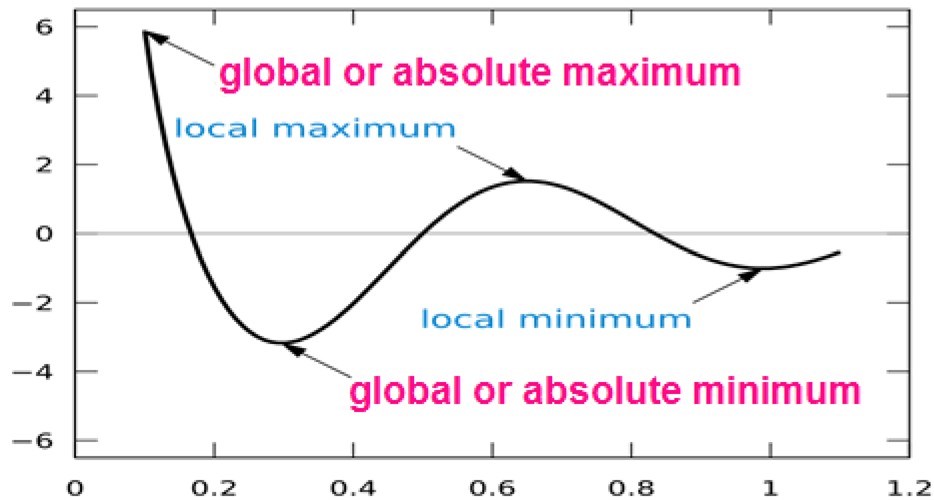
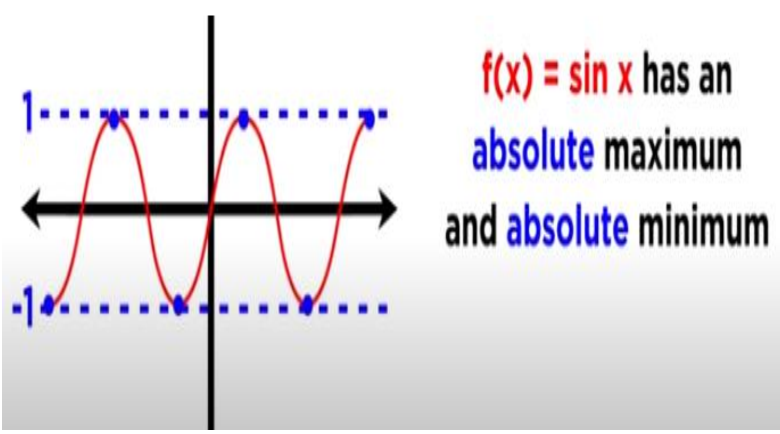
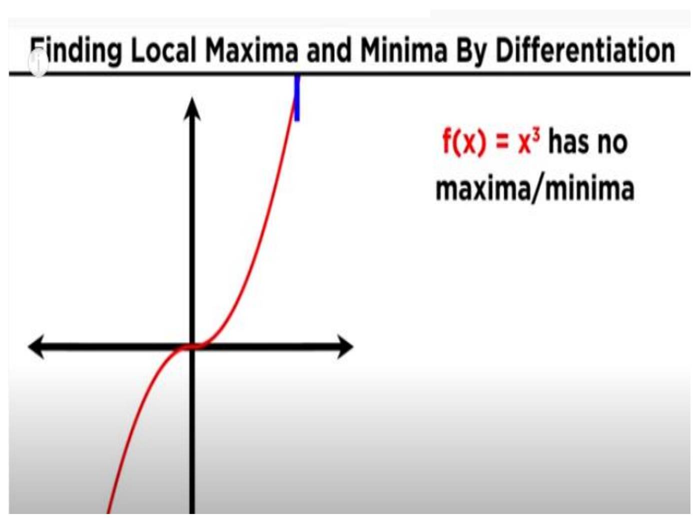
Maximum, minimum for derivatives of a function of one variable

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. The **maximum** and **minimum** of a function, Known collectively as **extreme**, are the largest and smallest value of the function, either with in a given range (the **local** or **relative** extreme) Or on the entire domain of a function (the **global** or **absolute** extreme).

