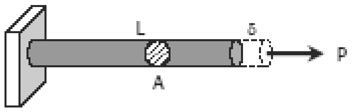
Strain

Simple Strain

Also known as unit deformation, strain is the ratio of the change in length caused by the applied force, to the original length.



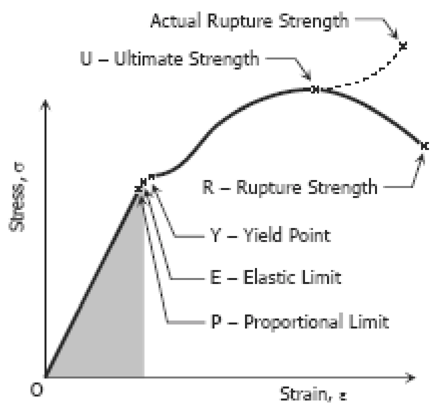
$$\varepsilon = \frac{\delta}{L}$$

where δ is the deformation and L is the original length, thus ε is dimensionless.

Stress-Strain Diagram

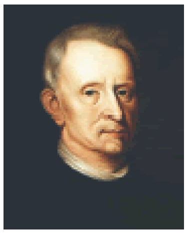
Suppose that a metal specimen be placed in tension-compression testing machine. As the axial load is gradually increased in increments, the total elongation over the gage length is measured at each increment of the load and this is continued until failure of the specimen takes place. Knowing the original cross-sectional area and length of the specimen, the normal stress σ and the strain ε can be obtained. The graph of these quantities with the stress σ along the y-axis and the strain ε along the x-axis is called the stress-strain diagram. The stress-strain diagram differs in form for various materials. The diagram shown below is that for a medium carbon structural steel.

Metallic engineering materials are classified as either ductile or brittle materials. A ductile material is one having relatively large tensile strains up to the point of rupture like structural steel and aluminum, whereas brittle materials has a relatively small strain up to the point of rupture like cast iron and concrete. An arbitrary strain of 0.05 mm/mm is frequently taken as the dividing line between these two classes.



PROPORTIONAL LIMIT (HOOKE'S LAW)

From the origin O to the point called proportional limit, the stress-strain curve is a straight line. This linear relation between elongation and the axial force causing was first noticed by Sir Robert Hooke in 1678 and is called Hooke's Law that within the proportional limit, the stress is directly proportional to strain or



Robert Hooke

$$\sigma \propto \varepsilon \text{ or } \sigma = k\,\varepsilon$$

The constant of proportionality k is called the Modulus of Elasticity E or Young's Modulus and is equal to the slope of the stress-strain diagram from O to P. Then

$$\sigma = E\,\varepsilon$$

ELASTIC LIMIT

The elastic limit is the limit beyond which the material will no longer go back to its original shape when the load is removed, or it is the maximum stress that may e developed such that there is no permanent or residual deformation when the load is entirely removed.

ELASTIC AND PLASTIC RANGES

The region in stress-strain diagram from O to P is called the elastic range. The region from P to R is called the plastic range.

YIELD POINT

Yield point is the point at which the material will have an appreciable elongation or yielding without any increase in load.

ULTIMATE STRENGTH

The maximum ordinate in the stress-strain diagram is the ultimate strength or tensile strength.

RAPTURE STRENGTH

Rapture strength is the strength of the material at rupture. This is also known as the breaking strength.

MODULUS OF RESILIENCE

Modulus of resilience is the work done on a unit volume of material as the force is gradually increased from O to P, in Nm/m^3. This may be calculated as the area under the stress-strain curve from the origin O to up to the elastic limit E (the shaded area in the figure). The resilience of the material is its ability to absorb energy without creating a permanent distortion.

MODULUS OF TOUGHNESS

Modulus of toughness is the work done on a unit volume of material as the force is gradually increased from O to R, in Nm/m^3. This may be calculated as the area under the entire stress-strain curve (from O to R). The toughness of a material is its ability to absorb energy without causing it to break.

WORKING STRESS, ALLOWABLE STRESS, AND FACTOR OF SAFETY

Working stress is defined as the actual stress of a material under a given loading. The maximum safe stress that a material can carry is termed as the allowable stress. The allowable stress should be limited to values not exceeding the proportional limit. However, since proportional limit is difficult to determine accurately, the allowable tress is taken as either the yield point or ultimate strength divided by a factor of safety. The ratio of this strength (ultimate or yield strength) to allowable strength is called the factor of safety.

AXIAL DEFORMATION

In the linear portion of the stress-strain diagram, the tress is proportional to strain and is given by

$$\sigma = E\varepsilon$$

since $\sigma = P / A$ and $\varepsilon e = \delta / L$, then $P / A = E \delta / L$. Solving for δ,

$$\delta = \frac{PL}{AE} = \frac{\sigma L}{E}$$

To use this formula, the load must be axial, the bar must have a uniform cross-sectional area, and the stress must not exceed the proportional limit. If however, the cross-sectional area is not uniform, the axial deformation can be determined by considering a differential length and applying integration.

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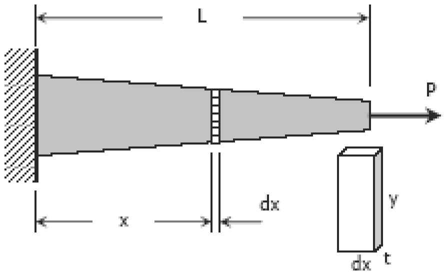


$$\delta = \frac{P}{E}\int_0^L \frac{dx}{A}$$

where A = ty and y and t, if variable, must be expressed in terms of x.

For a rod of unit mass ρ suspended vertically from one end, the total elongation due to its own weight is

$$\delta = \frac{\rho g L^2}{2E} = \frac{MgL}{2AE}$$

where ρ is in kg/m^3, L is the length of the rod in mm, M is the total mass of the rod in kg, A is the cross-sectional area of the rod in mm^2, and g = 9.81 m/s^2.

STIFFNESS, k

Stiffness is the ratio of the steady force acting on an elastic body to the resulting displacement. It has the unit of N/mm.

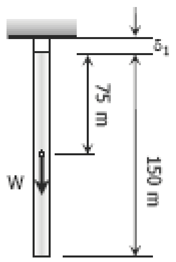
$$k = P / \delta$$

SOLVED PROBLEMS IN AXIAL DEFORMATION

Problem 206

A steel rod having a cross-sectional area of 300 mm^2 and a length of 150 m is suspended vertically from one end. It supports a tensile load of 20 kN at the lower end. If the unit mass of steel is 7850 kg/m^3 and E = 200 × 10^3 MN/m^2, find the total elongation of the rod.

