Strength of Materials

2^{nd} Stage (2021-2022)

Civil Engineering Department

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Syllabus

Chapter	Topic
0	Primary Introduction
1	Stress
2	Strain
3	Mechanical Properties of Materials
4	Axial Deformation
5	Shear Forces and Bending Moments diagrams
6	Torsion
7	Bending Stresses in Beams
8	Shear Stresses in Beams
9	Combined Loads
10	Stress Transformations
11	Beam Deflections

References

- **Philpot, T.A. (2017). *Mechanics of Materials: An Integrated Learning System*. 4th Ed. John Wiley & Sons, Inc.**

- **Hibbeler, R.C. (2011). *Mechanics of Materials*. 8th Ed. Pearson Prentice Hall.**

Outcomes of courses

- **To understand the concept of stress and strain.**
- **To be familiar with the numerical equations of stress and strain and the background theory.**
- **To know how to draw a free body diagram for the structural members.**
- **To analyse different types and directions of loadings.**
- **To assess the mechanical properties of materials.**
- **To be familiar with the behaviour of beams in bending and torsion.**
- **To analyse the behaviour of beams in shear and deflection.**
- **To understand the classifications of beams, e.g., determinate, indeterminate…etc.**
- **To be familiar with the stress transformation by using Mohr’s circle.**
- **To analyse axially compressed members, i.e., columns.**

Chapter 11 Beam Deflections

11.1 Introduction

When a beam with a straight longitudinal axis is loaded by lateral forces, the axis is deformed into a curve, called the **deflection curve** of the beam.

The calculation of deflections is an important part of structural analysis and design.

Deflections are sometimes calculated in order to verify that they are within tolerable limits. For instance, specifications for the design of buildings usually place upper limits on the deflections. Large deflections in buildings are unsightly (and even unnerving) and can cause cracks in ceilings and walls. In the design of machines and aircraft, specifications may limit deflections in order to prevent undesirable vibrations.

11.2 The Elastic Curve

Before the slope or the displacement at a point on a beam (or shaft) is determined, it is often helpful to sketch the deflected shape of the beam when it is loaded, in order to "visualize" any computed results and thereby partially check these results. The deflection curve of the longitudinal axis that passes through the centroid of each cross-sectional area of a beam is called the *elastic curve*. For most beams the elastic curve can be sketched without much difficulty. When doing so, however, it is necessary to know

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The relationship between the internal bending moment and the curvature of the elastic curve summarized the **moment–curvature** relationship:

$$\kappa = \frac{1}{\rho} = \frac{M}{EI_z}$$

This equation relates the radius of curvature ρ of the neutral surface of the beam to the internal bending moment M (about the z axis), the elastic modulus of the material E, and the moment of inertia of the cross-sectional area, I_z. Since E and I_z are always positive, the sign of ρ is consistent with the sign of the bending moment. As shown in Figure 11.1, a positive bending moment M creates a radius of curvature ρ that extends above the beam that is, in the positive v direction. When M is negative, ρ extends below the beam in a negative v direction. Figures 11.2-11.4 show the elastic curves of beams with various supports.

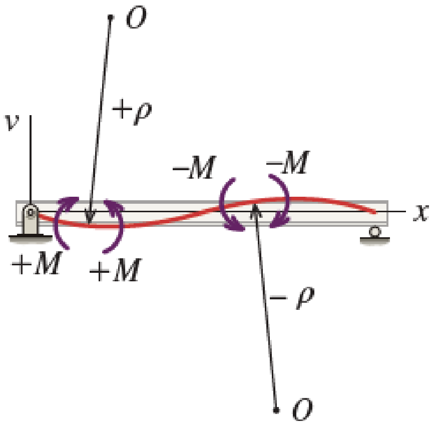

FIGURE 11.1 Radius of curvature ρ related to sign of M.

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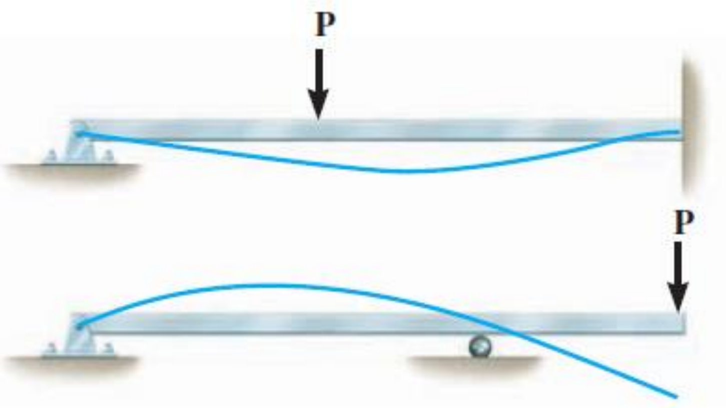