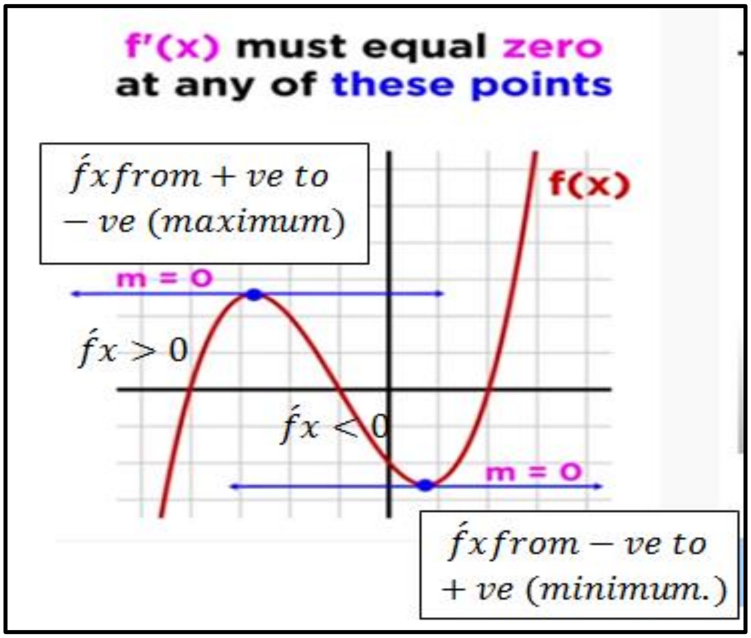


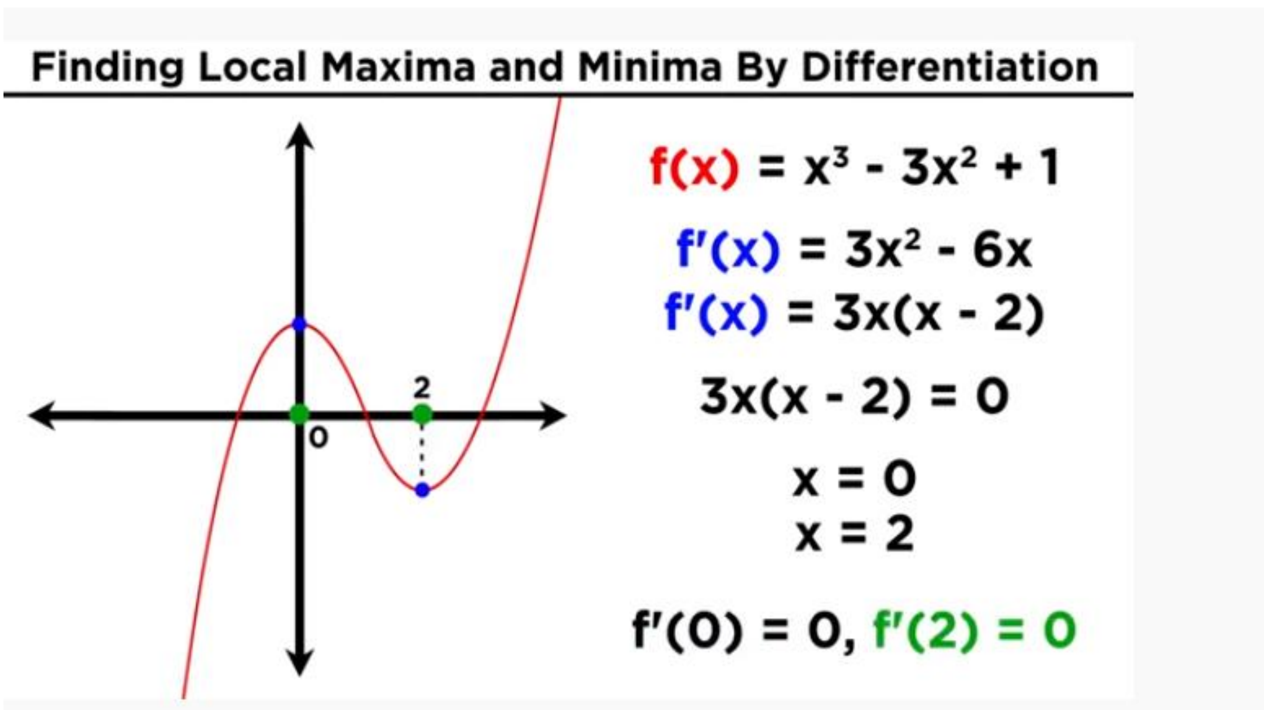




Critical Point

For functions of a single variable, we defined critical points as the values of the variable at which the function's derivative equals zero ($\acute{f}x = \frac{dy}{dx} = 0$) or do not exist.





Example: find the intervals where $f(x) = 4x^3 - x^4$ is increasing or decreasing, and find the local extreme.

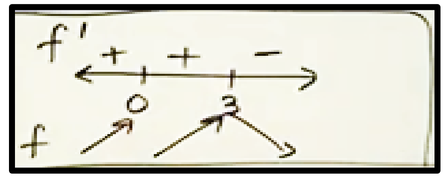
$\boldsymbol{f}(x) = 4x^3 - x^4$

$$\frac{dy}{dx} = f_x = 12x^2 - 4x^3 = 4x^2(3 - x)$$

Find critical point

$f'x = 4x^2(3 - x) = 0$

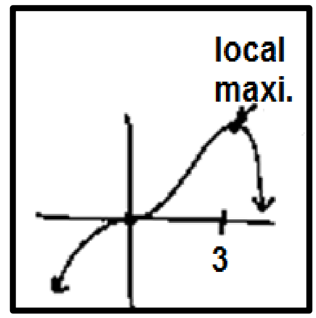
x= 0 and x=3



(- ∞, 0) $f'(-1) = 4(4) = +16$

(0, 3) $f'(1) = 4(2) = +8$

(3, ∞) $f'(4) = 64(-1) = -64$



f is increasing on (- ∞,0) and (0, 3)

f is decreasing on (3, ∞)

f (3) is a local maximum because f'(x)change sign from $+$ ve to $-$ ve

Maximum, minimum and saddle point of two or more variable

This application is important for functions of two or more variables, but, the introduction of more independent variables leads to more possible outcomes for the calculations.

Learning Objectives

- **Use partial derivatives to locate critical points for a function of two variables.**
- **Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.**
- **Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.**

