Strength of Materials
2nd Stage (2021-2022)
Civil Engineering Department
Dr. Ali Hassan Ali

Syllabus

Chapter	Topic
0	Primary Introduction
1	Stress
2	Strain
3	Mechanical Properties of Materials
4	Axial Deformation
5	Shear Forces and Bending Moments diagrams
6	Torsion
7	Bending Stress in Beams
8	Shear Stress in Beams
9	Deflection of Beams
10	Statically Indeterminate Beams
11	Stress Transformation
12	Strain Transformation
13	Combined Loads

References

- **Philpot, T.A. (2017). *Mechanics of Materials: An Integrated Learning System*. 4th Ed. John Wiley & Sons, Inc.**

- **Hibbeler, R.C. (2011). *Mechanics of Materials*. 8th Ed. Pearson Prentice Hall.**

Outcomes of courses

- To understand the concept of stress and strain.
- To be familiar with the numerical equations of stress and strain and the background theory.
- To know how to draw a free body diagram for the structural members.
- To analyse different types and directions of loadings.
- To assess the mechanical properties of materials.
- To be familiar with the behaviour of beams in bending and torsion.
- To analyse the behaviour of beams in shear and deflection.
- To understand the classifications of beams, e.g., determinate, indeterminate…etc.
- To be familiar with the stress transformation by using Mohr’s circle.
- To analyse axially compressed members, i.e., columns.

Chapter 5 Shear Force and Bending Moment Diagrams

5.1 Shear and Moment in Beams

To determine the stresses created by applied loads, it is first necessary to determine the internal shear force V and the internal bending moment M acting in the beam at any point of interest. The general approach for finding V and M is illustrated in Figure 5.1. In this figure, a simply supported beam with an overhang is subjected to two concentrated loads P_1 and P_2 as well as to a uniformly distributed load w. A free-body diagram is obtained by cutting a section at a distance x from pin support A. The cutting plane exposes an internal shear force V and an internal bending moment M. If the beam is in equilibrium, then any portion of the beam that we consider must also be in equilibrium. Consequently, the free body with shear force V and bending moment M must satisfy equilibrium. Thus, equilibrium considerations can be used to determine values for V and M acting at location x.

Because of the applied loads, beams develop internal shear forces V and bending moments M that vary along the length of the beam. For us to properly analyze the stresses produced in a beam, we must determine V and M at all locations along the beam span. These results are typically plotted as a function of x in what is known as a **shear-force and bending-moment diagram**. This diagram summarizes all shear forces and bending moments along the beam, making it straightforward to identify the maximum and minimum values for both V and M. These extreme values are required for calculating the largest stresses.

Chapter 5 Shear Force and Bending Moment Diagrams

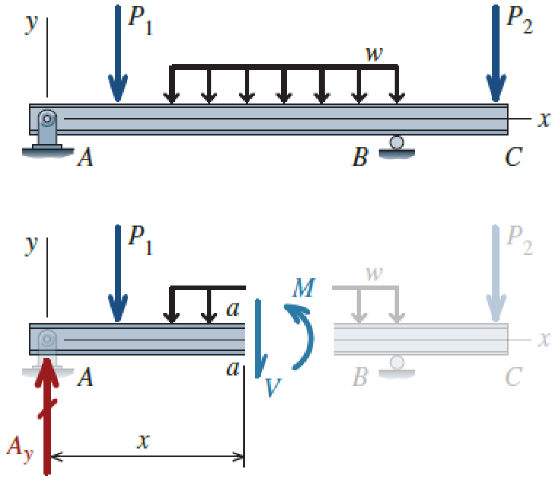


FIGURE 5.1 Method of sections applied to beams.

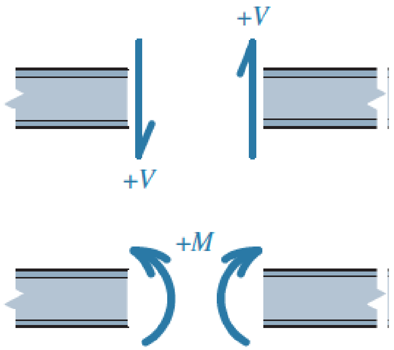


FIGURE 5.2 Sign conventions for internal shear force V and bending moment M.

Since many different loads may act on a beam, functions describing the variation of $V(x)$ and $M(x)$ may not be continuous throughout the entire span of the beam. Because of this consideration, shear-force and bending-moment functions must be determined for a number of intervals along the beam. In general, intervals are delineated by

(a) the locations of concentrated loads, concentrated moments, and support reactions or

(b) the span of distributed loads.

The examples that follow illustrate how shear-force and bending-moment functions can be derived for various intervals by the use of equilibrium considerations.

Sign Conventions for Shear-Force and Bending-Moment Diagrams. Before deriving internal shear-force and bending-moment functions, we must develop consistent sign conventions. These sign conventions are illustrated in Figure 5.2.

Chapter 5 Shear Force and Bending Moment Diagrams

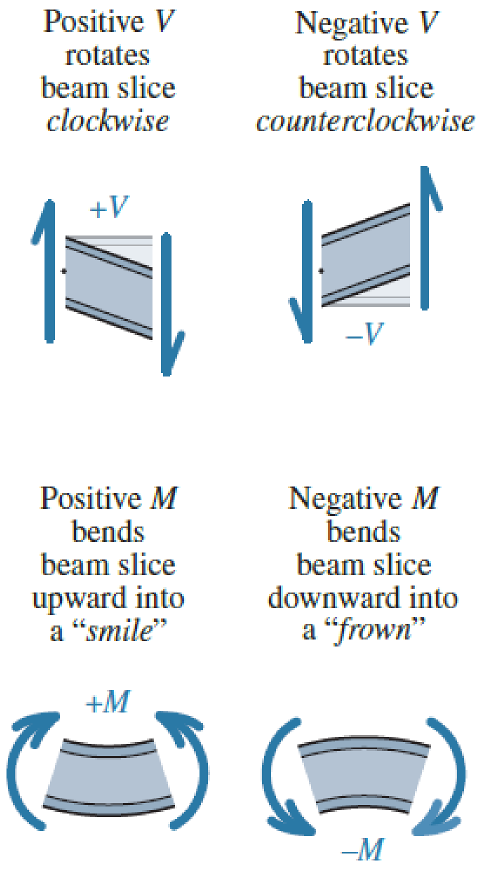


FIGURE 5.3 Sign conventions for V and M shown on beam slice.

A positive internal shear force V

- acts downward on the right-hand face of a beam.
- acts upward on the left-hand face of a beam.

A positive internal bending moment M

- acts counterclockwise on the right-hand face of a beam.
- acts clockwise on the left-hand face of a beam.

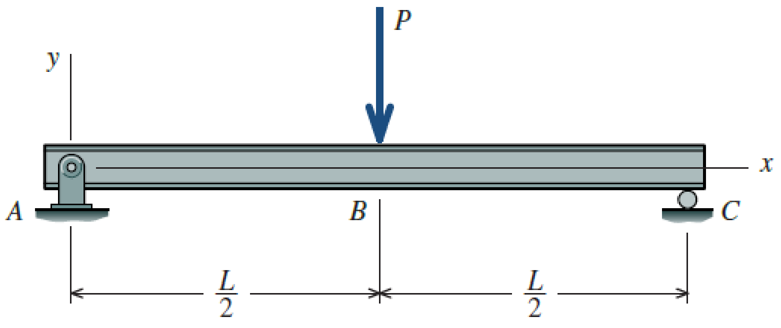
These sign conventions can also be expressed by the directions of V and M that act on a small slice of the beam. This alternative statement of the V and M sign conventions is illustrated in Figure 5.3.