Solution 206



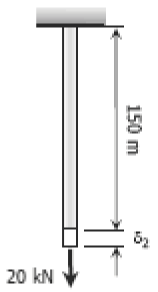
Let δ = total elongation
δ_1 = elongation due to its own weight
δ_2 = elongation due to applied load

$$\delta = \delta_1 + \delta_2$$

$$\delta_1 = \frac{PL}{AE}$$

Where: $P = W = 7850(1/1000)3(9.81)[300(150)(1000)]$
$P = 3465.3825$ N
$L = 75(1000) = 75\ 000$ mm
$A = 300\ mm^2$
$E = 200\ 000$ MPa

$$\delta_1 = \frac{3465.3825\,(75000)}{300\,(200\,000)} = 4.33\text{ mm}$$



$$\delta_2 = \frac{PL}{AE}$$

Where: $P = 20$ kN $= 20\ 000$ N
$L = 150$ m $= 150\ 000$ mm
$A = 300\ mm^2$
$E = 200\ 000$ MPa

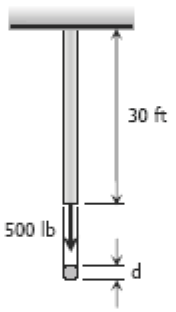
$$\delta_2 = \frac{20000(150000)}{300(200000)} = 50\text{ mm}$$

Total elongation:
$\delta = 4.33 + 50 = \mathbf{54.33\ mm}$

Problem 207

A steel wire 30 ft long, hanging vertically, supports a load of 500 lb. Neglecting the weight of the wire, determine the required diameter if the stress is not to exceed 20 ksi and the total elongation is not to exceed 0.20 in. Assume E = 29 × 10^6 psi.

Solution 207



Based on maximum allowable stress:

$$\sigma = \frac{P}{A}$$

$$20\,000 = \frac{500}{\frac{1}{4}\pi d^2}$$

$$d = 0.0318 \text{ in}$$

Based on maximum allowable deformation:

$$\delta = \frac{PL}{AE}$$

$$0.20 = \frac{500(30 \times 12)}{\frac{1}{4}\pi d^2 (29 \times 10^6)}$$

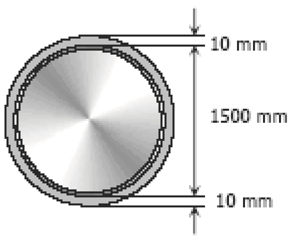
$$d = 0.0395 \text{ in}$$

Use the bigger diameter, d = **0.0395 in**

Problem 208

A steel tire, 10 mm thick, 80 mm wide, and 1500.0 mm inside diameter, is heated and shrunk onto a steel wheel 1500.5 mm in diameter. If the coefficient of static friction is 0.30, what torque is required to twist the tire relative to the wheel? Neglect the deformation of the wheel. Use E = 200 GPa.

Solution 208



$$\delta = \frac{PL}{AE}$$

Where:
- $\delta = \pi(1500.5 - 1500) = 0.5\pi$ mm
- P = T
- L = 1500π mm
- A = 10(80) = 800 mm^2
- E = 200 000 MPa

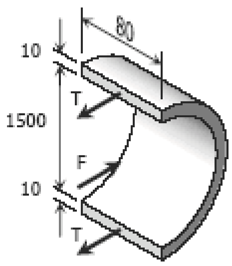
$$0.5\pi = \frac{T(1500\pi)}{800(200\,000)}$$

$$T = 53\,333.33 \text{ N}$$

$$F = 2T$$

$$p\,(1500)(80) = 2(53\,333.33)$$

$$p = 0.8889 \text{ MPa} \quad \rightarrow \text{internal pressure}$$



Total normal force, N:

$N = p \times$ contact area between tire and wheel

$$N = 0.8889 \times \pi(1500.5)(80)$$

$$N = 335\,214.92 \text{ N}$$

Friction resistance, f:

$$f = \mu N = 0.30(335\,214.92)$$

$$f = 100\,564.48 \text{ N} = 100.56 \text{ kN}$$

Torque = $f \times \frac{1}{2}$ (diameter of wheel)

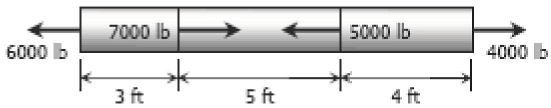
Torque = 100.56×0.75025

Torque = **75.44 kN·m**

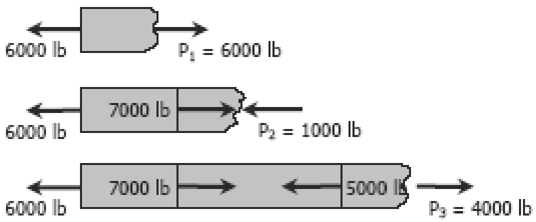
Problem 209

An aluminum bar having a cross-sectional area of 0.5 in^2 carries the axial loads applied at the positions shown in Fig. P-209. Compute the total change in length of the bar if E = 10 × 10^6 psi. Assume the bar is suitably braced to prevent lateral buckling.

Figure P-209 and P-210



Solution 209



P_1 = 6000 lb tension
P_2 = 1000 lb compression
P_3 = 4000 lb tension

$$\delta = \frac{PL}{AE}$$

$$\delta = \delta_1 - \delta_2 + \delta_3$$

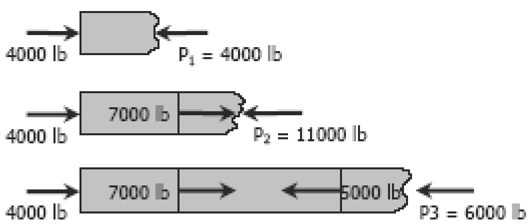
$$\delta = \frac{6000(3\times12)}{0.5(10\times10^6)} - \frac{1000(5\times12)}{0.5(10\times10^6)} + \frac{4000(4\times12)}{0.5(10\times10^6)}$$

δ = **0.0696 in (lengthening)**

Problem 210

Solve Prob. 209 if the points of application of the 6000-lb and the 4000-lb forces are interchanged.

Solution 210



P_1 = 4000 lb compression
P_2 = 11000 lb compression
P_3 = 6000 lb compression

$$\delta = \frac{PL}{AE}$$

$$\delta = -\delta_1 - \delta_2 - \delta_3$$

$$\delta = -\frac{4000(3\times12)}{0.5(10\times10^6)} - \frac{11000(5\times12)}{0.5(10\times10^6)} - \frac{6000(4\times12)}{0.5(10\times10^6)}$$

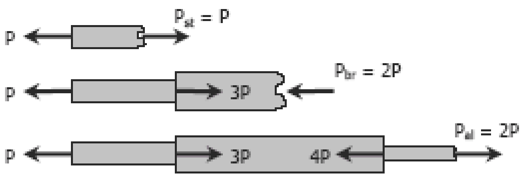
δ = −0.19248 in = **0.19248 in (shortening)**

Problem 211

A bronze bar is fastened between a steel bar and an aluminum bar as shown in Fig. P-211. Axial loads are applied at the positions indicated. Find the largest value of P that will not exceed an overall deformation of 3.0 mm, or the following stresses: 140 MPa in the steel, 120 MPa in the bronze, and 80 MPa in the aluminum. Assume that the assembly is suitably braced to prevent buckling. Use E_{st} = 200 GPa, E_{al} = 70 GPa, and E_{br} = 83 GPa.