- **Use partial derivatives to locate critical points for a function of two variables.**

For functions of two or more variables, the concept of critical point is essentially the same of one variable, except for the fact that we are now working with partial derivatives.

Definition critical points

Let **z=*f*(x,y)** be a function of two variables that is differentiable on an open set containing the point **(x_o,y_o)** is called a *critical point* of a function of two variables ***f*** if one of the two following conditions holds:

1. $f_x(x_o,y_o) = f_y(x_o,y_o) = 0$
2. Either $f_x(x_o, y_o)$ or fy(xo,y0) does not exist.

Example2: Find the critical points of each of the following functions:

a. $f(x,y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$

b. $g(x,y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution:

a. First, we calculate $f_x(x,y)$ and $f_y(x,y)$:

$$f_x(x,y) = \frac{1}{2}(-18x+36)(4y^2-9x^2+24y+36x+36)^{-1/2}$$
$$= \frac{-9x+18}{\sqrt{4y^2-9x^2+24y+36x+36}}$$
$$f_y(x,y) = \frac{1}{2}(8y+24)(4y^2-9x^2+24y+36x+36)^{-1/2}$$
$$= \frac{4y+12}{\sqrt{4y^2-9x^2+24y+36x+36}}$$

Next, we set each of these expressions equal to zero:

$$\frac{-9x+18}{\sqrt{4y^2-9x^2+24y+36x+36}} = 0$$
$$\frac{4y+12}{\sqrt{4y^2-9x^2+24y+36x+36}} = 0.$$

Then, multiply each equation by its common denominator:

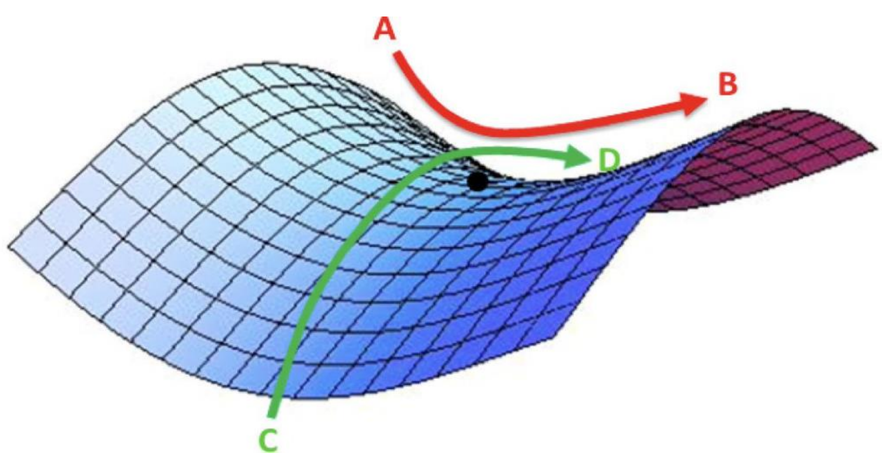
$$-9x+18=0$$
$$4y+12=0.$$

Therefore, $x=2$ and $y=-3$, so $(2,-3)$ is a critical point of f.

Saddle point:

Appoint (a,b) of a function f(x,y) is a saddle point if

- It is a critical point for ***f***(x,y)
- It is a crossing of two contour lines.
- The surface is shaped like a saddle around the critical point: concave up in one direction, concave down in another direction.



- **Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.**

The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** (D):

Let $z=f(x,y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the

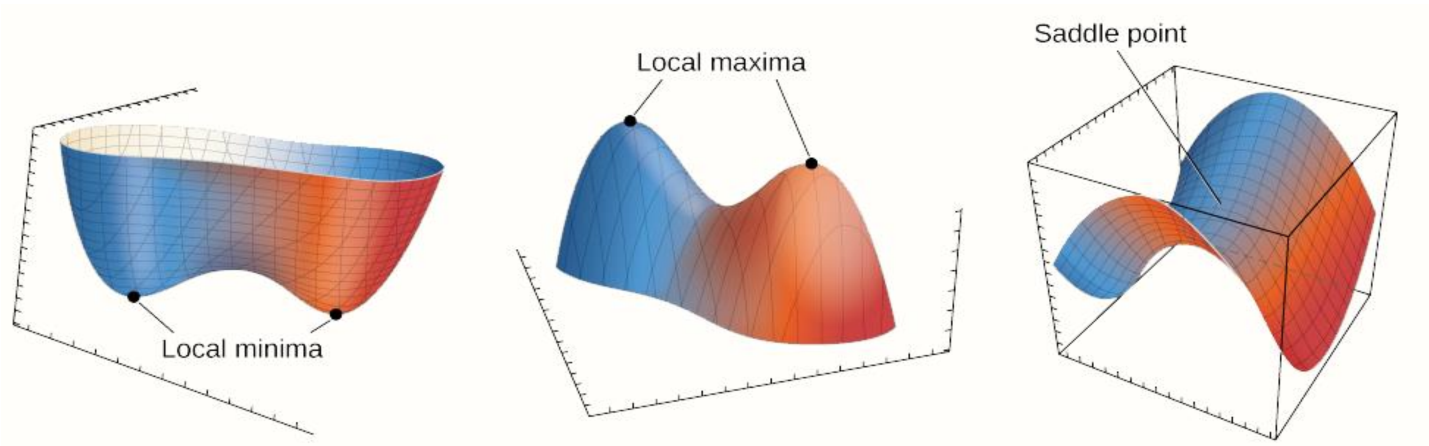
Point (x_0,y_0).