A positive internal shear force V causes a beam element to rotate clockwise.
A positive internal bending moment M bends a beam element concave upward.

A shear-force and bending-moment diagram will be created for each beam by plotting shear-force and bending-moment functions. To ensure consistency among the functions, it is very important that these sign conventions be observed.

EXAMPLE 5.1

Draw the shear-force and bending-moment diagrams for the simply supported beam shown.



Plan the Solution

First, determine the reaction forces at pin A and roller C. Then, consider two intervals along the beam span: between A and B, and between B and C. Cut a section in each interval and draw the appropriate free-body diagram (FBD), showing the unknown internal shear force V and internal bending moment M acting on the exposed surface. Write the equilibrium equations for each FBD, and solve them for functions describing the variation of V and M with location x along the span. Plot these functions to complete the shear-force and bending-moment diagrams.

SOLUTION

Support Reactions

Since this beam is symmetrically supported and symmetrically loaded, the reaction forces must also be symmetric. Therefore, each support exerts an upward force equal to $P/2$. Because no applied loads act in the x direction, the horizontal reaction force at pin support A is zero.

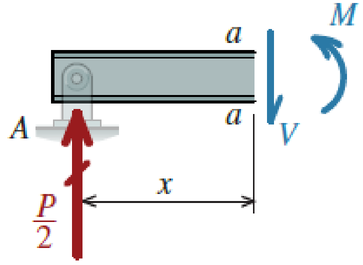
Shear and Moment Functions

In general, the beam will be sectioned at an arbitrary distance x from pin support A and all forces acting on the free body will be shown, including the unknown internal shear force V and internal bending moment M acting on the exposed surface.

Interval $0 \le x < L/2$: The beam is cut on section a–a, which is located an arbitrary distance x from pin support A. An unknown shear force V and an unknown bending moment M are shown on the exposed surface of the beam. Note that positive directions are assumed for both V and M. (See Figure 5.2 for sign conventions.)

Since no forces act in the x direction, the equilibrium equation $\Sigma F_x = 0$ is trivial. The sum of forces in the vertical direction yields the desired function for V:

$$\boxed{\Sigma F_y = \frac{P}{2} - V = 0 \qquad \therefore V = \frac{P}{2}} \qquad \text{(a)}$$

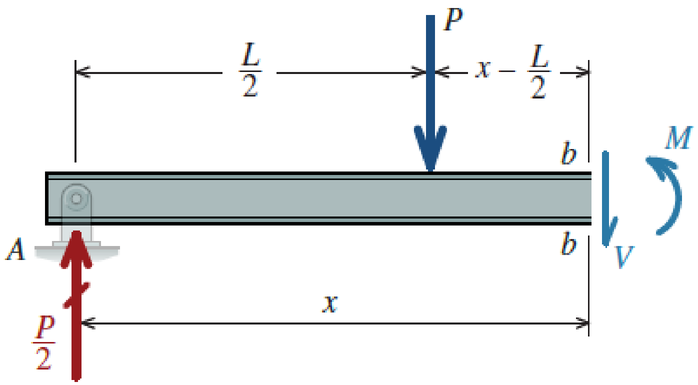


The sum of moments about section a–a gives the desired function for M:

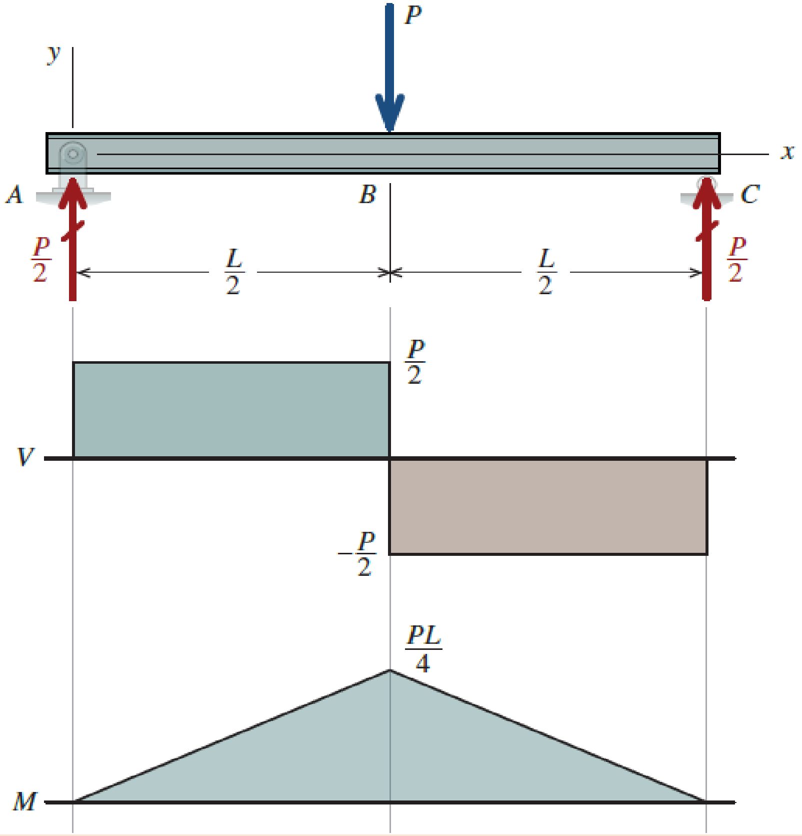
$$\boxed{\Sigma M_{a-a} = -\frac{P}{2}x + M = 0 \qquad \therefore M = \frac{P}{2}x} \qquad \text{(b)}$$

These results show that the internal shear force V is constant and the internal bending moment M varies linearly in the interval $0 \le x < L/2$.

Interval $L/2 \le x < L$: The beam is cut on section b–b, which is located an arbitrary distance x from pin support A. Section b–b, however, is located beyond B, where the concentrated load P is applied. As before, an unknown shear force V and an unknown bending moment M are shown on the exposed surface of the beam and positive directions are assumed for both V and M.



Chapter 5 Shear Force and Bending Moment Diagrams



The sum of forces in the vertical direction yields the desired function for V:

$$\Sigma F_y = \frac{P}{2} - P - V = 0 \qquad \therefore V = -\frac{P}{2} \tag{c}$$

The equilibrium equation for the sum of moments about section b–b gives the desired function for M:

$$\Sigma M_{b-b} = P\left(x - \frac{L}{2}\right) - \frac{P}{2}x + M = 0$$
$$\therefore M = -\frac{P}{2}x + \frac{PL}{2} \tag{d}$$

Again, the internal shear force V is constant and the internal bending moment M varies linearly in the interval $L/2 \le x < L$.

Plot the Functions

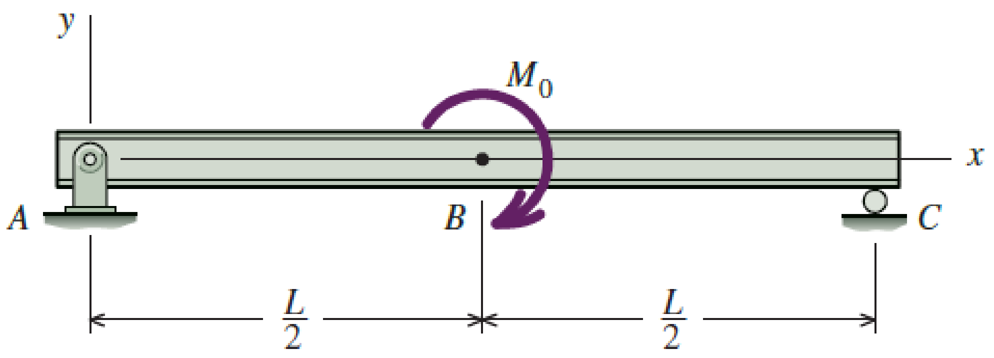
Plot the functions given in Equations (a) and (b) for the interval $0 \le x < L/2$, and the functions defined by Equations (c) and (d) for the interval $L/2 \le x < L$, to create the shear-force and bending-moment diagram shown.