Figure P -211

Solution 211



Based on allowable stresses:

Steel:

$P_{st} = \sigma_{st}A_{st}$

$P = 140(480) = 67\,200$ N

$P = 67.2$ kN

Bronze:

$P_{br} = \sigma_{br}A_{br}$

$2P = 120(650) = 78\,000$

$P = 39\,000$ N = 39 kN

Aluminum:

$P_{al} = \sigma_{al}A_{al}$

$2P = 80(320) = 25\,600$ N

$P = 12\,800$ N = 12.8 kN

Based on allowable deformation:

(steel and aluminum lengthens, bronze shortens)

$$\delta = \delta_{st} - \delta_{br} + \delta_{al}$$

$$3 = \frac{P(1000)}{480(200\,000)} - \frac{2P(2000)}{650(70\,000)} + \frac{2P(1500)}{320(83\,000)}$$

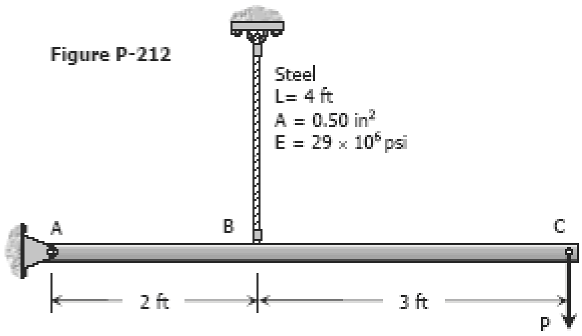
$$3 = \left(\tfrac{1}{96\,000} - \tfrac{1}{11\,375} + \tfrac{3}{26\,560}\right)P$$

$$P = 84\,610.99 \text{ N} = 84.61 \text{ kN}$$

Use the smallest value of P, P = **12.8 kN**

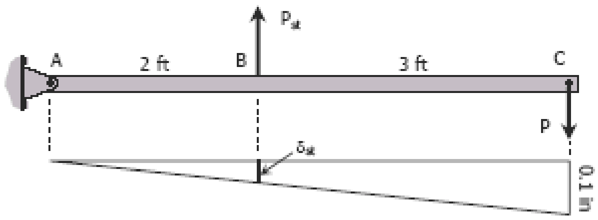
Problem 212

The rigid bar ABC shown in Fig. P-212 is hinged at A and supported by a steel rod at B. Determine the largest load P that can be applied at C if the stress in the steel rod is limited to 30 ksi and the vertical movement of end C must not exceed 0.10 in.



Solution 212

Free body and deformation diagrams:



Based on maximum stress of steel rod:

$\Sigma M_A = 0$

$5P = 2P_{st}$

$P = 0.4P_{st}$

$P = 0.4\sigma_{st}A_{st}$

$P = 0.4[30(0.50)]$

$P = 6$ kips

Based on movement at C:

$$\frac{\delta_{st}}{2} = \frac{0.1}{5}$$

$\delta_{st} = 0.04$ in

$$\frac{P_{st}L}{AE} = 0.04$$

$$\frac{P_{st}(4\times12)}{0.50(29\times10^6)} = 0.04$$

$P_{st} = 12\ 083.33$ lb

$\Sigma M_A = 0$

$5P = 2P_{st}$

$P = 0.4P_{st}$

$P = 0.4(12\ 083.33)$

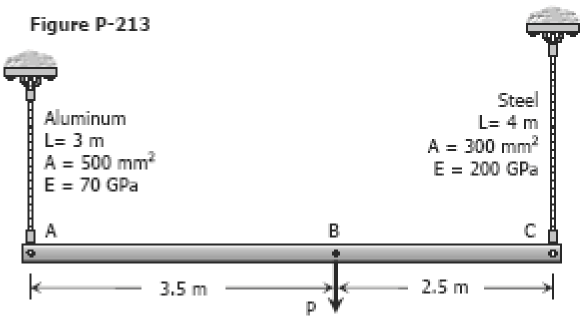
$P = 4833.33$ lb $= 4.83$ kips

Use the smaller value, $P =$ **4.83 kips**

Problem 213

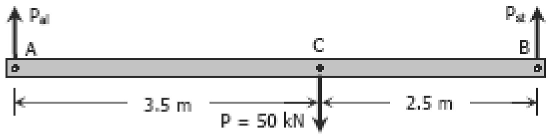
The rigid bar AB, attached to two vertical rods as shown in Fig. P-213, is horizontal before the load P is applied. Determine the vertical movement of P if its magnitude is 50 kN.

Figure P-213



Solution 213

Free body diagram:



For aluminum:

$[\Sigma M_B = 0]$ $\quad 6P_{al} = 2.5(50)$

$P_{al} = 20.83$ kN

$\left[\delta = \dfrac{PL}{AE}\right]_{al}$ $\quad \delta_{al} = \dfrac{20.83(3)1000^2}{500(70000)}$

$\delta_{al} = 1.78$ mm

For steel:

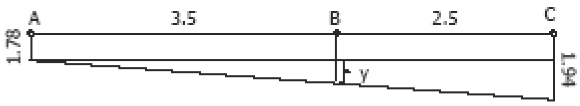
$[\Sigma M_A = 0]$ $\quad 6P_{st} = 3.5(50)$

$P_{st} = 29.17$ kN

$\left[\delta = \dfrac{PL}{AE}\right]_{st}$ $\quad \delta_{st} = \dfrac{29.17(4)1000^2}{300(200000)}$

