Headway models

Overview

Headway is a key input in calculating the overall route capacity of any transit system.
Vehicle time headway is a measure of the temporal space between two vehicles and is defined as the inter-arrival time difference between the leading vehicle and the following vehicle at a designated test point on a traffic lane.
Time headway (h_t) = the difference between the time when the front of a vehicle arrives at a point on the highway and the time the front of the next vehicle arrives at the same point (in seconds).

Relation between headway and capacity

The headway distribution describes which headways can be observed with which probability. One might argue that this is related to the flow since the average headway is the flow. The flow is determined by the demand. However, in the bottleneck, the flow is determined by the minimum headway at which drivers follow each other. Or in other words, the headway distribution determines the capacity in the bottleneck. Suppose the headway distribution is given by (h). Then, the average headway can be determined by:

$$\langle h \rangle = \int_0^{inf} hp(h)dh \qquad 1$$

This is a mathematical way to describe the average headway once the distribution is given. The flow in the active bottleneck, hence the capacity is the inverse of the mean headway, see Table 1:

$$c = 1/\langle h_{bottolneck} \rangle \qquad 2$$

The number of vehicles arriving in a certain period could be a useful measure. This holds for instance for traffic lights, where the number of arrivals per red period is relevant. As illustrated in Figure 1, there could be different lanes for different directions at a traffic light. The idea is that the traffic in one direction will not block the traffic in other directions, hence, the length should be long enough to allow the number of vehicles in the red period. The average number of vehicles in a red period can be determined from the flow. However, most requirements are that in p% of the red times (under a constant demand) the queue should not exceed the dedicated lane. In that case, the distribution of the number of arrivals in that period can form the basis for the calculations.

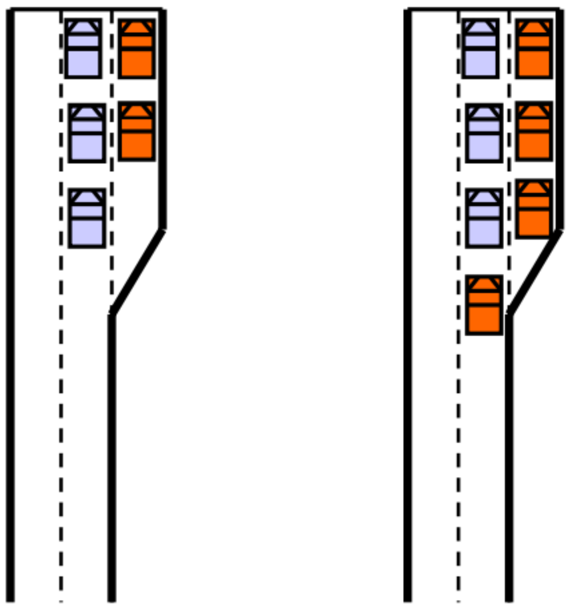

Figure 1: A queuing area. The orange-colored vehicles turn right, whereas the blue ones continue straight.

Table 1: The different processes and the underlying assumptions.

Process characteristic	Headway dist	Dist of nr of arrivals per interval
Independent arrivals	Exponential	Poisson
Correlated arrivals		Binomial
Negatively correlated arrivals		Negative binomial

Table 2: Overview of the means and variances of the different distributions. In this table, q is the flow in the observation period, and p is the probability of including the observation in the period.

Distribution	Mean	Variance	
Poisson	q	q	=mean
Binomial	np	$np(1-p)$	<mean
Negative binomial	$n(1-p)/p$	$n(1-p)/p^2$	>mean

Arrivals per interval

This section describes the number of arrivals per time interval. For different conditions, this distribution is different. Table 2 gives an overview of the distributions described in this section, and gives some characteristics. One should differentiate between the probability of several arrivals (a macroscopic characteristic, based on aggregating over a certain duration of time), described in this section, and the probability of a headway (a microscopic characteristic, described in the next section).