The maximum internal shear force is $V_{max} = \pm P/2$. The maximum internal bending moment is $M_{max} = PL/4$, and it occurs at $x = L/2$.

Notice that the concentrated load causes a discontinuity at its point of application. In other words, the shear-force diagram "jumps" by an amount equal to the magnitude of the concentrated load. The jump in this case is downward, which is the same direction as that of the concentrated load P.

Chapter 5 Shear Force and Bending Moment Diagrams

EXAMPLE 5.2



Draw the shear-force and bending-moment diagrams for the simple beam shown.

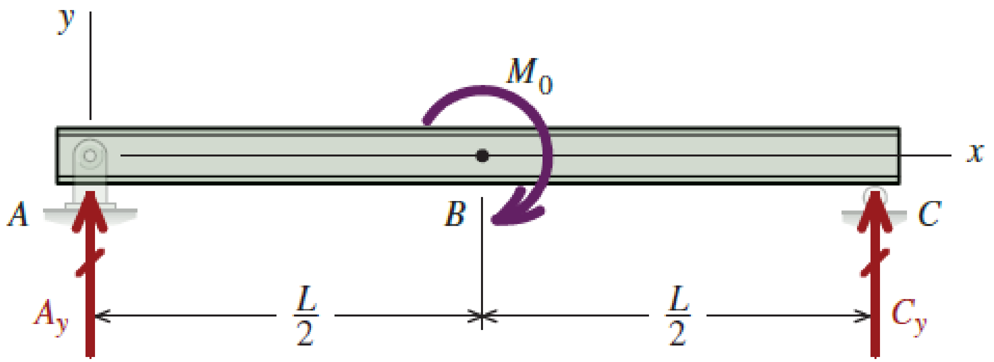
Plan the Solution

The solution process outlined in Example 5.1 will be used to derive *V* and *M* functions for this beam.

SOLUTION

Support Reactions

An FBD of the beam is shown. The equilibrium equations are as follows:



$$\Sigma F_y = A_y + C_y = 0$$

$$\Sigma M_A = -M_0 + C_y L = 0$$

From these equations, the beam reactions are

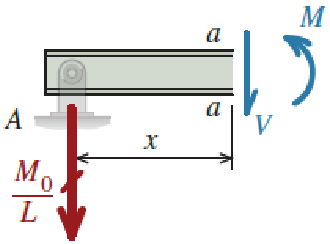
$$C_y = \frac{M_0}{L} \quad \text{and} \quad A_y = -\frac{M_0}{L}$$

The negative value for A_y indicates that this reaction force acts opposite to the direction assumed initially. Subsequent free-body diagrams will be revised to show A_y acting downward.

Interval $0 \le x < L/2$: Section the beam at an arbitrary distance x between A and B. Show the unknown shear force V and the unknown bending moment M on the exposed surface of the beam. Assume positive directions for both V and M, according to the sign convention given in Figure 5.2.

The sum of forces in the vertical direction yields the desired function for V:

$$\Sigma F_y = -\frac{M_0}{L} - V = 0 \qquad \therefore V = -\frac{M_0}{L} \tag{a}$$



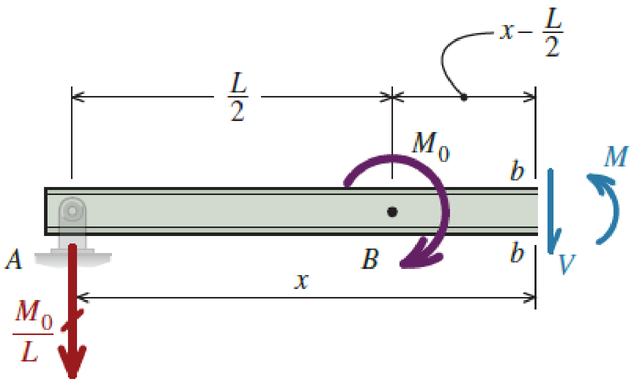
The sum of moments about section a–a gives the desired function for M:

$$\Sigma M_{a-a} = \frac{M_0}{L}x + M = 0 \qquad \therefore M = -\frac{M_0}{L}x \tag{b}$$

These results indicate that the internal shear force V is constant and the internal bending moment M varies linearly in the interval $0 \le x < L/2$.

Interval $L/2 \le x < L$: The beam is cut on section b–b, which is at an arbitrary location between B and C. The sum of forces in the vertical direction yields the desired function for V:

$$\Sigma F_y = -\frac{M_0}{L} - V = 0 \qquad \therefore V = -\frac{M_0}{L} \tag{c}$$



The equilibrium equation for the sum of moments about section b–b gives the desired function for M:

Chapter 5 Shear Force and Bending Moment Diagrams

$$\Sigma M_{b-b} = \frac{M_0}{L}x - M_0 + M = 0$$
$$\therefore M = M_0 - \frac{M_0}{L}x \qquad \text{(d)}$$

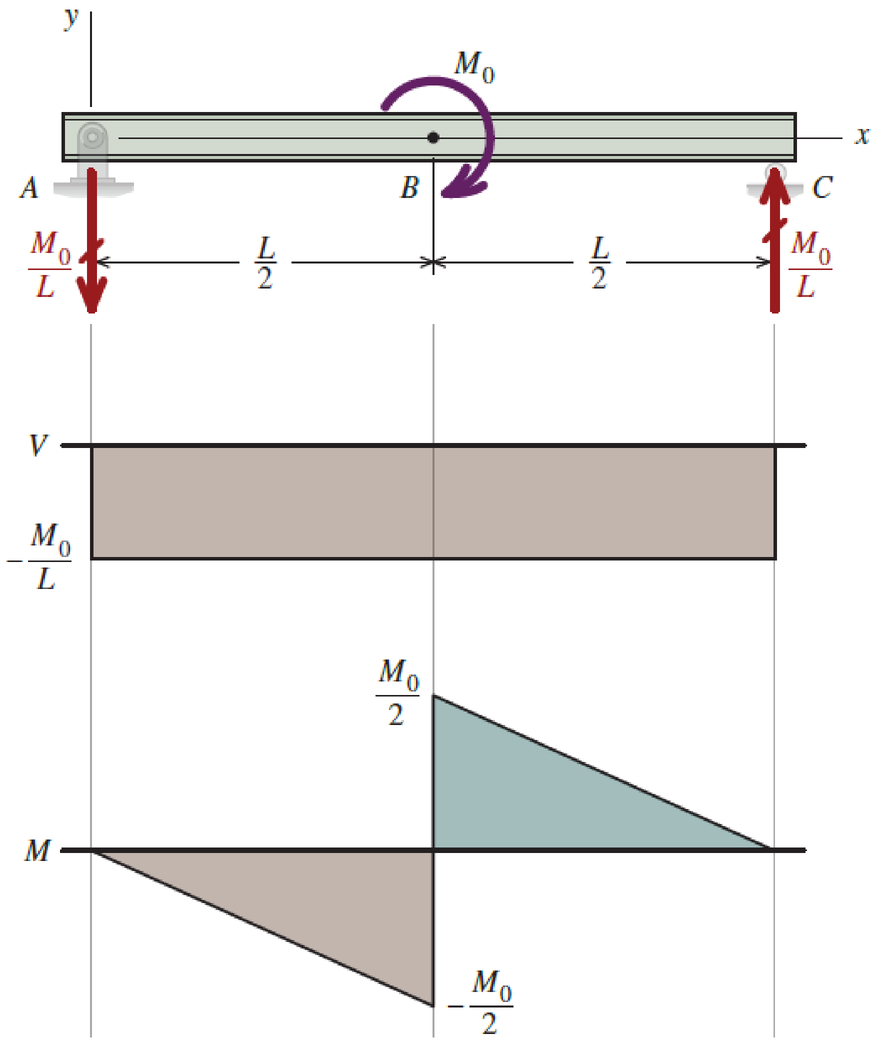
Again, the internal shear force V is constant and the internal bending moment M varies linearly in the interval $L/2 \leq x < L$.