1- Find the critical point, Suppose $f_x(x_o,y_o)=0$ and $f_y(x_o,y_o)=0$
2- Find f_{xx}, f_{yy} and f_{xy} at critical point (x_o,y_o)
3- Find the **discriminant** (D) from the equation

$$D=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-\left(f_{xy}(x_0,y_0)\right)^2.$$

Then:

i. If $D>0$ and $f_{xx}(x_0,y_0)>0$, then f has a local minimum at (x_0,y_0).
ii. If $D>0$ and $f_{xx}(x_0,y_0)<0$, then f has a local maximum at (x_0,y_0).
iii. If $D<0$, then f has a saddle point at (x_0,y_0).
iv. If $D=0$, then the test is inconclusive.



example3

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x,y) = 4x^2 + 9y^2 + 8x - 36y + 24$

b. $g(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution

a- $f(x,y)=4x^2+9y^2+8x-36y+24$

Step 1 of the problem-solving strategy involves finding the critical points of f. To do this, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set each of them equal to zero:

$$f_x(x,y) = 8x + 8$$
$$f_y(x,y) = 18y - 36.$$

Setting them equal to zero yields the system of equations

$$8x + 8 = 0$$
$$18y - 36 = 0.$$

The solution to this system is $x=-1$ and $y=2$. Therefore $(-1,2)$ is a critical point of ***f***.

Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of ***f***.

$$f_{xx}(x,y)=8$$
$$f_{xy}(x,y)=0$$
$$f_{yy}(x,y)=18.$$

Therefore, $D= f_{xx}(-1,2)\, f_{yy}(-1,2) - (f_{xy}(-1,2))^2=(8)(18)-(0)^2=144.$

Step 3 states to apply the four cases of the test to classify the function's behavior at this critical point.

Since $D>0$ and $f_{xx}(-1, 2)>0$, this corresponds to case 1.Therefore, f has a local minimum at $(-1, 2)$ as shown in the following figure.

$f(-1, 2) = 4(-1)^2 + 9(2)^2 + 8(-1) - 36(2) + 24 = -16$

b- $g(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

For **step 1**, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set each of them equal to zero:

$$g_x(x,y) = x^2 + 2y - 6$$
$$g_y(x,y) = 2y + 2x - 3.$$

Setting them equal to zero yields the system of equations

$$x^2 + 2y - 6 = 0$$
$$2y + 2x - 3 = 0.$$

To solve this system, first solve the second equation for y. This gives $y=\frac{3-2x}{2}$. Substituting this into the first equation gives

$$x^2 + 3 - 2x - 6 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0.$$

Therefore, $x=-1$ or $x=3$, Substituting these values into the equation $y=\frac{3-2x}{2}$ yields the critical points $(-1,\frac{5}{2})$ and $(3, -\frac{3}{2})$.

Step 2 involves calculating the second partial derivatives of g:

$$g_{xx}(x,y) = 2x$$
$$g_{xy}(x,y) = 2$$
$$g_{yy}(x,y) = 2.$$

Then, we find a general formula for D:

$D(x_0,y_0)=g_{xx}(x_0,y_0)\, g_{yy}(x_0,y_0)-(g_{xy}(x_0,y_0))^2=(2x_0)(2)-2^2=4x_0-4.$

Next, we substitute each critical point into this formula:

$$D\left(-1,\tfrac{5}{2}\right)=(2(-1))(2)-(2)^2=-4-4=-8$$
$$D\left(3,-\tfrac{3}{2}\right)=(2(3))(2)-(2)^2=12-4=8.$$

In step 3, we note that, applying Note to point $(-1,\frac{5}{2})$ a lead to case 3, which means that $(-1,\frac{5}{2})$ is a saddle point. Applying the theorem to point $(3, -\frac{3}{2})$ leads to case 1, which means that $(3, -\frac{3}{2})$ corresponds to a local minimum as shown in the following figure.