FIGURE 11.2

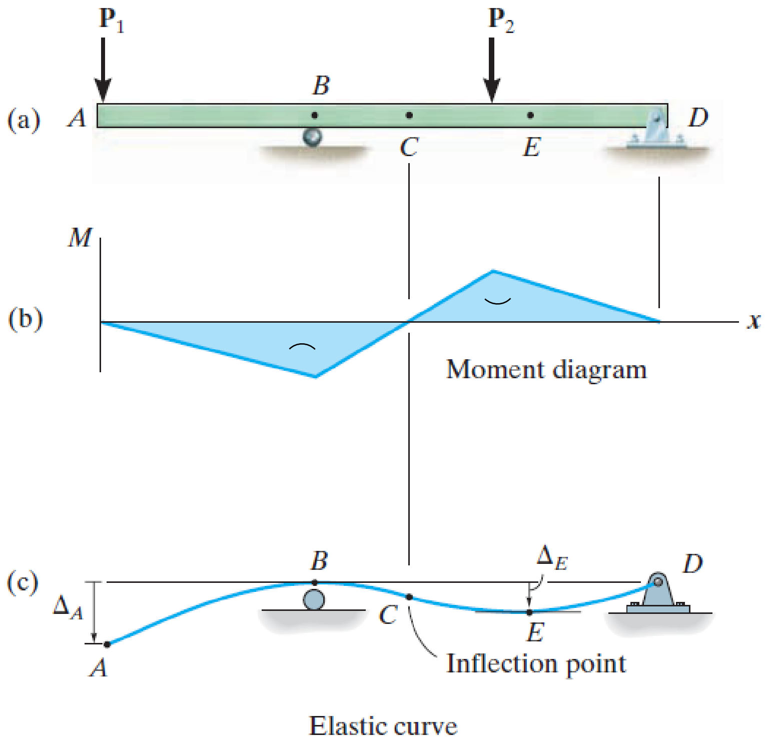

FIGURE 11.3

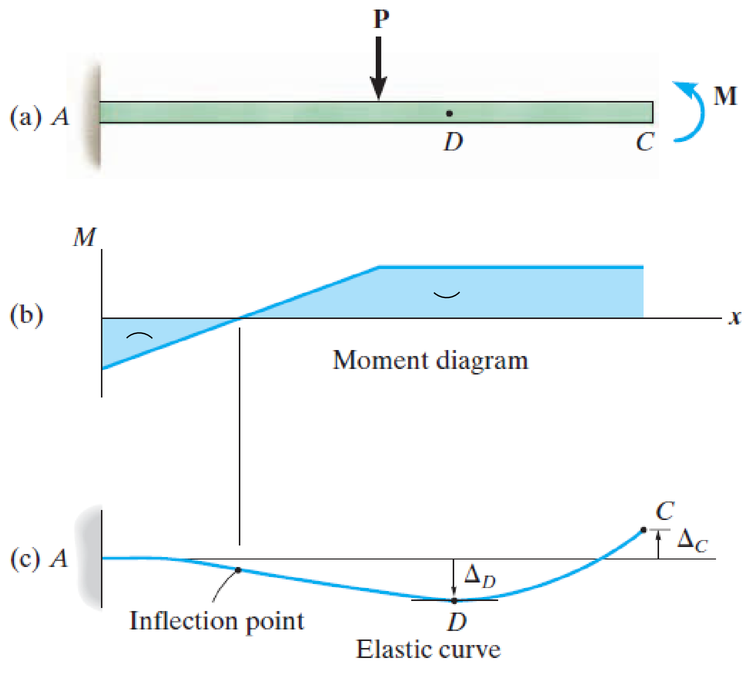

FIGURE 11.4

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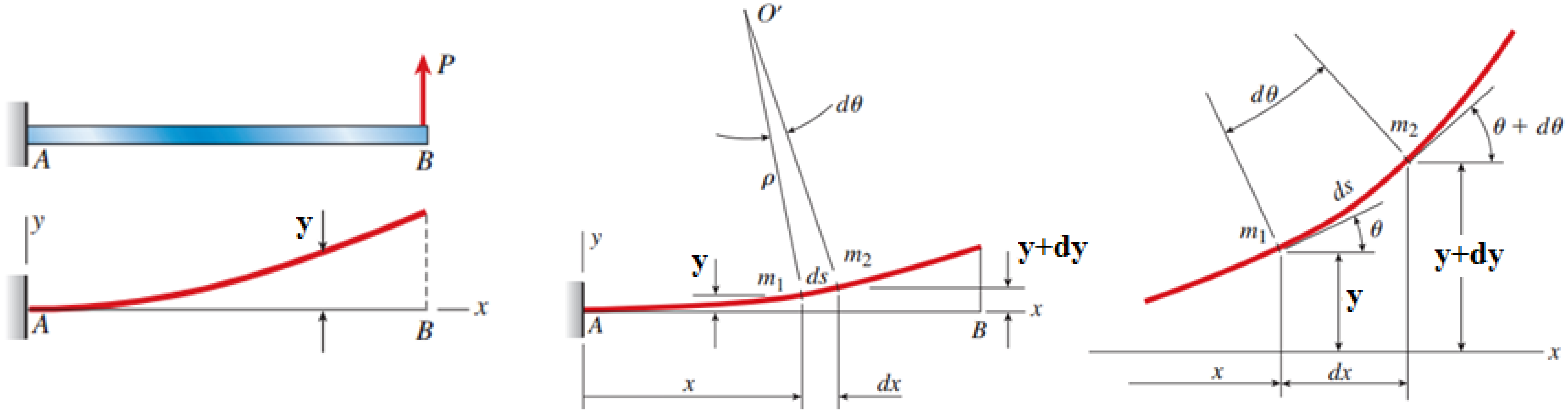
11.3 Evaluating Slope and Deflection Methods

Double Integration Method: In this method, for each segment, we integrate the differential equation of the bending moment twice to obtain a slope equation, a deflection equation , and two constants of integration.

Differential Equations of the Deflection Curve: consider a cantilever beam with a concentrated load acting upward at the free end. The axis of the beam deforms into a curve, as shown in the figure. The reference axes have their origin at the fixed end of the beam, with the x axis directed to the right and the y axis directed upward.



$\rho d\theta = ds$ $\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$ $\frac{d\,y}{dx} = \tan\theta$ $\cos\theta = \frac{dx}{ds}$ $\sin\theta = \frac{d\,y}{ds}$

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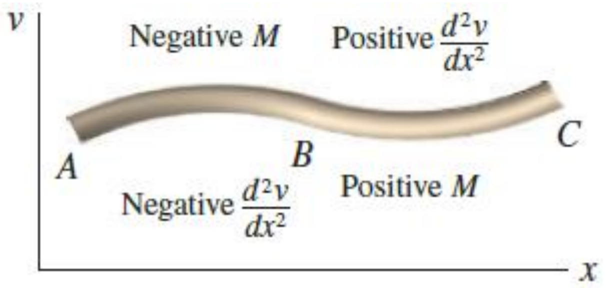
The changes in the structures are so small as to be unnoticed by a casual observer. Consequently, the deflection curves of most beams and columns have very small angles of rotation, very small deflections, and very small curvatures. Under these conditions we can make some mathematical approximations that greatly simplify beam analysis.

