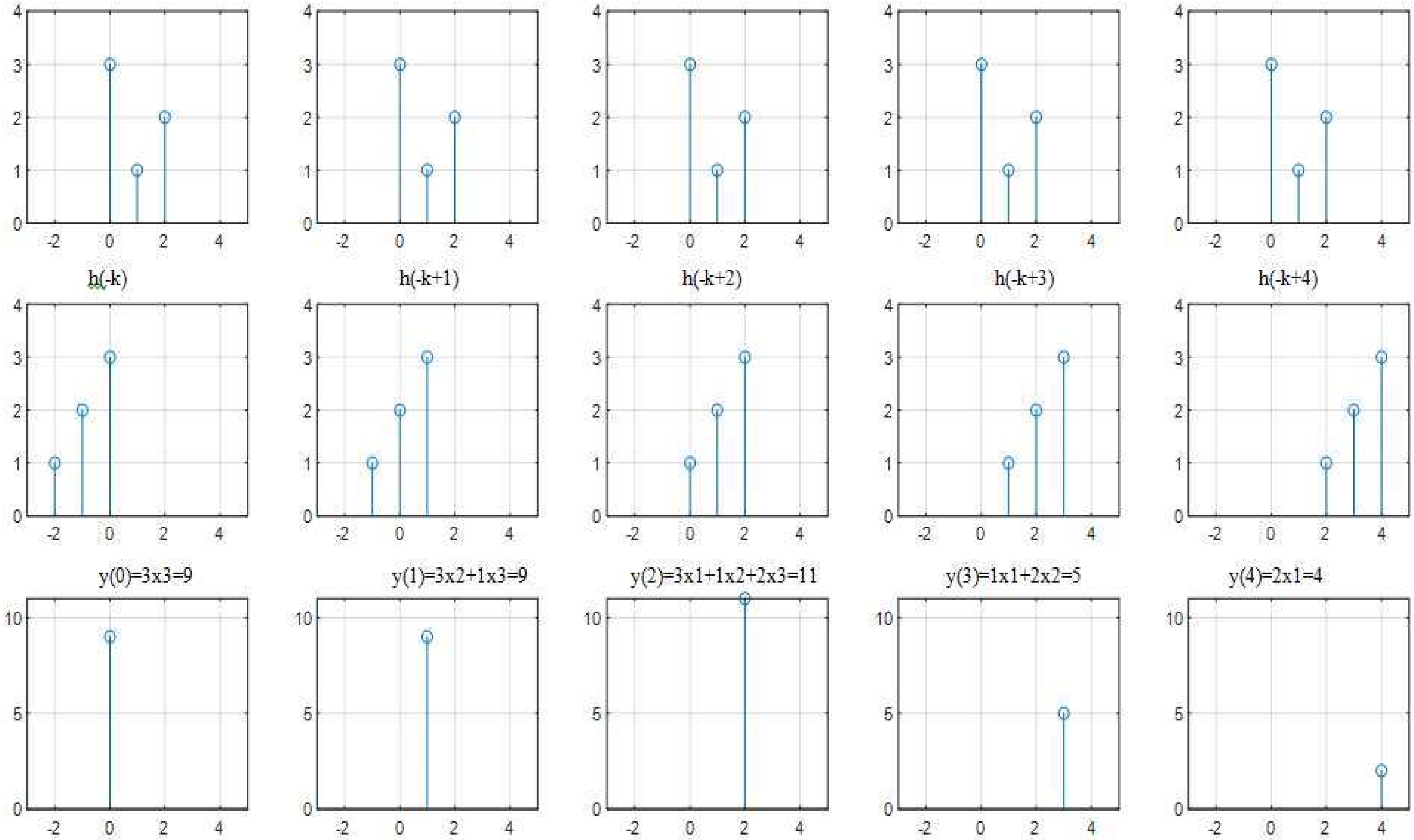


Table Lookup Method:

$y(0) = 9$

$y(1) = 9$

$y(2) = 11$

$y(3) = 5$

$y(4) = 2$

	y0)	y(1)	y(2)
	3	2	1
3	9	6	3
1	3	2	1
2	6	4	2

y(3)

y(4)

Matrix by Vector Method:

$$\boldsymbol{h} = [1,\ 2, -1,\ 1]$$
$$\boldsymbol{x} = [1,\ 1,\ 2,\ 1,\ 2,\ 2,\ 1,\ 1]$$

$$\boldsymbol{y} = H\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 5 \\ 3 \\ 7 \\ 4 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Linear Convolution and Circular Convolution:

Linear convolution:

$$x_1(n) \otimes x_2(n) = \sum_{k=-\infty}^{\infty} x_1(n-k)\, x_2(k) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Circular convolution:

$$x_1(n) \otimes_N x_2(n) = \sum_{k=0}^{N-1} x_1(\,(n-k) \mod N)\, x_2(k) = \sum_{k=0}^{N-1} x_1(k)\, x_2(\,(n-k) \mod N)$$

Note: If both $x_1(n)$ and $x_2(n)$ are of *finite length* N_1 and N_2 and defined on $[0\ N_1-1]$ and $[0\ N_2-1]$ respectively, the value of N needed so that circular and linear convolution are the same on [0 N-1] is: $\mathbf{N \geq N_1 + N_2 - 1}$

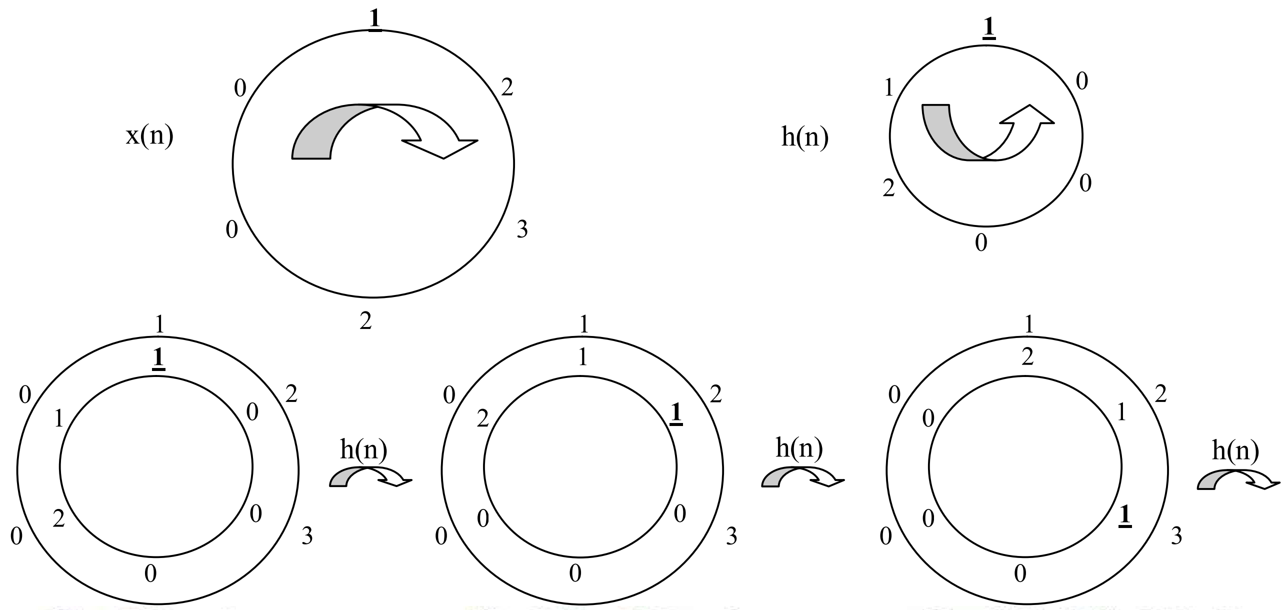
The circular convolution can be performed by ***direct, graphical***, and ***concentric circle*** methods.

Example: If $x(n)$ = [1 2 3 2], and h(n) = [1 1 2]. Find $y(n)$ such that linear and circular convolution are the same using concentric circle method.

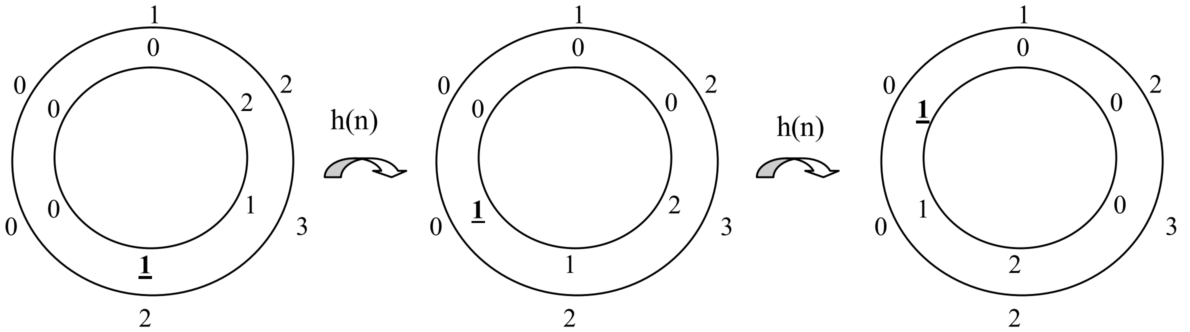
sol.