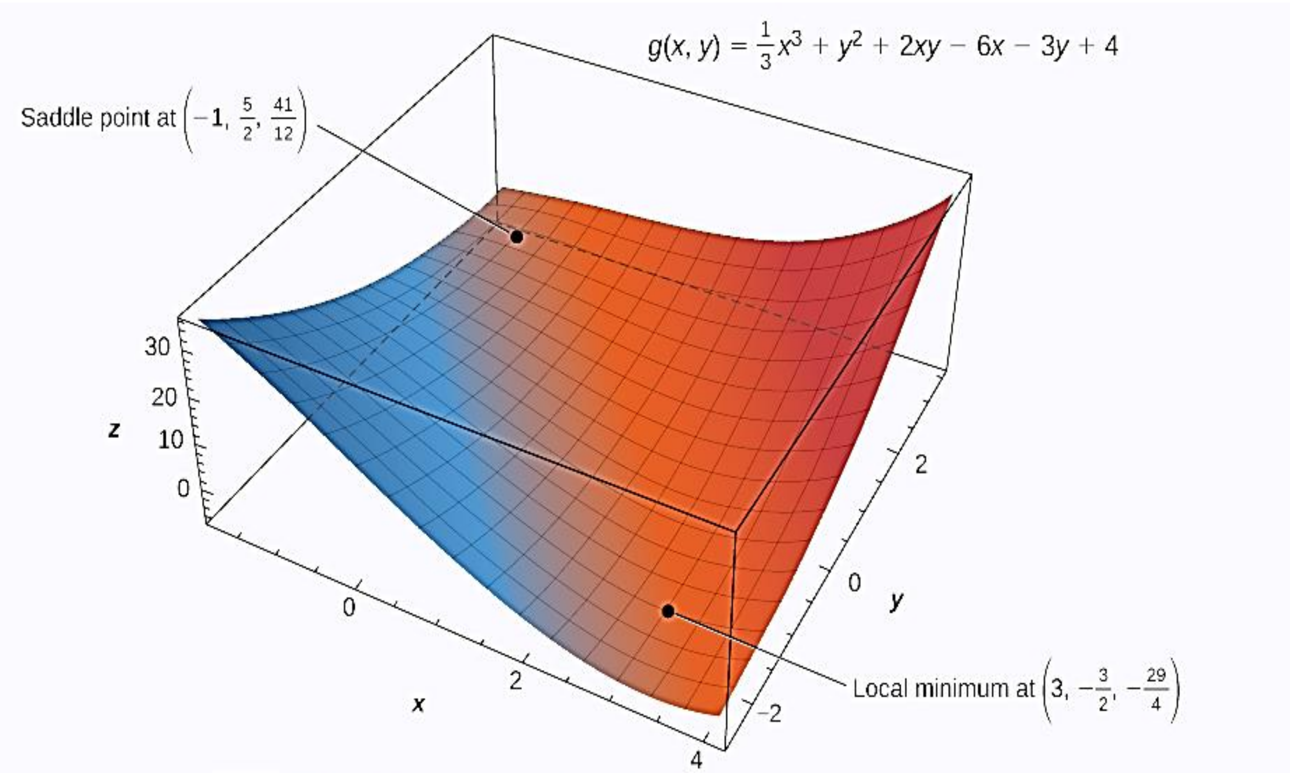


Figure: The function $g(x,y)$ has a local minimum and a saddle point.

- **Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.**

Let $z=f(x,y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3

Example 4: Use the problem-solving strategy for finding absolute extreme of a function to determine the absolute extreme of each of the following functions:

1- $f(x,y)=x^2-2xy+4y^2-4x-2y+24$ on the domain defined y $0\le x\le 4$ and $0\le y\le 2$

2- $g(x,y)=x2+y2+4x-6yg$ on the domain defined by $x2+y2\le 16$ **(H.W)**

Solution:

1- $f(x,y)=x^2-2xy+4y^2-4x-2y+24$

Step 1 involves finding the critical points of **f** on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$f_x(x,y)=2x-2y-4$$
$$f_y(x,y)=-2x+8y-2.$$