$$ds \approx dx \qquad \theta \approx \tan\theta = \frac{dy}{dx} \qquad \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \qquad \kappa = \frac{1}{\rho} = \frac{d\theta}{dx}$$

If the material of a beam is linearly elastic and follows Hooke's law, the curvature is:

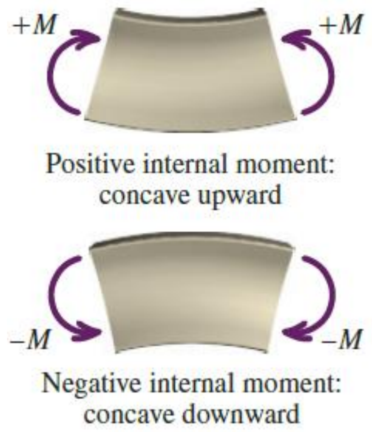
$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$



Thus, the basic **differential equation of the deflection curve** of a beam is:

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$EI\frac{d^2y}{dx^2} = M(x)$$



Sign Conventions:

The signs of the bending moment and the second derivative must be consistent. The beam slope changes from positive to negative in the segment from *A* to *B*; therefore, the second derivative is negative. For segment *BC*, both d^2v/dx^2 and *M* are seen to be positive.

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Relationship of Derivatives:

The successive derivatives of the elastic curve deflection v with the physical quantities that they represent in beam action are as follows:

$$\text{Deflection} = v$$

$$\text{Slope} = \frac{dv}{dx} = \theta$$

$$\text{Moment } M = EI\frac{d^2v}{dx^2}$$

$$\text{Shear } V = \frac{dM}{dx} = EI\frac{d^3v}{dx^3} \quad (\text{for } EI \text{ constant})$$

$$\text{Load } w = \frac{dV}{dx} = EI\frac{d^4v}{dx^4} \quad (\text{for } EI \text{ constant})$$

Slope and Deflection by Integration:

$$\text{Moment}: \; EI\frac{d^2v}{dx^2} = M(x)$$

$$\text{Slope (Rotation)}: EI\frac{dv}{dx} = \int M(x)\,dx$$

$$\text{Deflection}: EI\,v = \int EI\frac{dv}{dx}\,dx$$

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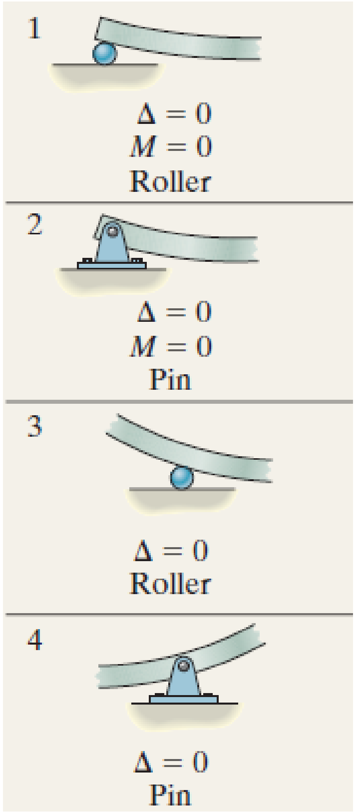
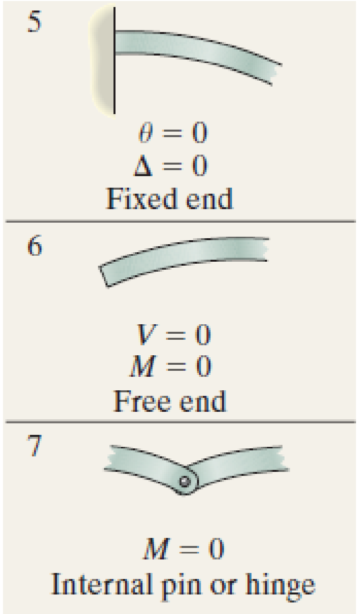
Boundary Conditions:

Boundary conditions are known slopes and deflections at the limits of the *bending moment equation* $M(x)$. The term "boundary" refers to the bounds of $M(x)$, not necessarily the bounds of the beam. Although boundary conditions are found at beam supports, only those supports within the bounds of the bending-moment equation should be considered.





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Procedure for Double-Integration Method

Calculating the deflection of a beam by the double-integration method involves several steps, and the following sequence is strongly recommended:

1. **Sketch:** Sketch the beam, including supports, loads, and the x–v coordinate system. Sketch the approximate shape of the elastic curve. Pay particular attention to the slope and deflection of the beam at the supports.
2. **Support reactions:** For some beam configurations, it may be necessary to determine support reactions before proceeding to analyze specific beam segments. For these configurations, determine the beam reactions by considering the equilibrium of the entire beam. Show these reactions in their proper direction on the beam sketch.
3. **Equilibrium:** Select the segment or segments of the beam to be considered. For each segment, draw a free-body diagram (FBD) that cuts through the beam segment at some distance x from the origin. On the FBD, show all loads acting on the beam. If distributed loads act on the beam, then that portion of the distributed load which acts on the FBD must be shown at the outset. Include the internal bending moment M acting at the cut surface of the beam, and always show M acting in the positive direction.
   The latter ensures that the bending-moment equation will have the correct sign. From the FBD, derive the bending-moment equation, taking care to note the interval to which it is applicable (e.g., $x_1 \leq x \leq x_2$).
4. **Integration:** For each segment, set the bending-moment equation equal to $EI\, d^2v/dx^2$. Integrate this differential equation twice, obtaining a slope equation dv/dx, a deflection equation v, and two constants of integration.