$\delta_{st} = 1.94$ mm

Movement diagram:



$$\frac{y}{3.5} = \frac{1.94 - 1.78}{6}$$

$y = 0.09$ mm

δ_B = vertical movement of P

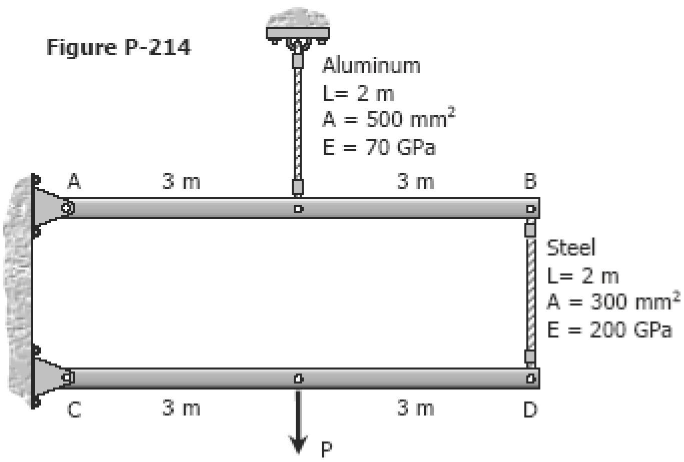
$\delta_B = 1.78 + y = 1.78 + 0.09$

$\delta_B = 1.87$ mm

Problem 214

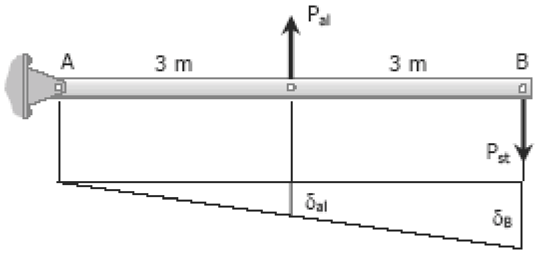
The rigid bars AB and CD shown in Fig. P-214 are supported by pins at A and C and the two rods. Determine the maximum force P that can be applied as shown if its vertical movement is limited to 5 mm. Neglect the weights of all members.

Figure P-214



Solution 214

$$[\Sigma M_A = 0] \qquad 3P_{al} = 6P_{st}$$
$$P_{al} = 2P_{st}$$



FBD and movement diagram of bar AB

By ratio and proportion:

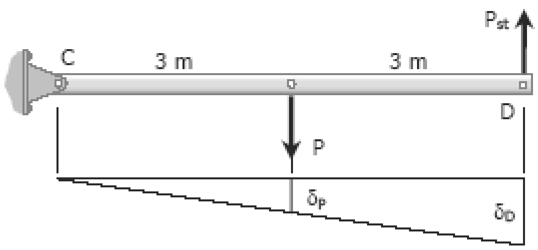
$$\frac{\delta_B}{6} = \frac{\delta_{al}}{3}$$

$$\delta_B = 2\delta_{al} = 2\left[\frac{PL}{AE}\right]_{al}$$

$$\delta_B = 2\left[\frac{P_{al}(2000)}{500(70\,000)}\right]$$

$$\delta_B = \frac{1}{8750}P_{al} = \frac{1}{8750}(2P_{st})$$

$\delta_B = \frac{1}{4375}P_{st}$ → movement of B



FBD and movement diagram of bar CD

Movement of D:

$$\delta_D = \delta_{st} + \delta_B = \left[\frac{PL}{AE}\right]_{st} + \frac{1}{4375}P_{st}$$

$$\delta_D = \frac{P_{st}(2000)}{300(200\,000)} + \frac{1}{4375}P_{st}$$

$$\delta_D = \frac{11}{42\,000}P_{st}$$

$$[\Sigma M_C = 0] \qquad 6P_{st} = 3P$$
$$P_{st} = \tfrac{1}{2}P$$

By ratio and proportion:

$$\frac{\delta_P}{3} = \frac{\delta_D}{6}$$

$$\delta_P = \tfrac{1}{2}\delta_D = \tfrac{1}{2}\left(\tfrac{11}{42\,000}P_{st}\right)$$

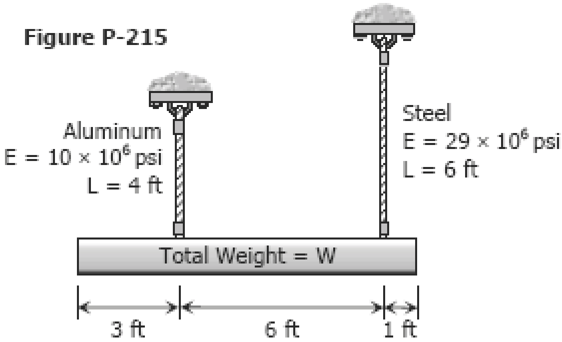
$$\delta_P = \frac{11}{84\,000}P_{st}$$

$$5 = \frac{11}{84\,000}\left(\tfrac{1}{2}P\right)$$

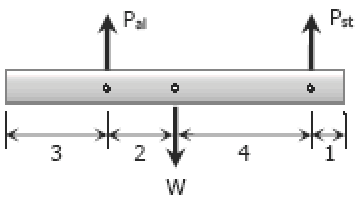
$P = 76\,363.64\text{ N} = \mathbf{76.4\text{ kN}}$

Problem 215

A uniform concrete slab of total weight W is to be attached, as shown in Fig. P-215, to two rods whose lower ends are on the same level. Determine the ratio of the areas of the rods so that the slab will remain level.



Solution 215



$[\Sigma M_{al} = 0] \qquad 6P_{st} = 2W$

$P_{st} = \frac{1}{3}W$

$[\Sigma M_{st} = 0] \qquad 6P_{al} = 4W$

$P_{al} = \frac{2}{3}W$

$$\delta_{st} = \delta_{al}$$

$$\left[\frac{PL}{AE}\right]_{st} = \left[\frac{PL}{AE}\right]_{al}$$

$$\frac{\frac{1}{3}W(6\times 12)}{A_{st}(29\times 10^6)} = \frac{\frac{2}{3}W(4\times 12)}{A_{al}(10\times 10^6)}$$

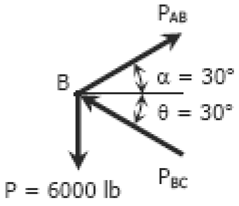
$$\frac{A_{al}}{A_{st}} = \frac{\frac{2}{3}W(4\times 12)(29\times 10^6)}{\frac{1}{3}W(6\times 12)(10\times 10^6)}$$

$$A_{al}/A_{st} = \mathbf{3.867}$$

Problem 216

As shown in Fig. P-216, two aluminum rods AB and BC, hinged to rigid supports, are pinned together at B to carry a vertical load P = 6000 lb. If each rod has a crosssectional area of 0.60 in^2 and E = 10 × 10^6 psi, compute the elongation of each rod and the horizontal and vertical displacements of point B. Assume α = 30° and θ = 30°.