N = 4 + 3 – 1 = 6 , then $x(n)$ = [1 2 3 2 0 0] and h(n) = [1 1 2 0 0 0]

$x(n)$ is arranged in clockwise direction ,while h(n) is arranged in the opposite clockwise direction (bold numbers). Each time, only h(n) will be shifted with the ***clockwise direction*** to find $y(n)$. ***Note:*** the reference point is * and, the arrows represent multiplication process. Finally, addition process is performed.



y(0) = 1(1) = 1 y(1)= 1(1) + 2(1) = 3 y(2)= 2(1)+2(1)+3(1)=7



y(3)= 2(2)+3(1)+2(1)=9 y(4)= 3(2)+2(1)= 8 y(5)= 2(2)= 4

Example: Use graphical method to find circular convolution of $x_1(n)$=[1 2 2] and $x_2(n)$=[0 1 2 3].

sol.

Applying the equation of circular convolution

$$y(n)=\sum_{k=0}^{3} x_1(k)\ x_2(\ (n-k) \bmod 4\ )$$

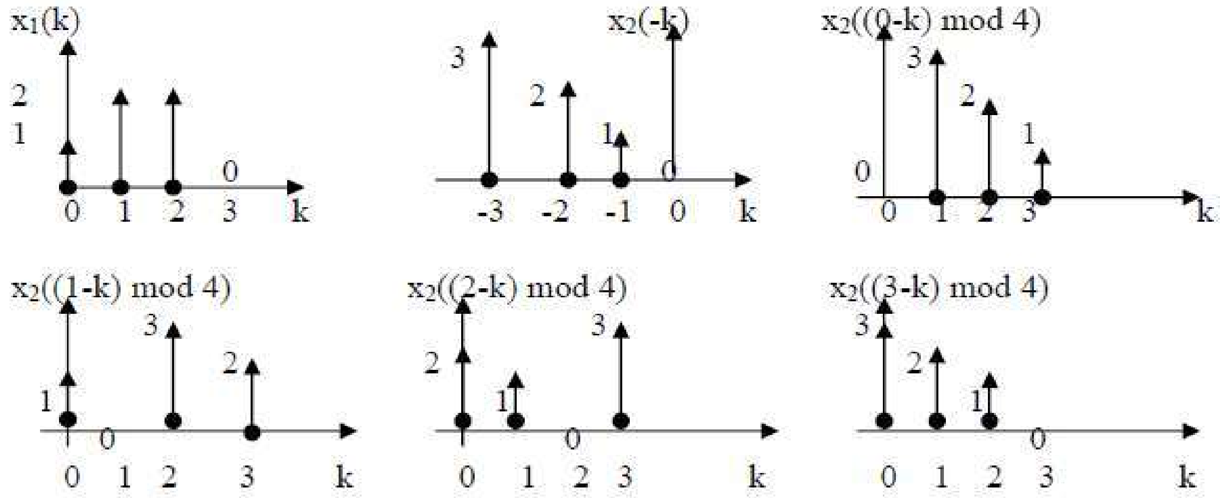
$$y(0)=\sum_{k=0}^{3} x_1(k)\ x_2(\ (-k) \bmod 4\ )$$

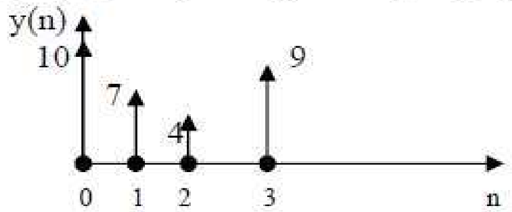
$$y(0) = x_1(0)\ x_2(-0 \bullet 4) + x_1(1)\ x_2(-1 \bullet 4) + x_1(2)\ x_2(-2 \bullet 4) + x_1(3)\ x_2(-3 \bullet 4)$$

$\bullet$ = mod addition

$y(0) = x_1(0)\ x_2(0) + x_1(1)\ x_2(3) + x_1(2)\ x_2(2) + x_1(3)\ x_2(1) = 1(0) + 2(3) + 2(2) + 0(1) = 10$

and so on





Deconvolution:

The digital Deconvolution can be performed by ***Iterative Approach***, ***Polynomial Approach***, and ***Graphical Method***. In the following subsection, the polynomial approach will be explained.

Polynomial Approach:

A long division process is applied between two polynomials. For causal system, the remainder is always *zero*.

If $y(n)$ = [12 10 14 6] and h(n) = [4 2]

Then $y = 12 + 10x + 14x^2 + 6x^3$, and $h = 4 + 2x$. Applying long division, we obtain

i/p = $3 + x + 3x^2$. Then $x(n)$ = [3 1 3]