Poisson

The first distribution function described here is the Poisson distribution. One will observe this distribution function once the arrivals are independent. The resulting probability is described by a so-called Poisson distribution function. Mathematically, this function is described by:

$$p(X = K) = \frac{q^k}{k!} e^{-q} \qquad 3$$

This equation gives the probability that k vehicles arrive if the average arrival rate per period is q. Hence note that one needs to rescale q to units of the number of vehicles per aggregation period!

For example, consider a flow of 600 veh/h with independent arrivals. What is the probability to have 7 vehicles arriving in a period of 30 seconds? Answer: we first rescale the flow to the same interval as the period of interest, i.e. 30 seconds. We find q=600 veh/h = 5 veh/30 seconds. Now, applying Equation 3, we find the probability of finding 7 vehicles in the observation interval:

$$p(X = 7) = \frac{q^7}{7!} e^{-5} = 0.104 \qquad 4$$

So in this example, there is a 10.4% chance that 7 vehicles arrive in this period.

Figure 2 shows examples of the Poisson distribution. Note that for low values of the flow (expected value smaller than 1), the probability is decreasing. If the flow is higher, there is a maximum probability of several arrivals that is at a higher value than 1.

Figure 3 shows the best fit of this distribution on real-life data. This distribution is accurate if the flow is low, and is not so good if the flow increases. This is because once the flow is high, the assumption of independent arrivals does not hold anymore. Once vehicles are bound by the minimum headway, the arrivals are not independent anymore. This restriction comes more into play once the flow is high.

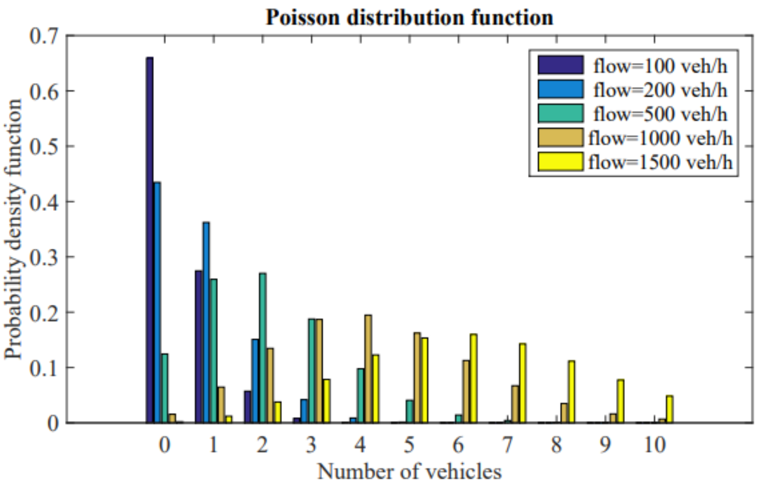

Figure 2: Example of the Poisson distribution for different flow values; the flow is indicated in veh/h, and these are the probabilities for arrivals in 15 seconds.

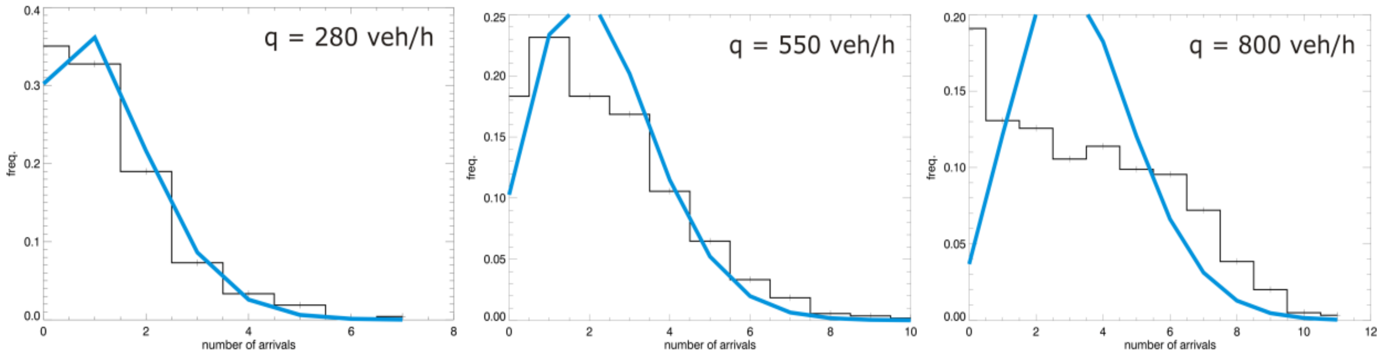

Figure 3: Illustration of the number of arrivals from real-world data.