Solution 216



$[\Sigma F_H = 0]$ $\quad P_{AB} \cos 30° = P_{BC} \cos 30°$

$P_{AB} = P_{BC}$

$[\Sigma F_V = 0]$ $\quad P_{AB} \sin 30° + P_{BC} \sin 30° = 6000$

$P_{AB}\,(0.5) + P_{AB}\,(0.5) = 6000$

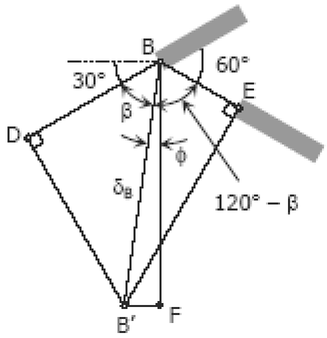
$P_{AB} = 6000$ lb tension

$P_{BC} = 6000$ lb compression

$$\delta = \frac{PL}{AE}$$

$$\delta_{AB} = \frac{6000(10 \times 12)}{0.6(10 \times 10^6)} = \mathbf{0.12\ in.}\ \text{lengthening}$$

$$\delta_{BC} = \frac{6000(6 \times 12)}{0.6(10 \times 10^6)} = \mathbf{0.072\ in.}\ \text{shortening}$$



Movement of B

$DB = \delta_{AB} = 0.12$ in

$BE = \delta_{BE} = 0.072$ in

$\delta_B = BB'$ = displacement of B

B' = final position of B after elongation

Triangle BDB':

$$\cos \beta = \frac{0.12}{\delta_B}$$

$$\delta_B = \frac{0.12}{\cos \beta}$$

Triangle BEB':

$$\cos (120° - \beta) = \frac{0.072}{\delta_B}$$

$$\delta_B = \frac{0.072}{\cos (120° - \beta)}$$

$\delta_B = \delta_B$

$$\frac{0.12}{\cos \beta} = \frac{0.072}{\cos (120° - \beta)}$$

$$\frac{\cos 120° \cos \beta + \sin 120° \sin \beta}{\cos \beta} = 0.6$$

$-0.5 + \sin 120° \tan \beta = 0.6$

$\tan \beta = 1.1/\sin 120°;\ \beta = 51.79°$

$\phi = 90 - (30° + \beta) = 90° - (30° + 51.79°)$

$\phi = 8.21°$

$$\delta_B = \frac{0.12}{\cos 51.79°}$$

$\delta_B = 0.194$ in

Triangle BFB':

$\delta_h = B'F = \delta_B \sin \phi = 0.194 \sin 8.21°$

$\delta_h = 0.0277$ in

$\delta_h = \mathbf{0.0023\ ft} \rightarrow$ horizontal displacement of B

$\delta_v = BF = \delta_B \cos \phi = 0.194 \cos 8.21°$

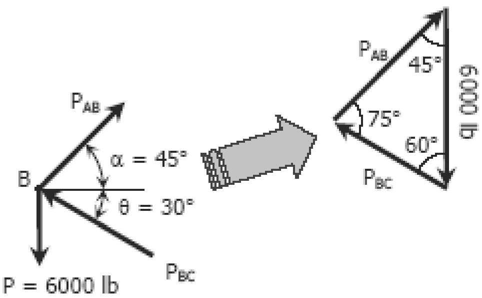
$\delta_v = 0.192$ in

$\delta_v = \mathbf{0.016\ ft}$ $\rightarrow$ vertical displacement of B

Problem 217

Solve Prob. 216 if rod AB is of steel, with E = 29 × 10^6 psi. Assume α = 45° and θ = 30°; all other data remain unchanged.

Solution 217



By Sine Law

$$\frac{P_{AB}}{\sin 60°} = \frac{6000}{\sin 75°}$$

$P_{AB} = 5379.45$ lb (Tension)

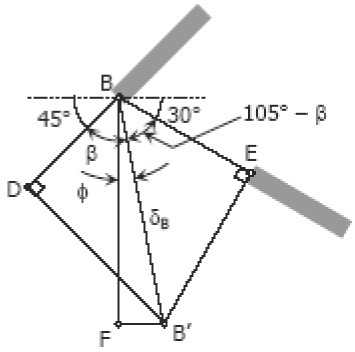
$$\frac{P_{BC}}{\sin 45°} = \frac{6000}{\sin 75°}$$

$P_{BC} = 4392.30$ lb (Compression)

$$\delta = \frac{PL}{AE}$$

$$\delta_{AB} = \frac{5379.45(10 \times 12)}{0.6(29 \times 10^6)} = \mathbf{0.0371\ in.}\ \text{(lengthening)}$$

$$\delta_{BC} = \frac{4392.30(6\times 12)}{0.6(10\times 10^6)} = \mathbf{0.0527\ in.}\ \text{(shortening)}$$



Movement of B

$DB = \delta_{AB} = 0.0371$ in
$BE = \delta_{BE} = 0.0527$ in
$\delta_B = BB'$ = displacement of B
B' = final position of B after deformation

Triangle BDB':

$$\cos\beta = \frac{0.0371}{\delta_B}$$

$$\delta_B = \frac{0.0371}{\cos\beta}$$

Triangle BEB':

$$\cos(105^\circ - \beta) = \frac{0.0527}{\delta_B}$$

$$\delta_B = \frac{0.0527}{\cos(105^\circ - \beta)}$$

$$\delta_B = \delta_B$$

$$\frac{0.0371}{\cos\beta} = \frac{0.0527}{\cos(105^\circ - \beta)}$$

$$\frac{\cos 105^\circ \cos\beta + \sin 105^\circ \sin\beta}{\cos\beta} = 1.4205$$

$$-0.2588 + 0.9659\tan\beta = 1.4205$$

$$\tan\beta = \frac{1.4205 + 0.2588}{0.9659}$$

$$\tan\beta = 1.7386$$

$$\beta = 60.1^\circ$$

$$\delta_B = \frac{0.0371}{\cos 60.1^\circ}$$

$$\delta_B = 0.0744 \text{ in}$$

$$\phi = (45^\circ + \beta) - 90^\circ$$
$$= (45^\circ + 60.1^\circ) - 90^\circ$$
$$= 15.1^\circ$$

Triangle BFB':

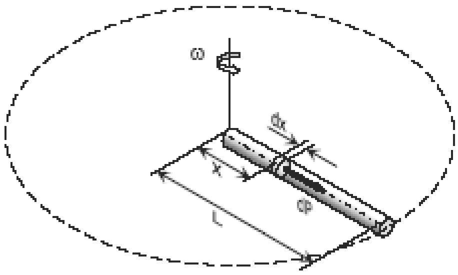
$\delta_h = FB' = \delta_B \sin\phi = 0.0744 \sin 15.1^\circ$
$\delta_h = 0.0194$ in
$\delta_h = \mathbf{0.00162\ ft}$ → horizontal displacement of B

$\delta_v = BF = \delta_B \cos\phi = 0.0744 \cos 15.1^\circ$
$\delta_v = 0.07183$ in
$\delta_v = \mathbf{0.00598\ ft}$ → vertical displacement of B

Problem 218

A uniform slender rod of length L and cross sectional area A is rotating in a horizontal plane about a vertical axis through one end. If the unit mass of the rod is ρ, and it is rotating at a constant angular velocity of ω rad/sec, show that the total elongation of the rod is $\rho\omega^2L^3/3E$.