Plot the Functions

Plot the functions given in Equations (a) and (b) for the interval $0 \leq x < L/2$, and the functions defined by Equations (c) and (d) for the interval $L/2 \leq x < L$, to create the shear-force and bending-moment diagram shown.

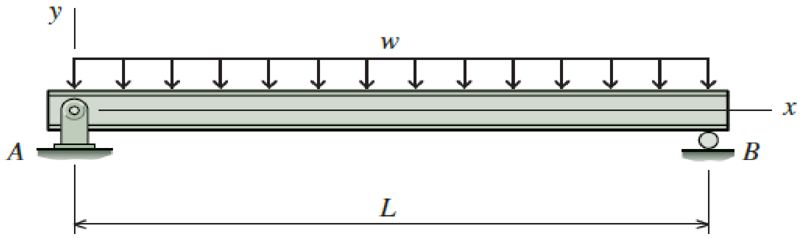
The maximum internal shear force is $V_{max} = -M_0/L$. The maximum internal bending moment is $M_{max} = \pm M_0/2$, and it occurs at $x = L/2$.

Notice that the concentrated moment does not affect the shear-force diagram at B. It does, however, create a discontinuity in the bending-moment diagram at the point of application of the concentrated moment: The bending-moment diagram "jumps" by an amount equal to the magnitude of the concentrated moment. *The clockwise concentrated external moment* M_0 *causes the bending-moment diagram to "jump" upward* at B by an amount equal to the magnitude of the concentrated moment.



Chapter 5 Shear Force and Bending Moment Diagrams

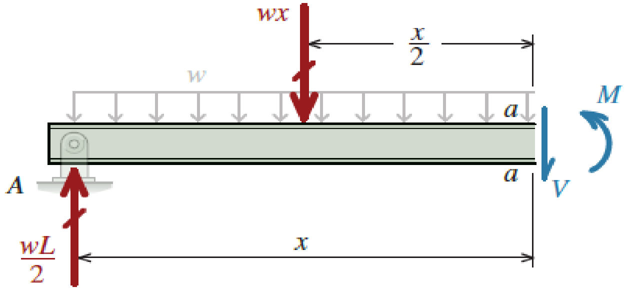
EXAMPLE 5.3



Draw the shear-force and bending-moment diagrams for the simply supported beam shown.

Plan the Solution

After the support reactions at pin A and roller B have been determined, cut a section at an arbitrary location x and draw the corresponding free-body diagram (FBD), showing the unknown internal shear force V and internal bending moment M acting on the cut surface. Develop the equilibrium equations for the FBD, and solve these two equations for functions describing the variation of V and M with location x along the span. Plot these functions to complete the shear-force and bending-moment diagrams.



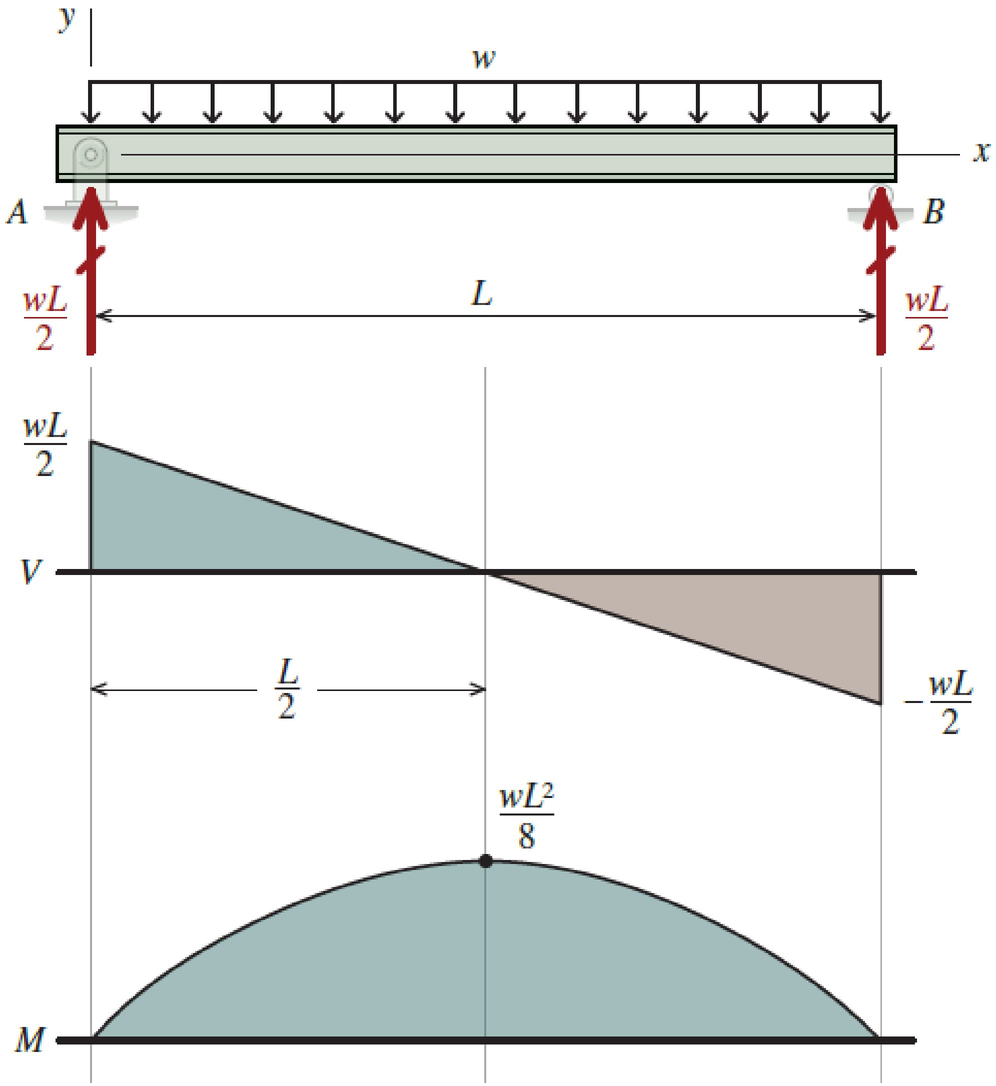
SOLUTION

Support Reactions

Since this beam is symmetrically supported and symmetrically loaded, the reaction forces must also be symmetric. The total load acting on the beam is wL; therefore, each support exerts an upward force equal to half of this load: $wL/2$.

Interval $0 \leq x < L$: Section the beam at an arbitrary distance x between A and B. **Make sure that the original distributed load w is shown on the FBD at the outset.** Show the unknown shear force V and the unknown bending moment M on the exposed surface of the beam. Assume positive directions for both V and M, according to the sign convention given in F Figure 5.2. 'he resultant of the uniformly distributed load w acting on a beam of length x is equal to wx. The resultant force acts at the middle of this loading (i.e., at the centroid of the rectangle that has width x and height w). The sum of forces in the vertical direction yields the

Chapter 5 Shear Force and Bending Moment Diagrams



desired function for V:

$$\Sigma F_y = \frac{wL}{2} - wx - V = 0$$
$$\therefore V = \frac{wL}{2} - wx = w\left(\frac{L}{2} - x\right) \qquad \text{(a)}$$

The shear-force function is linear (i.e., it is a first-order function), and the slope of this line is equal to $-w$ (which is the intensity of the distributed load).