Binomial

The binomial distribution can be used if there are correlations between the arrivals. For instance on busy roads, one can expect more vehicles to drive at a minimum headway. Whereas in the case of the Poisson distribution, the variance of the distribution was equal to the mean, in this case, the variance is smaller (since more drivers drive at a certain headway). The mathematical equation describing the function is:

$$P(X = K) = \binom{n}{k} P^K (1-p)^{n-K} \qquad 5$$

The idea behind the distribution is that one does n tries, each with an independent success rate of p. The number of successes is k. The mean of the distribution is np, see Table 2. A certain flow specifies the mean of the arrivals, which hence determines np. This gives the freedom to choose n or p, by which one can match the spread of the function. The number of observations in the distribution can never exceed n, so a reasonable choice of n would be the interval time divided by the minimum headway. Figure 4 shows examples of the binomial distribution function.

Note that the variance of the binomial function is smaller than the Poisson distribution for the same flow. This can be a reason to choose this function.

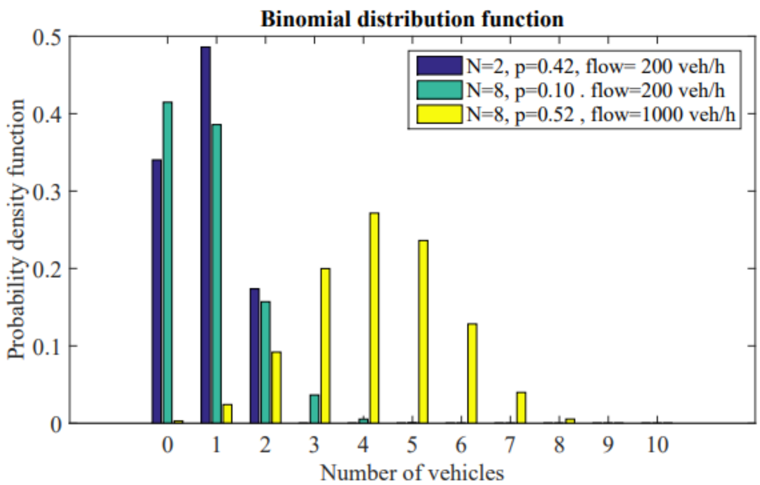

Figure 4: Examples of the binomial distribution function for the number of arrivals in an aggregation period.

Negative binomial

The negative binomial distribution can be used if there are negative correlations between the arrivals. In traffic, this happens for instance downstream of a signalized intersection. If one observes several vehicles at a short headway, one gets a larger probability that the net next headway will be large because the traffic light will switch to red.

The probability distribution function for the number of observed vehicles in an aggregation interval is:

$$(X = K) = \binom{K+r-1}{K} P^K (1-p)^r \qquad 6$$

This distribution describes when one observes an individual and independent process with a success rate of p. One observes so until r failures are observed. X is the stochastic indicating how many successes are observed.

Figure 5 shows the value of this function for different parameter sets. Note that the variance (and hence standard deviation) be set independently from the mean, like in the binomial distribution function. For this distribution (see Table 2), the mean is given by $n\ (1-p)/p$, and the variance is given by $(1-p)/p^2$, which is the mean divided by p. Since p is a probability and has a value between 0 and 1, we can derive that the variance is larger than the mean. A larger variance is what one would intuitively expect downstream of a signalized intersection. This characteristic can be used to have an idea of the distribution to use.