Setting them equal to zero yields the system of equations

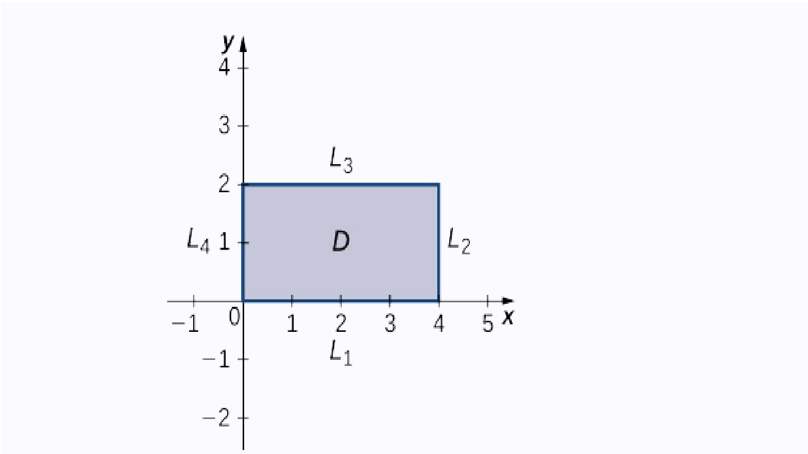
$$2x-2y-4=0$$

$$-2x+8y-2=0.$$

The solution to this system is $x=3$ and $y=1$. Therefore $(\mathbf{3, 1})$ is a critical

Point of $f(x,y)$. Calculating $f(\mathbf{3,1})$ gives $f(3,1)=17$.

The next step involves finding the extreme of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:



Graph of the domain of the function $f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$.

x	y	f(x,y)=x2−2xy+4y2−4x−2y+24
0	0	24
4	0	24
4	2	20
0	2	**36**
2	0	20
3	1	**17 (critical point)**