Solution 218



$$\delta = \frac{PL}{AE}$$

from the frigure:

$$d\delta = \frac{dP\ x}{AE}$$

Where:
dP = centrifugal force of differential mass
$dP = dM\ \omega^2 x = (\rho A\ dx)\omega^2 x$
$dP = \rho A\omega^2\ x\ dx$

$$d\delta = \frac{(\rho A\omega^2 x\ dx)\ x}{AE}$$

$$\delta = \frac{\rho\omega^2}{E}\int_0^L x^2\ dx = \frac{\rho\omega^2}{E}\left[\frac{x^3}{3}\right]_0^L$$

$$\delta = \frac{\rho\omega^2}{3E}\left[L^3 - 0^3\right]$$

$\delta = \rho\omega^2L^3/3E$ *ok!*

Problem 219

A round bar of length L, which tapers uniformly from a diameter D at one end to a smaller diameter d at the other, is suspended vertically from the large end. If w is the weight per unit volume, find the elongation of the rod caused by its own weight. Use this result to determine the elongation of a cone suspended from its base.

Solution 219

$$\delta = \frac{PL}{AE}$$

For the differential strip shown:

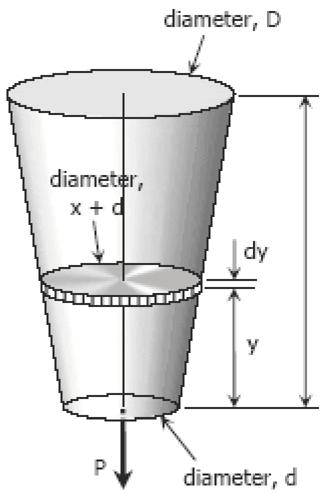
$\delta = d\delta$

P = weight carried by the strip

= weight of segment y

$L = dy$

A = area of the strip



For weight of segment y (Frustum of a cone):

$$P = wV_y$$

From section along the axis

$$\frac{x}{y} = \frac{D-d}{L}$$

$$x = \frac{D-d}{L}y$$

Volume for frustum of cone

$$V = \tfrac{1}{3}\pi h\,(R^2 + r^2 + Rr)$$

$$V_y = \tfrac{1}{3}\pi h\,[\,\tfrac{1}{4}(x+d)^2 + \tfrac{1}{4}d^2 + \tfrac{1}{2}(x+d)(\tfrac{1}{2}d)]$$

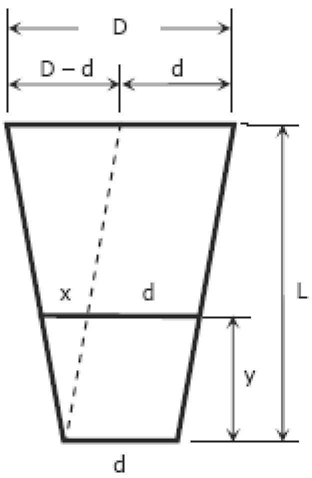
$$V_y = \tfrac{1}{12}\pi y\,[(x+d)^2 + d^2 + (x+d)d]$$

$$P = \tfrac{1}{12}\pi w\,[(x+d)^2 + d^2 + (x+d)d]\,y$$

$$P = \tfrac{1}{12}\pi w\,[x^2 + 2xd + d^2 + d^2 + xd + d^2]\,y$$

$$P = \tfrac{1}{12}\pi w\,[x^2 + 3xd + 3d^2]\,y$$

$$P = \frac{\pi w}{12}\left[\frac{(D-d)^2}{L^2}y^2 + \frac{3d(D-d)}{L}y + 3d^2\right]y$$



Section along the axis of the bar

Area of the strip:

$$A = \tfrac{1}{4}\pi(x+d)^2 = \frac{\pi}{4}\left(\frac{D-d}{L}y + d\right)^2$$

Thus,

$$\delta = \frac{PL}{AE}$$

$$d\delta = \frac{\frac{\pi w}{12}\left[\frac{(D-d)^2}{L^2}y^2 + \frac{3d(D-d)}{L}y + 3d^2\right]y\ dy}{\frac{\pi}{4}\left(\frac{D-d}{L}y + d\right)^2 E}$$

$$d\delta = \frac{4w}{12E}\left[\frac{\frac{(D-d)^2}{L^2}y^2 + \frac{3d(D-d)}{L}y + 3d^2}{\frac{(D-d)^2}{L^2}y^2 + \frac{2d(D-d)}{L}y + d^2}\right]y\ dy$$

$$d\delta = \frac{w}{3E}\left[\frac{\frac{(D-d)^2y^2 + 3Ld(D-d)y + 3L^2d^2}{L^2}}{\frac{(D-d)^2y^2 + 2Ld(D-d)y + L^2d^2}{L^2}}\right]y\ dy$$

$$d\delta = \frac{w}{3E}\left[\frac{(D-d)^2y^2 + 3Ld(D-d)y + 3L^2d^2}{(D-d)^2y^2 + 2Ld(D-d)y + L^2d^2}\right]y\ dy$$

Let: $a = D - d$; $b = Ld$

$$d\delta = \frac{w}{3E}\left[\frac{a^2y^2 + 3ab\,y + 3b^2}{a^2y^2 + 2ab\,y + b^2}\right]y\ dy$$

$$d\delta = \frac{w}{3E}\left[\frac{a^2y^2 + 3ab\,y + 3b^2}{(ay)^2 + 2(ay)b + b^2} \times \frac{a}{a}\right]y\ dy$$

$$d\delta = \frac{w}{3aE}\left[\frac{a^3y^3 + 3(a^2y^2)b + 3(ay)b^2}{(ay+b)^2}\right]dy$$

$$d\delta = \frac{w}{3aE}\left\{\frac{[(ay)^3 + 3(ay)^2b + 3(ay)b^2 + b^3] - b^3}{(ay+b)^2}\right\}dy$$

The quantity $(ay)^3 + 3(ay)^2b + 3(ay)b^2 + b^3$ is the expansion of $(ay + b)^3$

$$d\delta = \frac{w}{3aE}\left[\frac{(ay+b)^3 - b^3}{(ay+b)^2}\right]dy$$

$$d\delta = \frac{w}{3aE}\left[\frac{(ay+b)^3}{(ay+b)^2} - \frac{b^3}{(ay+b)^2}\right]dy$$

$$d\delta = \frac{w}{3aE}[(ay+b) - b^3(ay+b)^{-2}]\,dy$$

$$\delta = \frac{w}{3aE}\int_0^L [(ay+b) - b^3(ay+b)^{-2}]\,dy$$

$$\delta = \frac{w}{3aE}\left[\frac{(ay+b)^2}{2a} - \frac{b^3(ay+b)^{-1}}{-a}\right]_0^L$$

$$\delta = \frac{w}{3a^2E}\left[\frac{(ay+b)^2}{2} + \frac{b^3}{ay+b}\right]_0^L$$