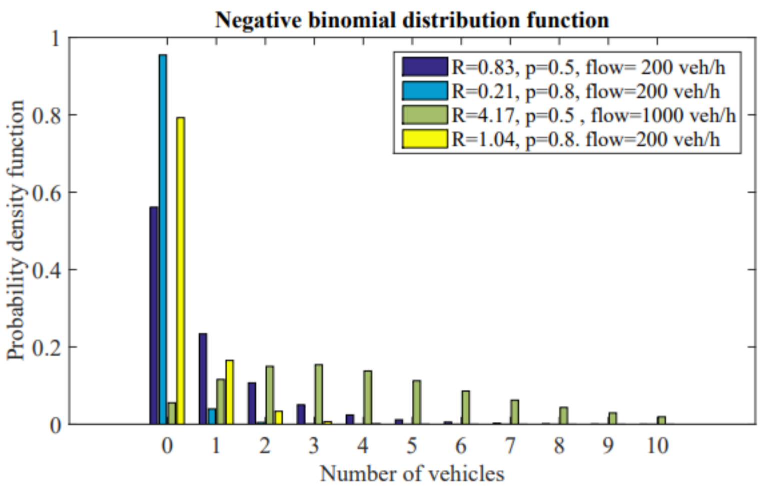

Figure 5: Example of the negative binomial function for the number of arrivals in an aggregation period.

Headway distributions
In this section, the exponential headway distribution is described, and used with independent arrivals, and composite headway models.

Exponential
The first distribution is the exponential distribution, shown in Figure 7. This is defined by:

$$p\left(h/_{q}\right) = q\ exp - qh \qquad 7$$

Note that this equation has a single parameter, the flow q. That is the inverse of the average headway. The underlying assumption of this distribution is that all drivers can choose their moment of arrival independently. Consider that each (infinitesimally small) time step a driver considers to leave with a fixed probability. One then gets an exponential distribution function for the headway.
This is a very good assumption on quiet roads when there are no interactions between the vehicles. The interactions occur once vehicles are limited in choosing their headway, mostly indicated by the minimum headway. An illustration of how this works out for a real-life case is shown in Figure 3 for the number of arrivals.

The only good way to test whether the exponential data describes the data one observes is to do a proper statistical test (e.g., a Kolmogorov–Smirnov test). There are also rules of thump. A characteristic of this distribution is that the standard deviation is equal to the mean. If one has data of which one thinks the arrivals are independent and this criterion is satisfied, the arrivals are very likely to be exponentially distributed.
This distribution function for the headways matches the Poisson distribution function for the number of arrivals.

Composite headway models
Whereas the exponential distribution function works well for low flows, for higher flows the distribution function is not very good. For these situations, so-called composite headway distributions are being used (see for more information Hoogendoorn (2005)). The basic idea is that a fraction of the traffic Φ is driving freely following a headway distribution function $P_{\text{free}}\ (h)$. The other fraction of the traffic 1−Φ is driving constraint, i.e., is following their leader, and has a headway distribution function $P_{\text{constraint}}\ (h)$.

In a composite headway distribution, these two distribution functions are combined. The combined distribution function can hence be expressed as:

$$P(h) = \emptyset P_{free}(h) + (1 - \emptyset)P_{constraint}(h) \qquad 8$$

A plot of the headway distribution function (in fact, a survival function of the headway, i.e. 1- the cumulative distribution function) is shown in Figure 7. The vertical axis is logarithmic. Note that in this axis the exponential distribution function is a straight line. That is what is observed for the large headways. For the smaller headways (in the figure, for less than 7 seconds) this does no longer hold. This is due to the limitation of the following distance. As the figure shows, this can be determined graphically.