The sum of moments about section a–a gives the desired function for M:

$$\Sigma M_{a-a} = -\frac{wL}{2}x + wx\frac{x}{2} + M = 0$$
$$\therefore M = \frac{wL}{2}x - \frac{wx^2}{2} = \frac{wx}{2}(L - x) \qquad \text{(b)}$$

The internal bending moment M is a quadratic function (i.e., a second-order function).

Plot the Functions

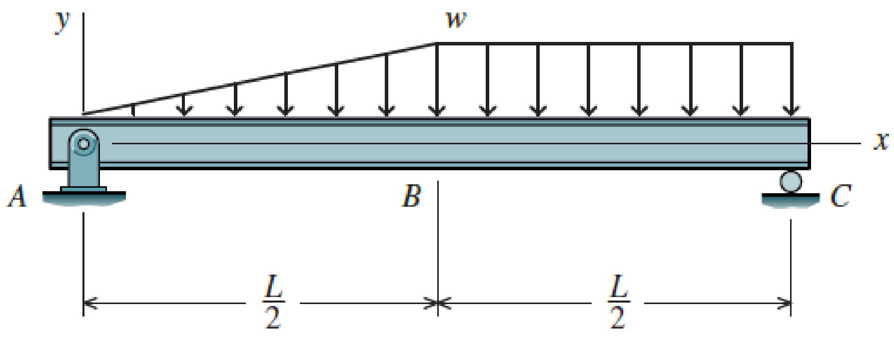
Plot the functions given in Equations (a) and (b) to create the shear-force and bending-moment diagram shown.

The maximum internal shear force is $V_{max} = \pm wL/2$, and it is found at A and B. The maximum internal bending moment is $M_{max} = wL^2/8$, and it occurs at $x = L/2$.

Note that the maximum bending moment occurs at a location where the shear force V is equal to zero.

EXAMPLE 5.4

Draw the shear-force and bending-moment diagrams for the simply supported beam shown.



Plan the Solution

After determining the support reactions at pin A and roller C, cut sections between A and B (in the linearly distributed loading) and between B and C (in the uniformly distributed loading). Draw the appropriate free-body diagrams, work out the equilibrium equations for each FBD, and solve the equations for functions describing the variation of V and M with location x along the span. Plot these functions to complete the shear-force and bending-moment diagrams.

SOLUTION

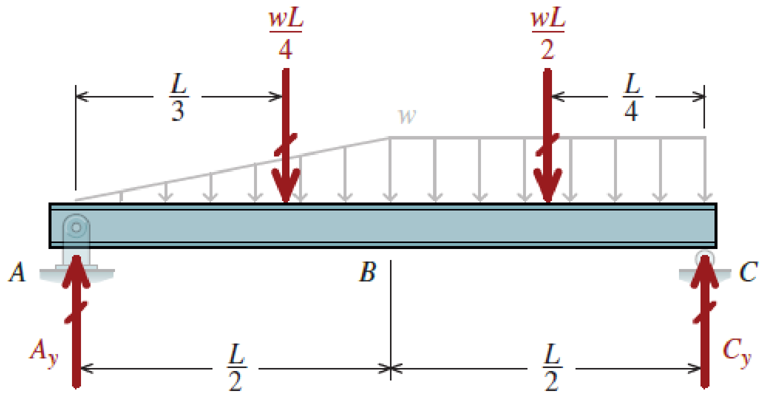
Support Reactions

The FBD for the entire beam is shown. The resultant force of the linearly distributed loading is equal to the area of the triangle that has base $L/2$ and height w:

$$\frac{1}{2}\left(\frac{L}{2}\right)w = \frac{wL}{4}$$

The resultant force acts at the centroid of this triangle, which is located at two-thirds of the base dimension, measured from the point of the triangle:

$$\frac{2}{3}\left(\frac{L}{2}\right) = \frac{L}{3}$$



Equilibrium equations for the beam can be written as

$$\Sigma F_y = A_y + C_y - \frac{wL}{4} - \frac{wL}{2} = 0 \quad \text{and} \quad \Sigma M_A = C_y L - \frac{wL}{4}\left(\frac{L}{3}\right) - \frac{wL}{2}\left(\frac{3L}{4}\right) = 0$$

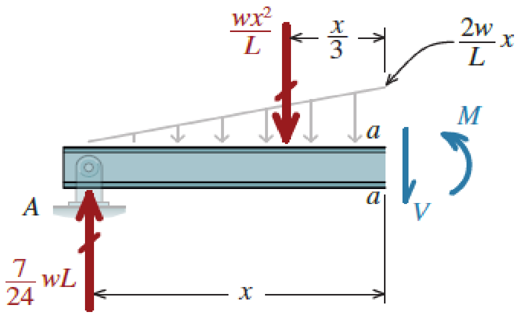
which can be solved to determine the reaction forces:

$$A_y = \frac{7}{24}wL \quad \text{and} \quad C_y = \frac{11}{24}wL$$

Interval $0 \le x < L/2$: Section the beam at an arbitrary distance x between A and B. **Make sure that you replace the original linearly distributed load on the FBD.** A new resultant force for the linearly distributed load must be derived specifically for this FBD.

The slope of the linearly distributed load is equal to $w/(L/2) = 2w/L$. Accordingly, the height of the triangular loading at section a–a is equal to the product of this slope and the distance x—that is, $(2w/L)x$. Therefore, the resultant of the linearly distributed

Chapter 5 Shear Force and Bending Moment Diagrams

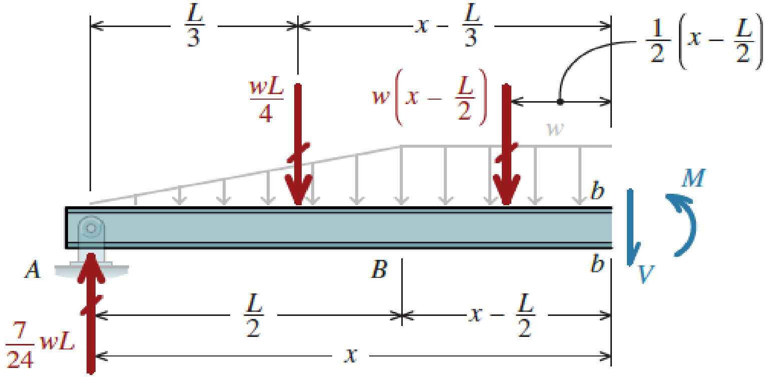


load that is acting on this FBD is $(1/2)x\,[(2w/L)x] = (wx^2/L)$, and it acts at a distance $x/3$ from section a–a.

V and M functions applicable for the interval $0 \le x < L/2$ can be derived from the equilibrium equations for the FBD:

$$\Sigma F_y = \frac{7}{24}wL - \frac{wx^2}{L} - V = 0 \qquad \therefore\; V = -\frac{wx^2}{L} + \frac{7}{24}wL \tag{a}$$

$$\Sigma M_{a-a} = -\frac{7}{24}wLx + \frac{wx^2}{L}\left(\frac{x}{3}\right) + M = 0 \qquad \therefore\; M = -\frac{wx^3}{3L} + \frac{7}{24}wLx \tag{b}$$



The shear-force function is quadratic (i.e., a second-order function), and the bending-moment function is cubic (i.e., a third-order function).