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5. **Boundary and continuity conditions:** List the boundary conditions that are applicable to the bending-moment equation. If the analysis involves two or more beam segments, list the continuity conditions also. Remember that two conditions are required in order to evaluate the two constants of integration produced in each beam segment.
6. **Evaluate constants:** Use the boundary and continuity conditions to evaluate all constants of integration.
7. **Elastic curve and slope equations:** Replace the constants of integration arrived at in step 4 with the values obtained from the boundary and continuity conditions found in step 6. Check the resulting equations for dimensional homogeneity.
8. **Deflections and slopes at specific points:** Calculate the deflection at specific points when required.

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In simply supported beam subjected to concentrated load at x=L/2, the max. deflection occurs at $x = L/2$ where $dv/dx = 0$.

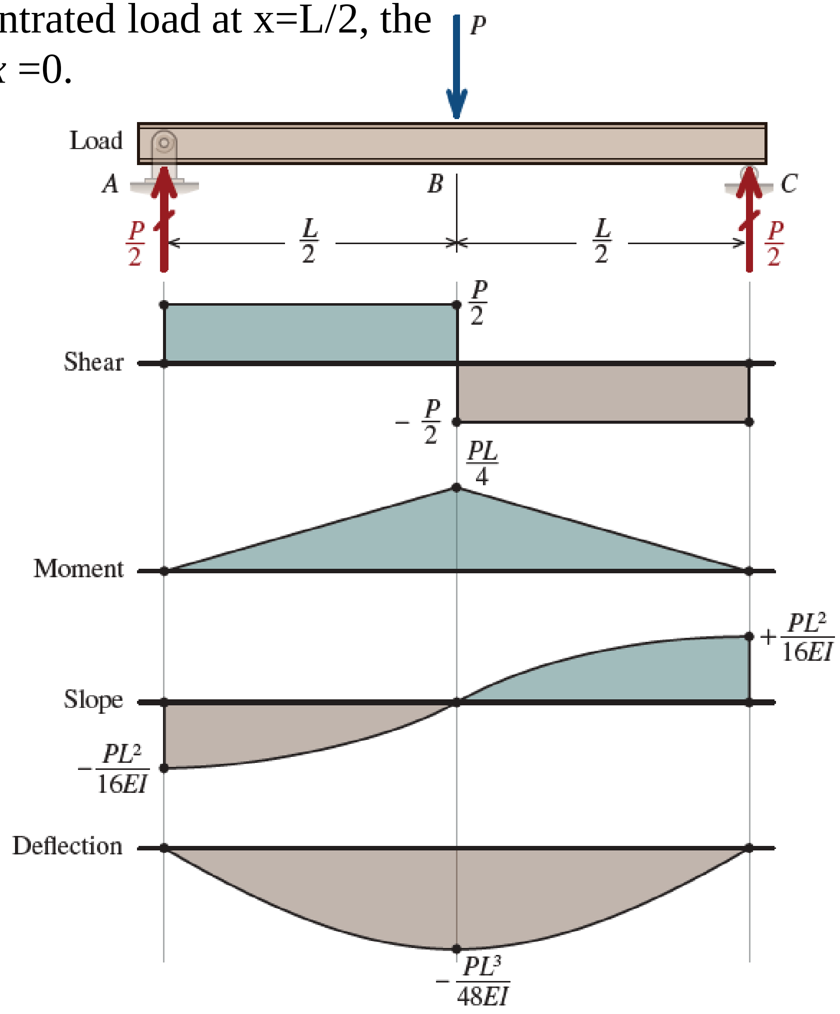


FIGURE 11.5 Relationship among beam diagrams.

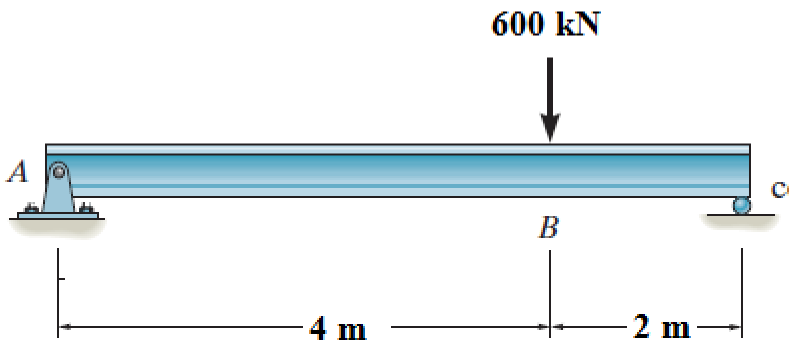
Example 11.1: The beam is loaded and supported as shown in Fig. Assume *EI* constant of the beam, determine:

a- the equation of the elastic curve in term of *x*, *E* and *I*.

b- the rotation (slope) at points *A*, *B* and *C*.

c- the deflection at point *B*.

d- the max deflection.



Solution:

Find the support reactions:

↻ $\Sigma M @ A = 0$

$600 * 4 - C_y * 6 = 0\ ,\ C_y = 400$ kN ↑

↑ $\Sigma F_y = 0$

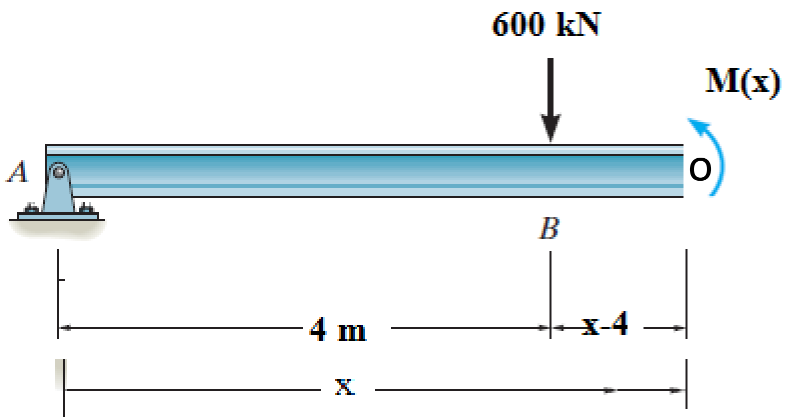
$400 - 600 + A_y = 0,\ A_y = 200$ kN ↑

Equilibrium:

↻ $\Sigma M @ o = 0$

$200 * x - 600 * (x - 4) - M(x) = 0$

$M(x) = 200x - 600(x - 4)$



Integration:

$$EI\,\frac{d^2v}{dx^2} = M(x) = 200\,x - 600\,(x-4) \ldots\ldots\ldots\ldots 1$$

$$EI\,\frac{dv}{dx} = Rotation = 100\,x^2 - 300\,(x-4)^2 + C1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

$$EI\,v = deflection = \frac{100}{3}\,x^3 - 100\,(x-4)^3 + C1x + C2 \ldots\ldots\ldots\ldots 3$$

Boundary Conditions:

At $x = 0$, $v = 0$, sub in Eq. 3

$EI * 0 = (100/3) * (0^3) – 100 * (0 – 4)^3 + C_1 * 0 + C_2$, $C_2 = 0$

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At $x = 6$ m, $v = 0$, sub in Eq. 3

$EI * 0 = (100/3) * (6^3) – 100 * (6 – 4)^3 + C_1 * 6$, $C_1 = -1066.66$

a- equation of the elastic curve:

$$EI\, v = deflection = \frac{100}{3} x^3 – 100\,(x – 4)^3 – 1066.66x$$

b- the rotation (slope) at points *A*, *B* and *C*.

$$EI \frac{dv}{dx} = Rotation = 100\, x^2 – 300\,(x – 4)^2 – 1066.66$$

Rotation at A, $x = 0$

$$EI \frac{dv}{dx} = 100\,(0)^2 – 300\,(0 – 4)^2 – 1066.66$$

$\theta_A = -1066.66/EI$

Rotation at B, $x = 4$ m

$$EI \frac{dv}{dx} = 100\,(4)^2 – 300\,(4 – 4)^2 – 1066.66$$

$\theta_B = 533.34/EI$

Rotation at C, $x = 6$ m

$EI\frac{dv}{dx} = 100\,(6)^2 - 300\,(6-4)^2 - 1066.66$

$\theta_C = 1333.34/EI$

c- deflection at *B*, *x* = 4 m

$EI\,v = \frac{100}{3}(4)^3 - 100\,(4-4)^3 - 1066.66(4)$

$v_B = -2133.31/EI$

d- Maximum deflection

At the max. deflection, the slope ($\theta = 0$), so $EI\frac{dv}{dx} = 0$

$EI\frac{dv}{dx} = 0 = 100\,x^2 - 300\,(x-4)^2 - 1066.66$

Assume Vmax at $0 \le x \le 4$

$0 = 100\,x^2 - 300\,(x-4)^2 - 1066.66$, $x = 3.26$ ok. Sub in Eq.3

0

$EI\,v_{max} = \frac{100}{3}(3.26)^3 - 100\,(3.26-4)^3 - 1066.66(3.26)$

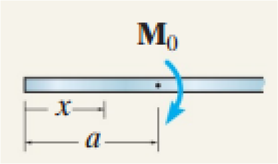
$v_{max} = -2322.45/EI$

Basic Load Represented by Discontinuity Function:

The integration procedures used to derive the elastic curve equations are relatively straightforward if the beam loading can be expressed as single continuous function acting over the entire length of the beam. However, the integration procedures can become quite complicated and tedious for beams that carry multiple concentrated loads or segmented distributed loads.

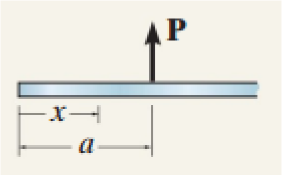
1- Concentrated moment

$M(x) = M_0 \, (x\text{-}a)^0$



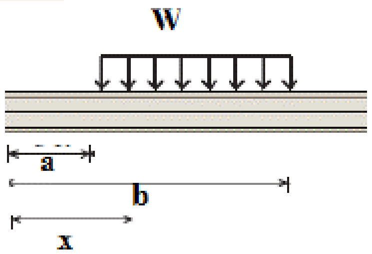
2- Concentrated load

$M(x) = P \, (x\text{-}a)^1$

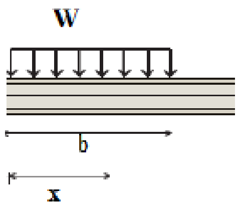


3- Uniform distributed load

$M(x) = \frac{w}{2}(x-b)^2 - \frac{w}{2}(x-a)^2$

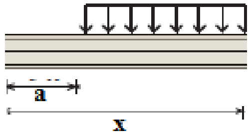


If a=0



$M(x) = \frac{w}{2}(x-b)^2 - \frac{w}{2}(x)^2$

If b=x



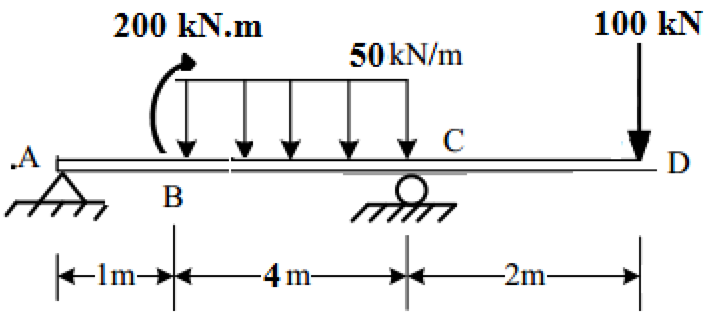
$M(x) = -\frac{w}{2}(x-a)^2$

Example 11.2: for the beam and loading shown in the figure compute:

a- the slope of the beam at (C).

b- the deflection of the beam at (B).

Assume a constant value of $EI = 5*10^{13}$ N.mm^2 for the beam.