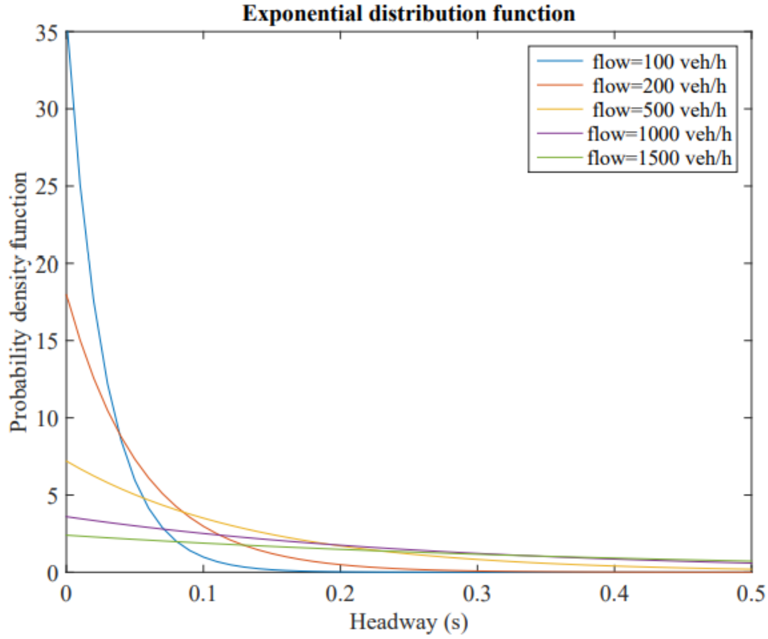

Figure 6: The probability density function for headways according to the exponential distribution for different flow values.

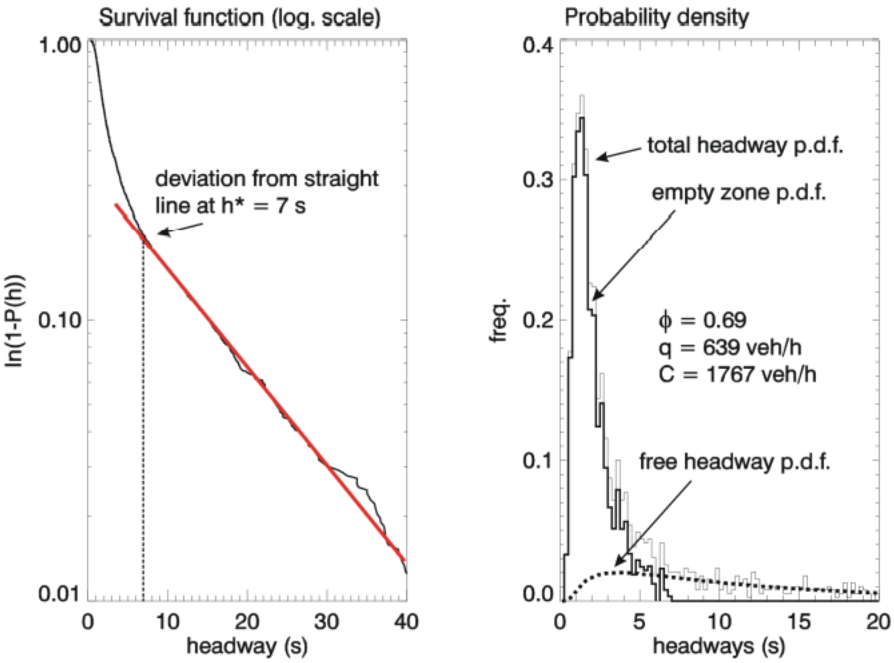

Figure 7: Example of the composite headway distribution and its estimation for real-life data.

Critical gap

The critical gap is the smallest gap (time gap) that a vehicle accepts to go into. Some examples are given in the next section. The relation with the inflow capacity is discussed in the next section.

Situations

Examples of situations where the critical gap is relevant are turning onto a road where one has to give priority, overtaking, merging into the traffic stream, or lane changing. Some of them are shown in Figure 8 (b).