Line L1,

y=0 and

$f(x,0) = x^2 - 2x(0) + 4(0)^2 - 4x - 2(0) + 24$

$f(x,0) = x^2 - 4x + 24 \qquad 0 \le x \le 4$

$f(0,0) = 24 \qquad f(4,0) = 24$

$f'x = \mathbf{2x - 4 = 0} \longrightarrow \mathbf{x = 2}$

$\mathbf{f(2,0) = 20}$

Line L2,

x=4 and

$f(4,y) = 4^2 - 2(4)(y) + 4y^2 - 4(4) - 2y + 24 = 16 - 8y + 4y^2 - 16 - 2y + 24$

$f(4,y) = 4y^2 - 10y + 24 \qquad 0 \le y \le 2$

$f(4,0) = 24 \qquad f(4,2) = 20$

Line L3,

y=2 and

$f(x,2)= x^2-2x(2) +4(2)^2-4x-2(2)+24$

$f(x,0)= x^2-8x+36 \qquad 0\leq x\leq 4$

$f(0,2) = 36 \qquad f(4,2) = 20$

Line L4,

x=0 and

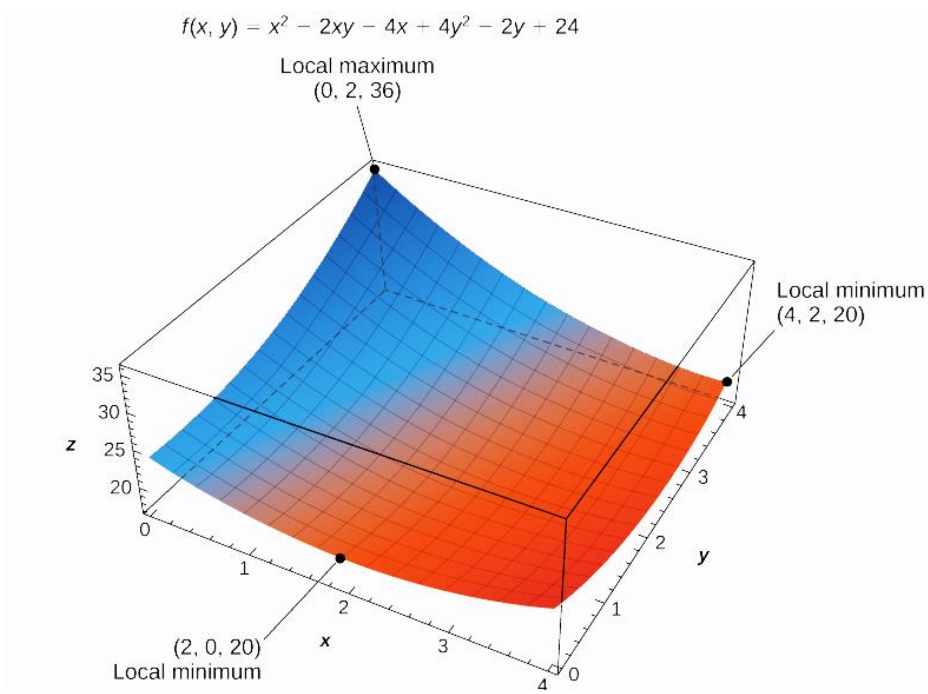
$f(0,y)= 0^2-2(0)(y) +4y^2-4(0)-2y+24$

$f(4,y)= 4y^2-2y+24 \qquad 0\leq y\leq 2$

$f(0,0) = 24 \qquad f(0,2) = 36$

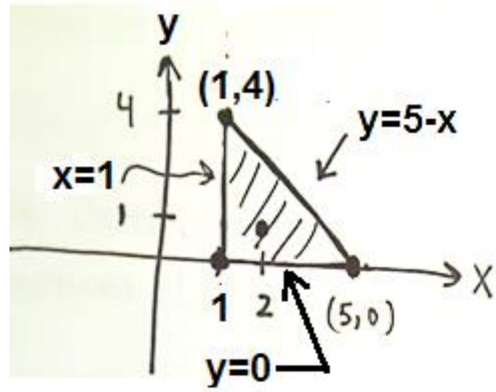
The absolute maximum value is 36, which occurs at (0, 2) and the absolute minimum value is 17, which occurs at critical point (3, 1).

The local minimum value is 20, which occurs at both (4, 2) and (2, 0) as shown in the following figure.



Example 5: Find the absolute maximum and minimum of:

F(x,y)= 3+ xy – x – 2y ; Domain is the closed triangular region with vertices(1,0),(5,0), (1,4).



x	y	*f*(x,y)= 3+ xy – x – 2y
1	0	2
5	0	-2
1	4	-2
2	1	**1** **(critical point)**
3	2	2

Solution:

Step 1: find critical point *f*x=0 ; *f*y=0

*f*x = y-1=0 ; *f*y= x-2=0

y= 1; x=2 critical point (2, 1)

Step2: draw the triangular boundaries

X=1

Y = 0

Find the equation of slope line; from two points (1, 4) and (5, 0)

y-yo= m (x-xo) take point (5,0) as (xo,yo)

y-0= $\frac{y1-yo}{x1-xo}\ (x-5)$

y= $\frac{4-0}{1-5}\ (x-5) \rightarrow\ y = -\frac{4}{4}\ (x-5) = -1(x-5)$

y= 5-x

Step 3: find the value of **f(x,y)** at boundaries

Along line x=1

f(1,y)= 3 +y – 1 – 2y = 2 – y 0≤ y≤ 4

f(1,0) = 2-y=2-0=2

f(1,4) = 2- y= 2- 4 = -2

Along line y =0

$f(x,0) = 3 + x(0) - x - 2(0) = 3 - x \quad 1 \le x \le 5$

$f(1,0) = 3 - x = 3 - 1 = 2$

$f(5,0) = 3 - x = 3 - 5 = -2$

Along y = 5 – x $f(x,y) = 3 + xy - x - 2y)$

$\mathbf{f(x, 5-x) = 3 + x(5-x) - x - 2(5-x) = 3 - x2 + 5x - x + 2x - 10}$

$\mathbf{f(x,5-x) = -x2 + 6x - 7}$ **1≤ x≤ 5 (من الرسم)**

$f(1,4) = -1 + 6 - 7 = -2$

$f(5, 0) = -25 + 30 - 7 = -2$

g(x) = $\mathbf{-x2 + 6x - 7}$

$\acute{g}_x = -2x + 6 = 0 \rightarrow -2x = -6 \quad x = 3$

$g(3) = -3^2 + 6(3) - 7 = 2$

$f(x, 5-x)$ at $x = 3$

$f(3, 2) = 2$

From table

Absolute maximum *f*(x,y) or z = 2

Absolute minimum *f*(x,y) or z = -2