$$\delta = \frac{w}{3a^2E}\left\{\left[\tfrac{1}{2}(aL+b)^2 + \frac{b^3}{aL+b}\right] - \left[\tfrac{1}{2}b^2 + \frac{b^3}{b}\right]\right\}$$

$$\delta = \frac{w}{3a^2E}\left\{\tfrac{1}{2}(aL+b)^2 + \frac{b^3}{aL+b} - \tfrac{3}{2}b^2\right\}$$

$$\delta = \frac{w}{3a^2E}\left[\frac{(aL+b)^3 + 2b^3 - 3b^2(aL+b)}{2(aL+b)}\right]$$

$$\delta = \frac{w}{6a^2E}\left[\frac{(aL)^3 + 3(aL)^2b + 3(aL)b^2 + b^3 + 2b^3 - 3ab^2L - 3b^3}{aL+b}\right]$$

$$\delta = \frac{w}{6a^2E}\left[\frac{a^3L^3 + 3a^2bL^2}{aL+b}\right] \text{ ;Note: } a = D - d \ \& \ b = Ld$$

$$\delta = \frac{w}{6(D-d)^2E}\left[\frac{(D-d)^3L^3 + 3(D-d)^2(Ld)L^2}{(D-d)L + (Ld)}\right]$$

$$\delta = \frac{w}{6(D-d)^2E}\left\{\frac{(D-d)L^3[(D-d)^2 + 3d(D-d)]}{LD - Ld + Ld}\right\}$$

$$\delta = \frac{wL^3}{6(D-d)E}\left[\frac{(D-d)^2 + 3d(D-d)}{LD}\right]$$

$$\delta = \frac{wL^3}{6(D-d)E}\left[\frac{D^2 - 2Dd + d^2 + 3Dd - 3d^2}{LD}\right]$$

$$\delta = \frac{wL^3}{6(D-d)E}\left[\frac{D^2 + Dd - 2d^2}{LD}\right]$$

$$\delta = \frac{wL^3}{6(D-d)E}\left[\frac{D(D+d) - 2d^2}{LD}\right]$$

$$\delta = \frac{wL^3}{6(D-d)E}\left[\frac{D(D+d)}{LD}\right] - \frac{wL^3}{6(D-d)E}\left[\frac{2d^2}{LD}\right]$$

$$\boldsymbol{\delta = \frac{wL^2(D+d)}{6E(D-d)} - \frac{wL^2d^2}{3ED(D-d)}}$$

For a cone:

$D = D$ and $d = 0$

$$\delta = \frac{wL^2\cancel{(D+0)}}{6E\cancel{(D-0)}} - \cancel{\frac{wL^2(0)^2}{3ED(D-0)}}$$

$$\boldsymbol{\delta = \frac{wL^2}{6E}}$$

SOLVED PROBLEMS IN STRAIN AND AXIAL DEFORMATION

Problem 203

The following data were recorded during the tensile test of a 14-mm-diameter mild steel rod. The gage length was 50 mm.

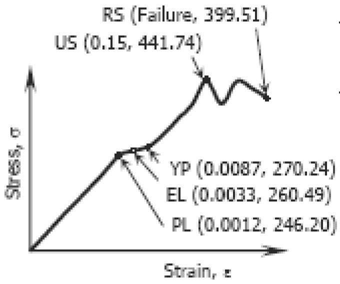
Load (N)	Elongation (mm)	Load (N)	Elongation (mm)
0	0	46200	1.25
6310	0.010	52400	2.50
12600	0.020	58500	4.50
18800	0.030	68000	7.50
25100	0.040	59000	12.50
31300	0.050	67800	15.50
37900	0.060	65000	20.00
40100	0.163	61500	Fracture
41600	0.433		

Plot the stress-strain diagram and determine the following mechanical properties: (a) proportional limits; (b) modulus of elasticity; (c) yield point; (d) ultimate strength; and (e) rupture strength.

Solution 203

Area, $A = \frac{1}{4}\pi(14)^2 = 49\pi$ mm²;   Length, $L = 50$ mm

Strain = Elongation/Length;   Stress = Load/Area



Stress-Strain Diagram
(not drawn to scale)

PL = Proportional Limit
EL = Elastic Limit
YP = Yield Point
US = Ultimate Strength
RS = Rupture Strength

Load (N)	Elongation (mm)	Strain (mm/mm)	Stress (MPa)
0	0	0	0
6310	0.010	0.0002	40.99
12600	0.020	0.0004	81.85
18800	0.030	0.0006	122.13
25100	0.040	0.0008	163.05
31300	0.050	0.001	203.33
37900	0.060	0.0012	246.20
40100	0.163	0.0033	260.49
41600	0.433	0.0087	270.24
46200	1.250	0.025	300.12
52400	2.500	0.05	340.40
58500	4.500	0.09	380.02
68000	7.500	0.15	441.74
59000	12.500	0.25	383.27
67800	15.500	0.31	440.44
65000	20.000	0.4	422.25
61500	Failure		399.51

From stress-strain diagram:

(a) Proportional Limit = **246.20 MPa**

(b) Modulus of Elasticity

E = slope of stress-strain diagram within proportional limit

$$E = \frac{246.20}{0.0012} = 205\,166.67 \text{ MPa}$$

$E = 205.2$ GPa

(c) Yield Point = **270.24 MPa**

(d) Ultimate Strength = **441.74 MPa**

(e) Rupture Strength = **399.51 MPa**

Problem 204

The following data were obtained during a tension test of an aluminum alloy. The initial diameter of the test specimen was 0.505 in. and the gage length was 2.0 in.

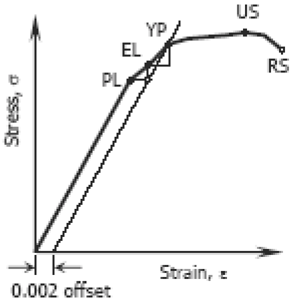
Load (lb)	Elongation (in.)	Load (lb)	Elongation (in.)
0	0	14 000	0.020
2 310	0.00220	14 400	0.025
4 640	0.00440	14 500	0.060
6 950	0.00660	14 600	0.080
9 290	0.00880	14 800	0.100
11 600	0.0110	14 600	0.120
12 600	0.0150	13 600	Fracture

Plot the stress-strain diagram and determine the following mechanical properties: (a) proportional limit; (b) modulus of elasticity; (c) yield point; (d) yield strength at 0.2% offset; (e) ultimate strength; and (f) rupture strength.