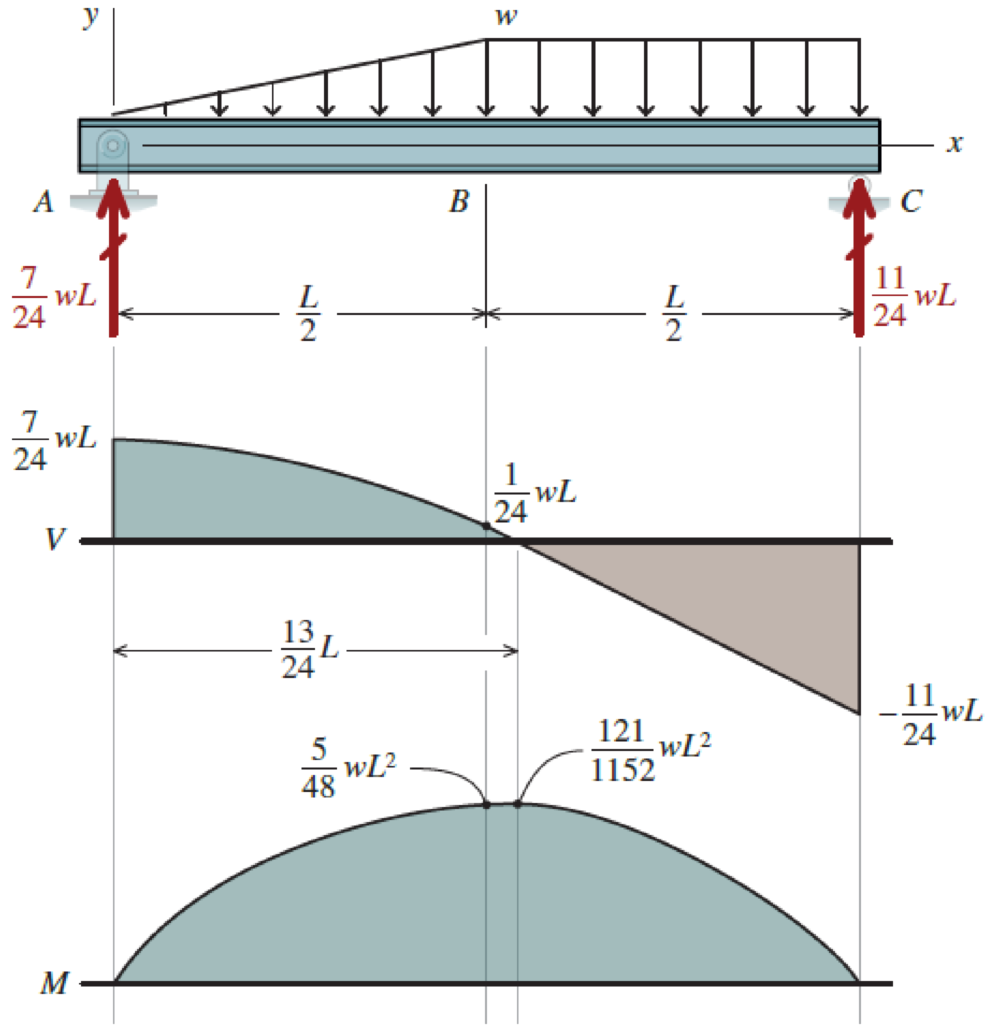
Interval $L/2 \le x < L$: Section the beam at an arbitrary distance x between B and C. **Make sure that you replace the original distributed loads on the FBD before deriving the V and M functions.**

On the basis of this FBD, the equilibrium equations can be written as follows:

$$\Sigma F_y = \frac{7}{24}wL - \frac{wL}{4} - w\left(x - \frac{L}{2}\right) - V = 0 \qquad \therefore\; V = \frac{7}{24}wL - \frac{wL}{4} - w\left(x - \frac{L}{2}\right) \tag{c}$$

$$\Sigma M_{a-a} = -\frac{7}{24}wLx + \frac{wL}{4}\left(x - \frac{L}{3}\right) + w\left(x - \frac{L}{2}\right)\frac{1}{2}\left(x - \frac{L}{2}\right) + M = 0$$
$$\therefore\; M = \frac{7}{24}wLx - \frac{wL}{4}\left(x - \frac{L}{3}\right) - \frac{w}{2}\left(x - \frac{L}{2}\right)^2 \tag{d}$$

Chapter 5 Shear Force and Bending Moment Diagrams



These equations can be simplified to

$$V = w\left(\frac{13}{24}L - x\right) \quad \text{and}$$

$$M = \frac{w}{24}(-12x^2 + 13Lx - L^2)$$

The shear-force function is linear (i.e., a first-order function), and the bending-moment function is quadratic (i.e., a second-order function), between B and C.

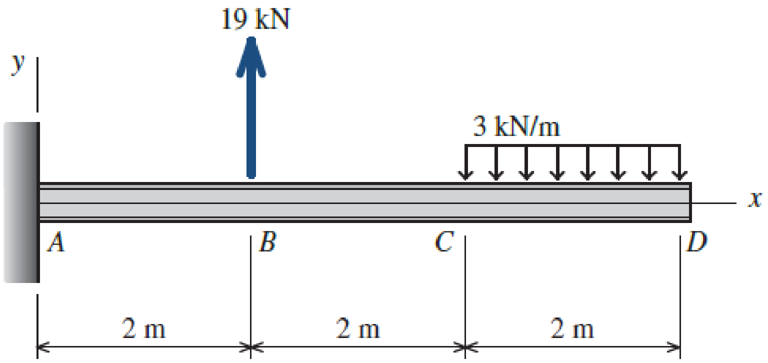
Plot the Functions

Plot the V and M functions to create the shear-force and bending-moment diagram shown.

Notice that the maximum bending moment occurs at a location where the shear force V is equal to zero.

EXAMPLE 5.5

Draw the shear-force and bending-moment diagrams for the cantilever beam shown.



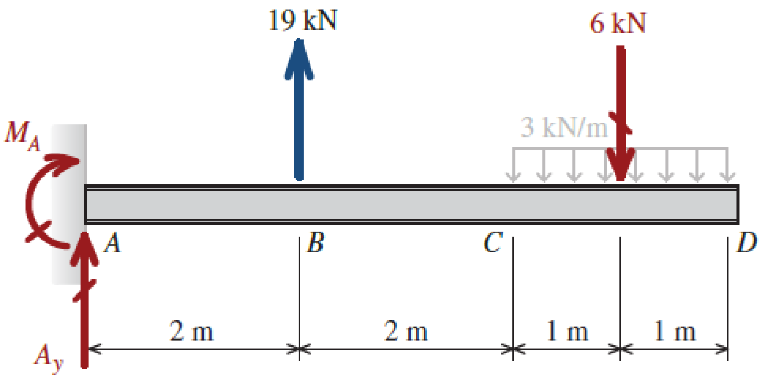
Plan the Solution

Initially, determine the reactions at fixed support A. Three sections will need to be considered, one each for the intervals between AB, BC, and CD. For each section, draw the appropriate free-body diagram, develop the equilibrium equations, and solve the equations for functions describing the variation of V and M with location x along the span. Plot these functions to complete the shear-force and bending-moment diagrams.

SOLUTION

Support Reactions

An FBD of the entire beam is shown. Since no forces act in the x direction, the reaction force $A_x = 0$ will be omitted from the FBD. The nontrivial equilibrium equations are as follows:



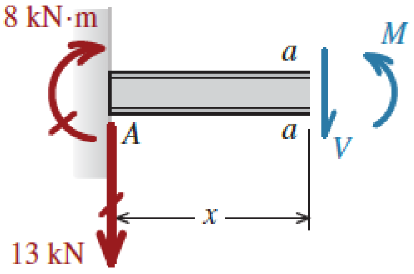
$$\Sigma F_y = A_y + 19 \text{ kN} - 6 \text{ kN} = 0$$

$$\Sigma M_A = -M_A + (19 \text{ kN})(2 \text{ m}) - (6 \text{ kN})(5 \text{ m}) = 0$$

From these equations, the beam reactions are found to be

$$A_y = -13 \text{ kN} \quad \text{and} \quad M_A = 8 \text{ kN}\cdot\text{m}$$

Since A_y is negative, it really acts downward. The correct direction of this reaction force will be shown in subsequent free-body diagrams.