Solution:

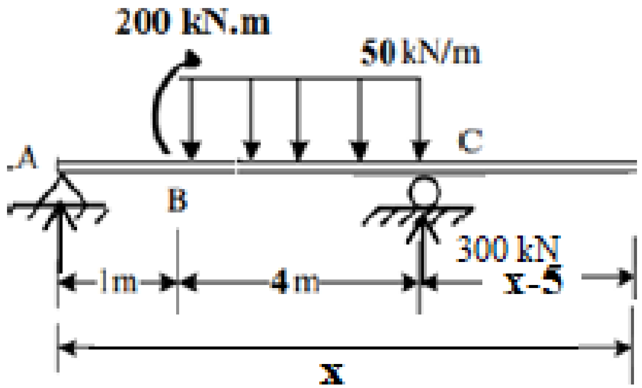
1- Find the support reactions:

$\curvearrowright \Sigma M @ A = 0$

$100 * 7 + 200 + 50 * 4 * 3 - C_y * 5 = 0 \,,\, C_y = 300 \text{ kN} \uparrow$

$\uparrow \Sigma F_y = 0$

$300 - 50 * 4 - 100 + A_y = 0, A_y = 0 \text{ kN}$



2- Equilibrium: $a = 1$ m , $b = 5$ m للحمل المنتشر

$$M(x) = M_0 (x-a)^0 + \frac{w}{2}(x-b)^2 - \frac{w}{2}(x-a)^2$$

$$M(x) = 200 (x-1)^0 + \frac{50}{2}(x-5)^2 - \frac{50}{2}(x-1)^2 + 300 (x-5)$$

$$EI \frac{dv}{dx} = 200 (x-1) + \frac{50}{6}(x-5)^3 - \frac{50}{6}(x-1)^3 + 150 (x-5)^2 + C_1$$

$$EI\, v = 100 (x-1)^2 + \frac{50}{24}(x-5)^4 - \frac{50}{24}(x-1)^4 + 50 (x-5)^3 + C_1 x + C_2$$

3- Boundary conditions:

At $x_A = 0$, $v_A = 0$

$EI\,v = 100\,(x-1)^2 + \frac{50}{24}(x-5)^4 - \frac{50}{24}(x-1)^4 + 50\,(x-5)^3 + C_1x + C_2$

$0 = \frac{200}{2}(0-1)^2 + \frac{50}{24}(0-5)^4 - \frac{50}{24}(0-1)^4 + 50\,(0-5)^3 + C_1{*}0 + C_2,\ C_2 = 0$

At $x_C = 5$ m , $v_C = 0$

$0 = 100\,(5-1)^2 + \frac{50}{24}(5-5)^4 - \frac{50}{24}(5-1)^4 + 50\,(5-5)^3 + C_1{*}5,\ C_1 = -213.33$

a- Slope at *C*:

$EI\,\frac{dv}{dx} = 200\,(x-1) + \frac{50}{6}(x-5)^3 - \frac{50}{6}(x-1)^3 + 150\,(x-5)^2 - 213.33$

At $x_C = 5$ m

$EI\,\theta_c = 200\,(5-1) + \frac{50}{6}(5-5)^3 - \frac{50}{6}(5-1)^3 + 150\,(5-5)^2 - 213.33$

$\theta_c = \frac{53.34}{50000} = 0.00107$ rad

b- Deflection at *B*:

$EI\,v = 100\,(x-1)^2 + \frac{50}{24}(x-5)^4 - \frac{50}{24}(x-1)^4 + 50\,(x-5)^3 - 213.33x$

At $x_B = 1$ m, $EI\,v_B = 100\,(1-1)^2 + \frac{50}{24}(1-5)^4 - \frac{50}{24}(1-1)^4 + 50\,(1-5)^3 - 213.33{*}1$, $v_B = -213.33/50000 = 0.0043$ m $= 4.3$ mm

Example 11.3: A beam is loaded and supported as shown in the Figure. Assume *EI* is constant for the maximum deflection.

Solution:

$M(\mathrm{x}) = - P * x$

$EI \frac{d^2v}{dx^2} = M(\mathrm{x}) = - Px$

$EI \frac{dv}{dx} = - \frac{P}{2} x^2 + C_1$

$EI\, v = - \frac{P}{6} x^3 + C_1 x + C_2$

At $x = L, \frac{dv}{dx} = 0, v = 0$

$EI \frac{dv}{dx} = 0 = - \frac{P}{2} L^2 + C_1, C_1 = \frac{P}{2} L^2$

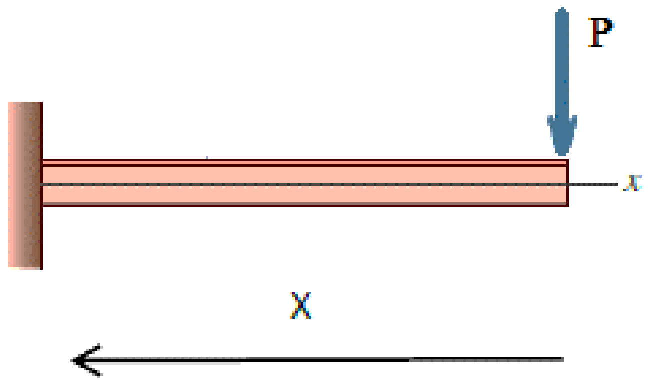
At $x = L, v = 0$

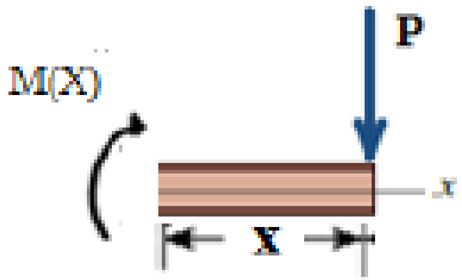
$EI\, v = 0 = - \frac{P}{6} L^3 + \frac{P}{2} L^3 + C_2, C_2 = - \frac{P}{3} L^3$

At $x_{\mathrm{free}} = 0, v = v_{\mathrm{max}}$

$EI\, v = - \frac{P}{6} x^3 + \frac{P}{2} L^2 x - \frac{P}{3} L^3$

$v_{\mathrm{max}} = - \frac{P}{3EI} L^3$





Example 11.4: A beam is loaded and supported as shown in the Figure. Assume *EI* is constant for the maximum deflection.