Solution 204

Area = $\frac{1}{4}\pi(0.505)^2 = 0.0638\pi$ in^2; Length, $L = 2.0$ in.
Strain = Elongation/Length; Stress = Load/Area

Load (lb)	Elongation (in)	Strain (in/in)	Stress (psi)
0	0	0	0
2310	0.0022	0.0011	11532.92
4640	0.0044	0.0022	23165.70
6950	0.0066	0.0033	34698.62
9290	0.0088	0.0044	46381.32
11600	0.011	0.0055	57914.24
12600	0.015	0.0075	62906.85
14000	0.02	0.01	69896.49
14400	0.025	0.0125	71893.54
14500	0.06	0.03	72392.80
14600	0.08	0.04	72892.06
14800	0.1	0.05	73890.58
14600	0.12	0.06	72892.06
13600	Fracture		67899.45



PL (0.0055, 57914.24)
EL (0.0075, 62906.85)
YP (0.01, 69896.49)
US (0.05, 73890.58)
RS (Failure, 67899.45)

From stress-strain diagram:

(a) Proportional Limit = **57,914.24 psi**

(b) Modulus of Elasticity:

$$E = \frac{57914.24}{0.0055} = 10{,}529{,}861.82 \text{ psi}$$

$E = 10{,}529.86$ ksi

(c) Yield Point = **69,896.49 psi**

(d) Yield Strength at 0.2% Offset:

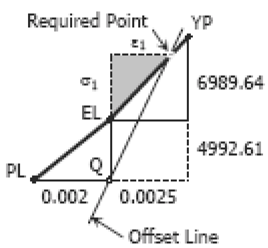
Strain of Elastic Limit

= ε at PL + 0.002

= 0.0055 + 0.002

= 0.0075 in/in

The offset line will pass through Q(See figure):



Slope of 0.2% offset

= E = 10,529,861.82 psi

Test for location

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$10{,}529{,}861.82 = \frac{6989.64 + 4992.61}{\text{run}}$$

run = 0.00113793 < 0.0025, therefore, the required point is just before *YP*.

Slope of *EL* to *YP*

$$\frac{\sigma_1}{\varepsilon_1} = \frac{6989.64}{0.0025}$$

$$\frac{\sigma_1}{\varepsilon_1} = 2\,795\,856$$

$$\varepsilon_1 = \frac{\sigma_1}{2\,795\,856}$$

For required point

$$E = \frac{4992.61 + \sigma_1}{\varepsilon_1}$$

$$10\,529\,861.82 = \frac{4992.61 + \sigma_1}{\frac{\sigma_1}{2\,795\,856}}$$

$$3.7662\sigma_1 = 4992.61 + \sigma_1$$

$$\sigma_1 = 1804.84 \text{ psi}$$

Yield Strength at 0.2% Offset

$= EL + \sigma_1$

$= 62906.85 + 1804.84$

= 64,711.69 psi

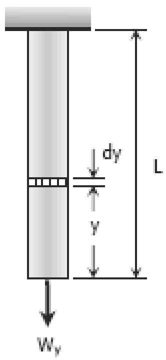
(e) Ultimate Strength = **73,890.58 psi**

(f) Rupture Strength = **67,899.45 psi**

Problem 205

A uniform bar of length L, cross-sectional area A, and unit mass ρ is suspended vertically from one end. Show that its total elongation is $\delta = \rho g L^2 / 2E$. If the total mass of the bar is M, show also that $\delta = MgL/2AE$.

Solution 205



$$\delta = \frac{PL}{AE}$$

From the figure:
$\delta = d\delta$
$P = W_y = (\rho A y)g$
$L = dy$

$$d\delta = \frac{(\rho A y)g\;dy}{AE}$$

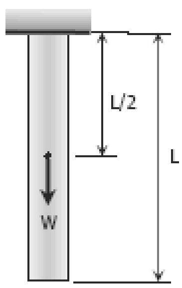
$$\delta = \frac{\rho g}{E}\int_0^L y\,dy = \frac{\rho g}{E}\left[\frac{y^2}{2}\right]_0^L$$

$$\delta = \frac{\rho g}{2E}\left[L^2 - 0^2\right] = \rho g L^2/2E \qquad \textit{ok!}$$

Given the total mass M:
$\rho = M/V = M/AL$

$\delta = \rho g L^2/2E = (M/AL)(gL^2/2E)$
$\delta = MgL/2AE$ *ok!*

Another Solution:



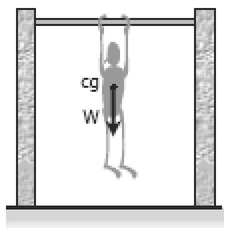
The weight will act at the center of gravity of the bar:

$$\delta = \frac{PL}{AE}$$

Where: $P = W = (\rho A L)g$
$L = L/2$

$$\delta = \frac{[(\rho A L)g](L/2)}{AE}$$

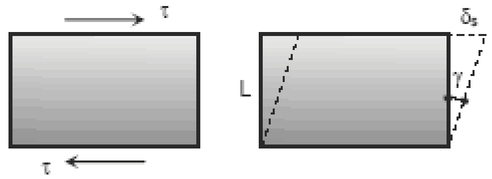
$$\delta = \frac{\rho g L^2}{2E} \qquad \textit{ok!}$$



For you to feel the situation, position yourself in pull-up exercise with your hands on the bar and your body hang freely above the ground. Notice that your arms suffer all your weight and your lower body fells no stress (center of weight is approximately just below the chest). If your body is the bar, the elongation will occur at the upper half of it.

Shearing Deformation

Shearing forces cause shearing deformation. An element subject to shear does not change in length but undergoes a change in shape.



The change in angle at the corner of an original rectangular element is called the **shear strain** and is expressed as

$$\gamma = \frac{\delta_s}{L}$$

The ratio of the shear stress τ and the shear strain γ is called the **modulus of elasticity** in shear or modulus of rigidity and is denoted as G, in MPa.

$$G = \frac{\tau}{\gamma}$$

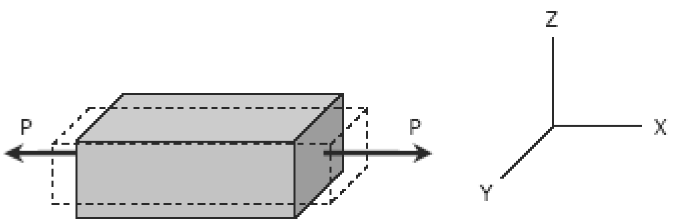
The relationship between the shearing deformation and the applied shearing force is

$$\delta_s = \frac{VL}{A_s G} = \frac{\tau L}{G}$$

where V is the shearing force acting over an area A_s.

Poisson's Ratio

When a bar is subjected to a tensile loading there is an increase in length of the bar in the direction of the applied load, but there is also a decrease in a lateral dimension perpendicular to the load. The ratio of the sidewise deformation (or strain) to the longitudinal deformation (or strain) is called the Poisson's ratio and is denoted by ν. For most steel, it lies in the range of 0.25 to 0.3, and 0.20 for concrete.



$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$