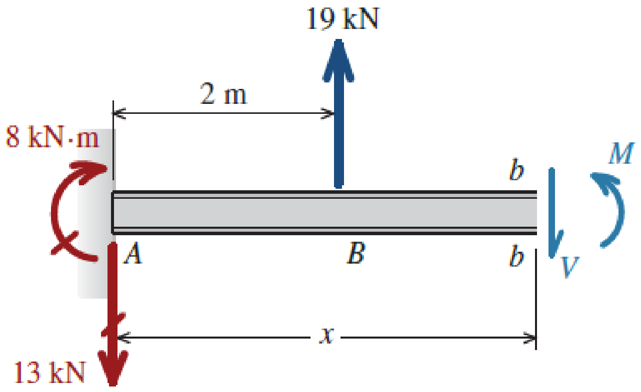


Interval $0 \le x < 2$ m: Section the beam at an arbitrary distance x between A and B. The FBD for this section is shown. From the equilibrium equations for this FBD, determine the desired functions for V and M:

$$\Sigma F_y = -13\text{ kN} - V = 0 \qquad \therefore V = -13\text{ kN} \tag{a}$$

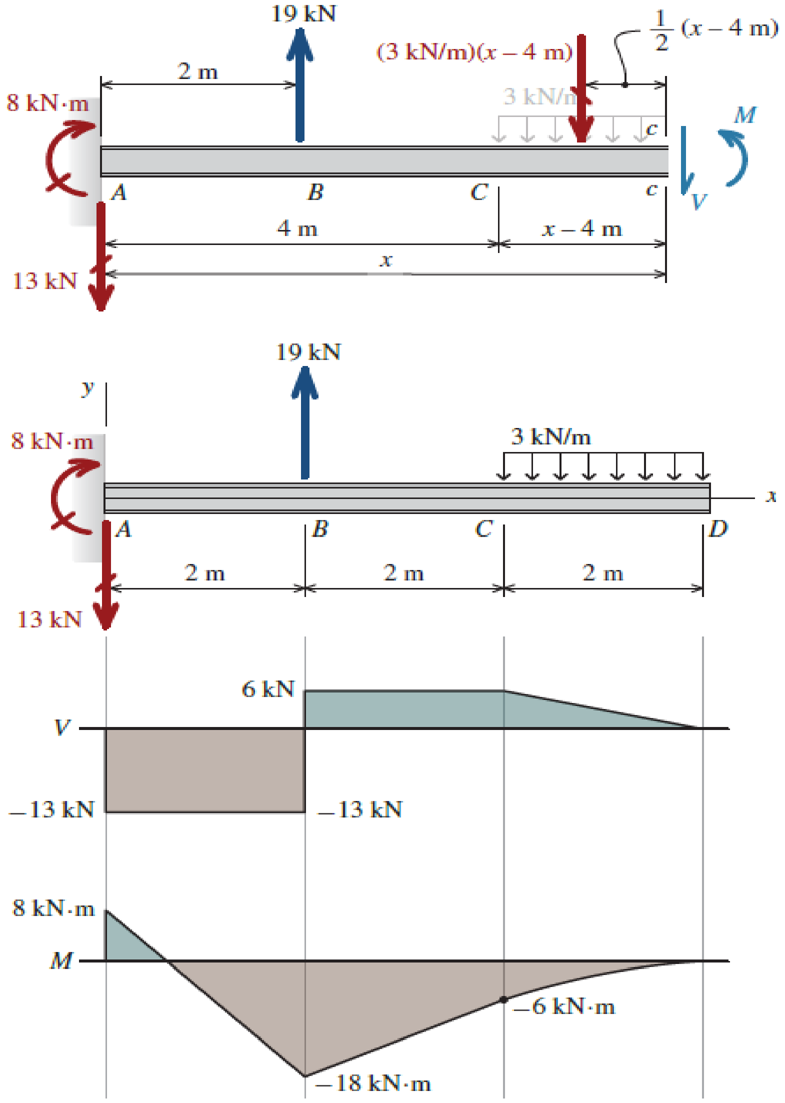
$$\Sigma M_{a-a} = (13\text{ kN})x - 8\text{ kN}\cdot\text{m} + M = 0 \qquad \therefore M = -(13\text{ kN})x + 8\text{ kN}\cdot\text{m} \tag{b}$$

Interval $2\text{ m} \le x < 4$ m: From a section cut between B and C, determine the desired shear and moment functions:



$$\Sigma F_y = -13\text{ kN} + 19\text{ kN} - V = 0 \qquad \therefore V = 6\text{ kN} \tag{c}$$

$$\Sigma M_{b-b} = (13\text{ kN})x - (19\text{ kN})(x - 2\text{ m}) - 8\text{ kN}\cdot\text{m} + M = 0 \qquad \therefore M = (6\text{ kN})x - 30\text{ kN}\cdot\text{m} \tag{d}$$



Interval 4 m ≤ x < 6 m: From a section cut between C and D, determine the desired shear and moment functions:

$$\begin{aligned}\Sigma F_y &= -13\text{ kN} + 19\text{ kN} \\ &\quad - (3\text{ kN/m})(x - 4\text{ m}) - V = 0 \\ \therefore V &= (3\text{ kN/m})x + 18\text{ kN}\end{aligned} \tag{e}$$

$$\begin{aligned}\Sigma M_{c-c} &= (13\text{ kN})x - (19\text{ kN})(x - 2\text{ m}) \\ &\quad + (3\text{ kN/m})(x - 4\text{ m})\frac{(x-4\text{ m})}{2} \\ &\quad - 8\text{ kN}\cdot\text{m} + M = 0 \\ \therefore M &= -(1.5\text{ kN/m})x^2 + (18\text{ kN})x - 54\text{ kN}\cdot\text{m}\end{aligned} \tag{f}$$

Plot the Functions

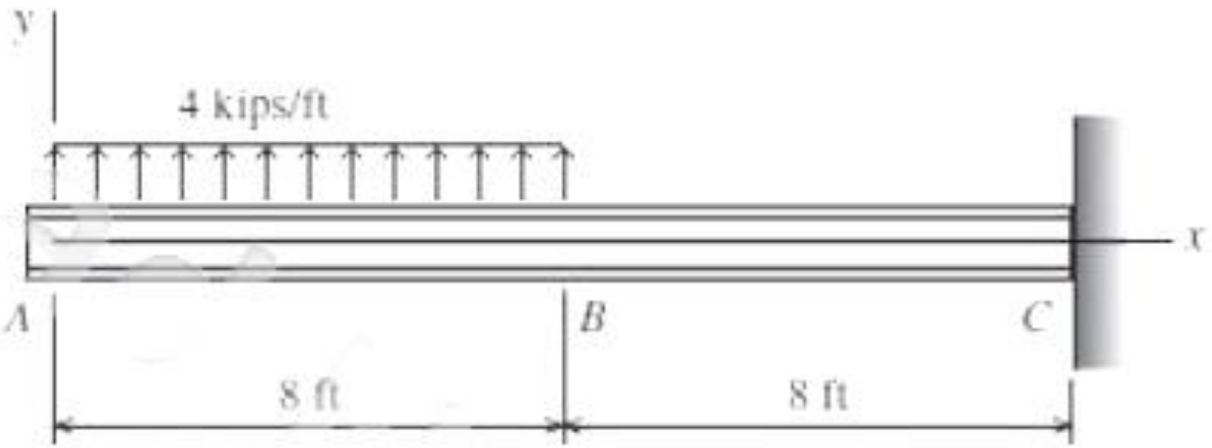
Plot the functions given in Equations (a) through (f) to construct the shear-force and bending-moment diagram shown.