Solution:

$M(x) = -\frac{w}{2}x^2$

$EI\frac{d^2v}{dx^2} = M(x) = -\frac{w}{2}x^2$

$EI\frac{dv}{dx} = -\frac{w}{6}x^3 + C_1$

$EI\,v = -\frac{w}{24}x^4 + C_1x + C_2$

At $x = L$, $\frac{dv}{dx} = 0$

$EI\frac{dv}{dx} = 0 = -\frac{w}{6}L^3 + C_1$, $C_1 = \frac{w}{6}L^3$

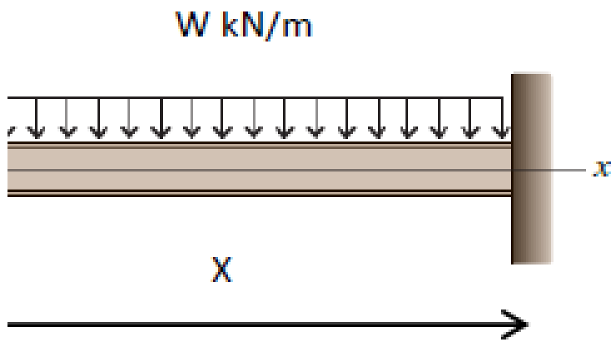
At $x = L$, $v = 0$

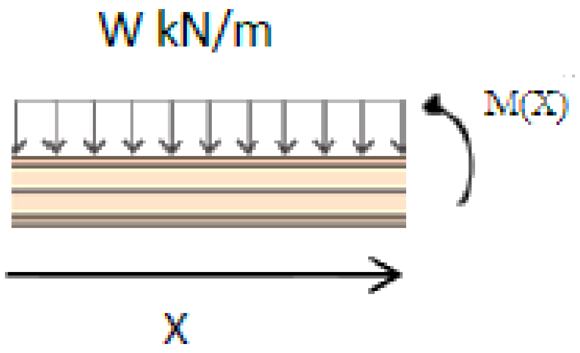
$EI\,v = 0 = -\frac{w}{24}L^4 + \frac{w}{6}L^4 + C_2$, $C_2 = -\frac{w}{8}L^4$

At free end $x = 0$, $v = v_{max}$

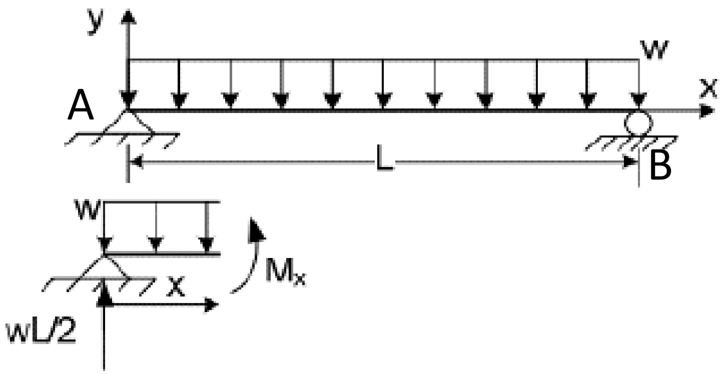
$EI\,v = -\frac{w}{24}x^4 + C_1x - \frac{w}{8}L^4$

$v_{max} = -\frac{w}{8EI}L^4$





Example 11.5: find the maximum deflection for the simply supported beam. Assume *EI* is constant.



Solution:

$EI\frac{d^2y}{dx^2} = M(\text{x}) = \frac{wLx}{2} - \frac{w}{2}x^2$

Integrate both sides of the above equation, yields:

$EI\frac{dy}{dx} = \frac{wL}{4}x^2 - \frac{w}{6}x^3 + C_1$

Apply the boundary conditions (B.C.):

At $x = L$, $y_B = 0$, $C_1 = -\frac{w}{24}L^3$

$EI\frac{dy}{dx} = \frac{w}{4}Lx^2 - \frac{w}{6}x^3 - \frac{w}{24}L^3$ (Rotation equation)

At $x = 0$ or L, $v_{max} = \theta_{max} = -\frac{w}{24EI}L^3$

Integrate both sides of the above equation, yields:

$EI\,y = \frac{w}{12}Lx^3 - \frac{w}{24}x^4 - \frac{w}{24}L^3x + C_2$

At $x_A = 0$, $y_A = 0$, $C_2 = 0$

$EI\,y = \frac{w}{12}Lx^3 - \frac{w}{24}x^4 - \frac{w}{24}L^3x$ (Deflection equation)

At $x = \frac{L}{2}$, $y_{max} = \Delta_{max} = -\frac{5w}{384EI}L^4$

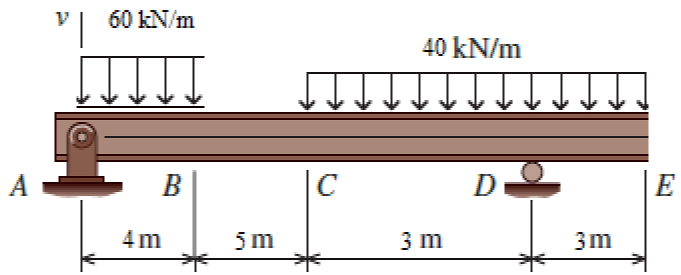
Example 11.6: for the beam shown in Figure, compute:

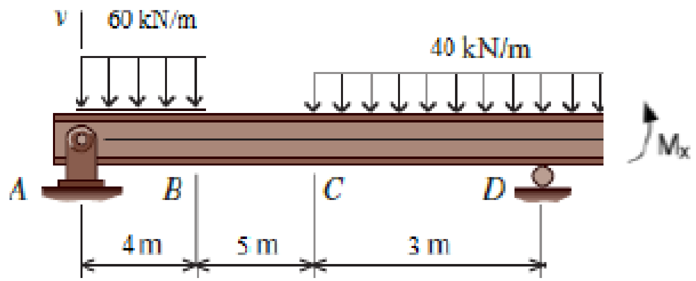
a- the slope of the beam at A.

b- the deflection of the beam at B.