Notice that the shear-force diagram is constant in intervals *AB* and *BC* (i.e., it is a zero-order function) and linear in interval *CD* (i.e., it is a first-order function). The bending-moment function is linear in intervals *AB* and *BC* (i.e., it is a first-order function) and quadratic in interval *CD* (i.e., it is a second-order function).

H.W.27 For the cantilever beam and loading shown in Figure .

(a) derive equations for the shear force V and the bending moment M for any location in the beam. (Place the origin at point A.)

(b) plot the shear-force and bending-moment diagrams for the beam, using the derived functions.



FIGURE

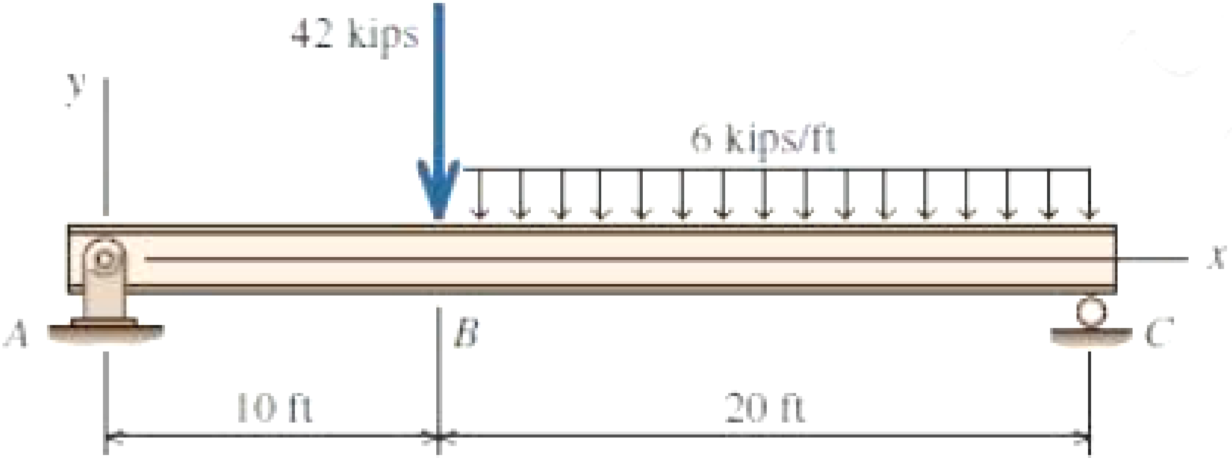
Chapter 5 Shear Force and Bending Moment Diagrams

H.W.28 For the simply supported beam subjected to the loading shown in Figure .

(a) derive equations for the shear force V and the bending moment M for any location in the beam. (Place the origin at point A.)

(b) plot the shear-force and bending-moment diagrams for the beam, using the derived functions.

(c) report the maximum bending moment and its location.

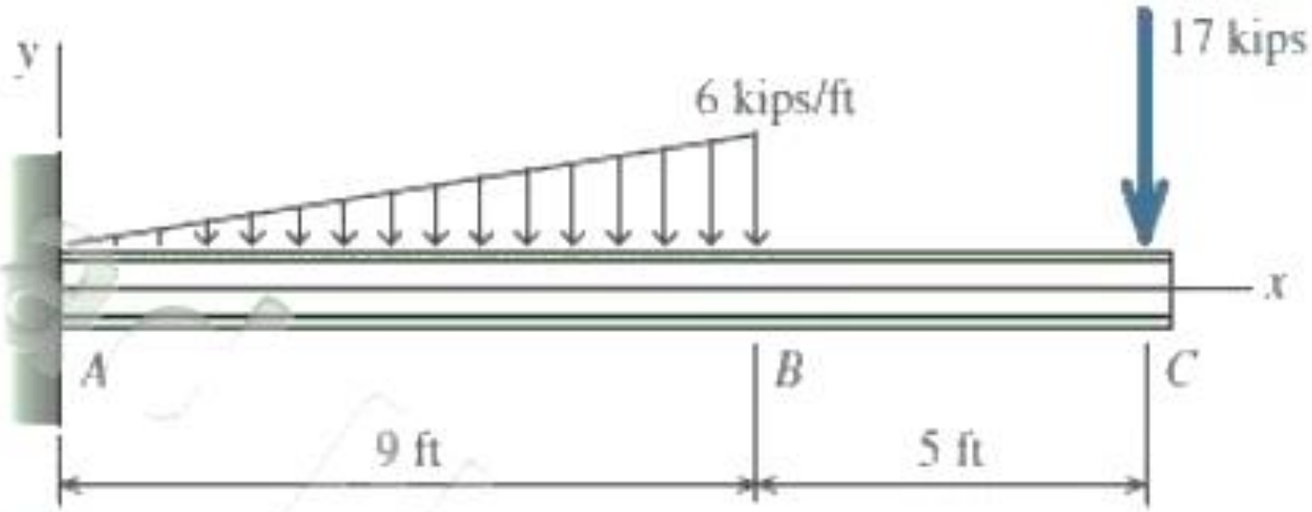


FIGURE

H.W.29 For the cantilever beam and loading shown in Figure

(a) derive equations for the shear force V and the bending moment M for any location in the beam.

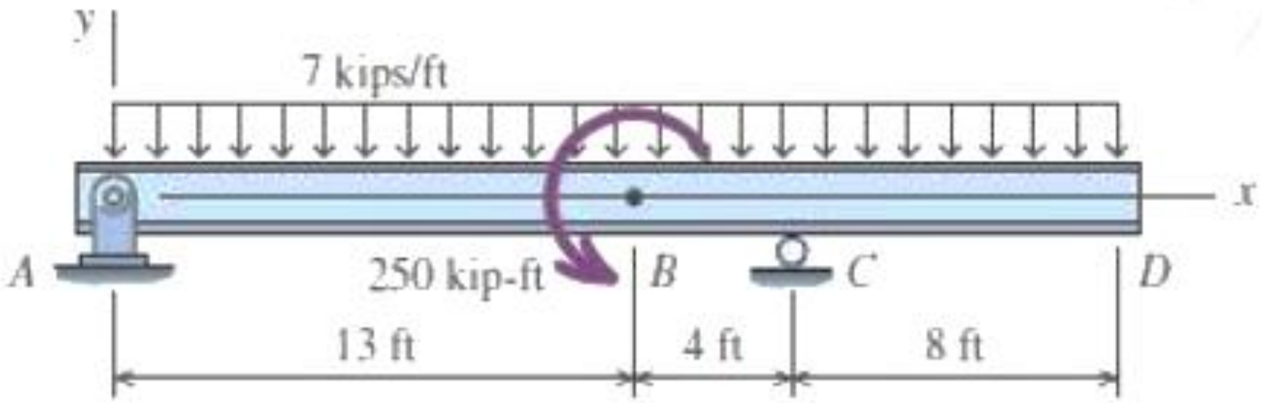
(b) plot the shear-force and bending-moment diagrams for the beam, using the derived functions.



FIGURE

H.W.30 For the simply supported beam subjected to the loading shown in Figure

(a) derive equations for the shear force V and the bending moment M for any location in the beam.
(b) plot the shear-force and bending-moment diagrams for the beam, using the derived functions.
(c) report the maximum positive bending moment, the maximum negative bending moment, and their respective locations.



FIGURE