Assume a constant value of EI = 125000 kN.m2 for the beam.

Solution:





1- Find the support reactions:

↻ $\Sigma M @ A = 0$

$60 * 4 * 2 + 40 * 6 * 12 - D_y * 12 = 0\ ,\ D_y = 280 \text{ kN}$ ↑

↑ $\Sigma F_y = 0$

$-60 * 4 - 40 * 6 + 280 + A_y = 0,\ A_y = 200 \text{ kN}$

2- Equilibrium:

$$M(x) = 200\,(x) + \frac{60}{2}(x-4)^2 - \frac{60}{2}(x)^2 + 280\,(x-12) - \frac{40}{2}(x-9)^2$$

$$EI\frac{dv}{dx} = 100\,(x)^2 + 10\,(x-4)^3 - 10\,(x)^3 + 140\,(x-12)^2 - \frac{20}{3}(x-9)^3 + C_1$$

$$EI\,v = \frac{100}{3}(x)^3 + \frac{10}{4}(x-4)^4 - \frac{10}{4}(x)^4 + \frac{140}{3}(x-12)^3 - \frac{20}{12}(x-9)^4 + C_1x + C_2$$

3- Boundary conditions:

At $x_A = 0,\ v_A = 0$

$$0 = \frac{100}{3}(0)^3 + \frac{10}{4}(0-4)^4 - \frac{10}{4}(0)^4 + \frac{140}{3}(0-12)^3 - \frac{20}{12}(0-9)^4 + C_1{*}0 + C_2\ ,\ C_2 = 0$$

Continue.

At $x_D = 12$, $v_D = 0$

$$0 = \frac{100}{3}(12)^3 + \frac{10}{4}(12 - 4)^4 - \frac{10}{4}(12)^4 + \frac{140}{3}(12 - 12)^3 - \frac{20}{12}(12 - 9)^4 + C_1 * 12, \quad C_1 = -1322.08$$

a- the slope of the beam at *A*:

At $x_A = 0$

$$EI\frac{dv}{dx} = 100\,(0)^2 + 10\,(0 - 4)^3 - 10\,(0)^3 + 140\,(0 - 12)^2 - \frac{20}{3}(0 - 9)^3 - 1322.08$$

$$\theta_A = -\frac{1322.08}{125000} = -0.01058 \text{ rad}$$

b- the deflection of the beam at *B*:

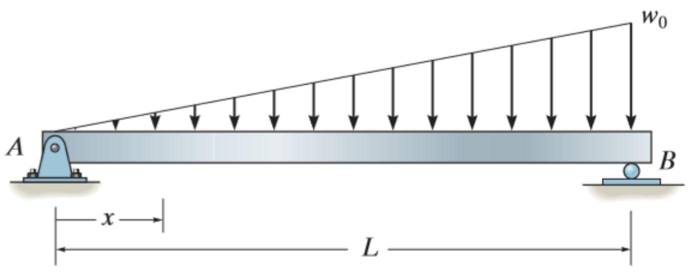
At $x_B = 4$ m

$$EI\, v_B = \frac{100}{3}(4)^3 + \frac{10}{4}(4 - 4)^4 - \frac{10}{4}(4)^4 + \frac{140}{3}(4 - 12)^3 - \frac{20}{12}(4 - 9)^4 - 1322.08 * 4$$

$$v_B = -\frac{3795}{125000} = -0.03036 \text{ m} = 30.4 \text{ mm} \downarrow$$

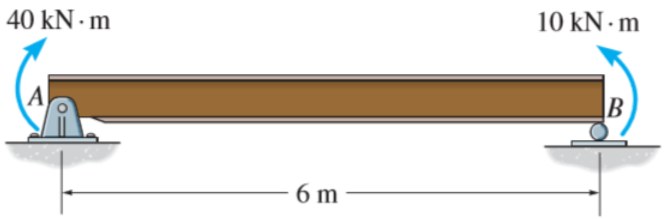
Chapter 11 Beam Deflections

H.W1 The beam is subjected to the linearly varying distributed load. Determine the maximum deflection of the beam. EI is constant.



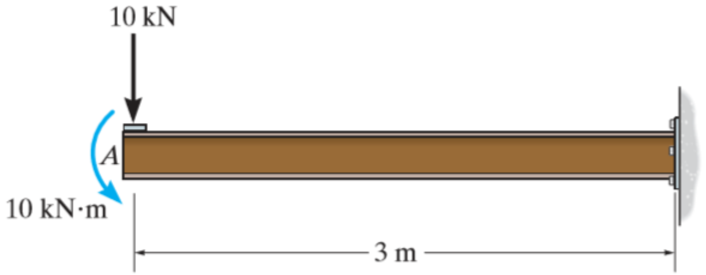
H. W1

H.W2 Determine the maximum deflection of the simply supported beam. $E = 200$ GPa and $I = 39.9(10^{-6})$ m^4.



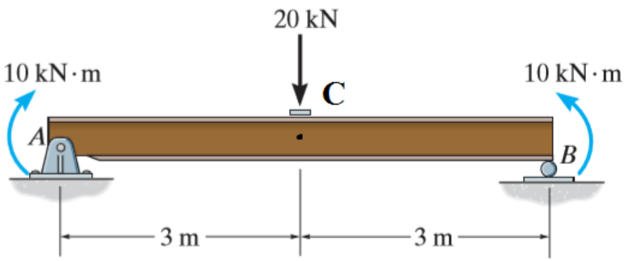
H.W2

H.W3 Determine the slope and deflection of end A of the cantilevered beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.



H.W3

H.W4 Determine the slope of the simply supported beam at A and deflection at C. $E = 200$ GPa and $I = 39.9(10^{-6})$ m^4.



H.W4