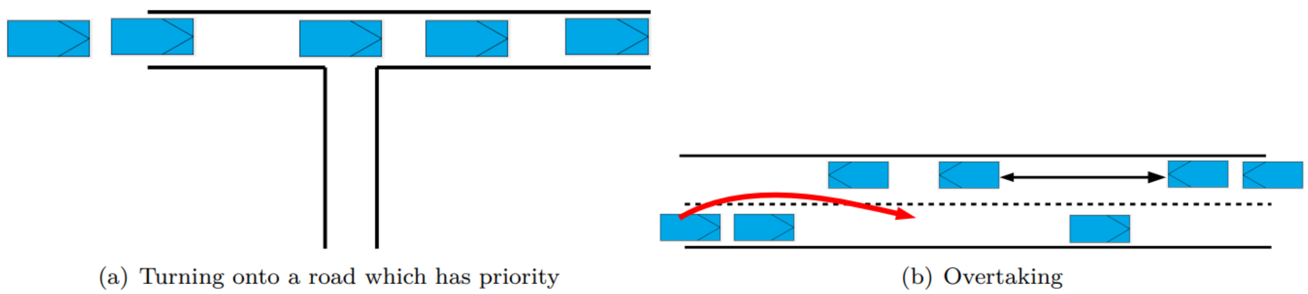

Figure 8: Example of overtaken.

The critical gap is defined in the flow. Note that the critical gap differs per person, as well as per the situation. One might have a small critical gap to merge into a traffic stream moving at a slow speed.
For overtaking, the critical gap (measured in headway in the opposite traffic stream) needs to be larger because the traffic is moving in the opposite direction. For instance, if a driver spends 3 seconds in the lane of the opposing traffic for an overtaking maneuver, during that time the opposing vehicles also move in the direction of the overtaking vehicle. To find the critical (time) gap, one needs to consider the distance the overtaken uses at the other lane (overtaking times speed) – this is the space the overtaken would need if the opposing vehicles were stationary. To this, one should add the distance the opposing traffic moves in the overtaking time (overtaking time times the speed of the opposing vehicles). This combined distance gap (plus arguably a safety margin) can be translated into a time gap. Since one measures the gap as headway in the opposing traffic, one needs to divide the required space gap by the speed of the opposing traffic.

Inflow capacity

Consider the situation as in Figure 8(a). Consider that the main line traffic (traffic from left to right) has a headway distribution of (h), and there is an infinite line of vehicles waiting to enter the road, all with a critical gap gc. One can calculate how many vehicles can enter the road per unit of time, as will be shown below.

We know that for certain gaps, vehicles can enter. The first step is to find the number of gaps that pass per unit of time. That equals the number of vehicles (each vehicle causes one new gap). The number of vehicles per unit of time is the flow, which is the inverse of the mean of the headway, which in turn can be calculated from the headway distribution.

Then, for all headways larger than the critical gap, but smaller than twice the critical gap one vehicle can enter. The frequency that this happens is the flow (=gap rate) times the probability that the gap is this size, determined by integrating the probability density function of the headway over the right headways:

$$q_{in\ 1_{vehicle}} = q \int_g C^{2gc} P(h)dh \tag{9}$$

The probability that in one gap two vehicles can enter is the integral of the probability density function for the headways from twice the critical gap to three times the critical gap. The rate at which gaps these gaps occur is the flow q, so the rate at which these gaps occur is $q \int_{2gc}^{3gc} p(h)dh$ per gap, two vehicles enter, so the matching inflow is:

$$q_{in\ 2_{vehicle}} = 2q \int_{2gc}^{3gc} p(h)dh \tag{10}$$

One can continue this reasoning and find an equation for the inflow:

$$q_{in_{total}} = \sum_n (q_{in_{n\ vehicles}}) = \sum_n nq \int_{ngc}^{(n+1)gc} p(h)dh \tag{11}$$

This can numerically be evaluated.

Typically, one would expect the headway in the order of twice the minimum headway. Namely, after merging, one would like to have at least a minimum headway upstream (lag gap) and a minimum headway downstream (lead gap).

In the calculation of the maximum inflow, the headway distribution function plays an important role. As an extreme example, suppose that all headways are equal at 99% of the critical gap. Then, the flow is at about half the capacity of the road, but the inflow is 0. For different distribution functions, it can be calculated what the maximum inflow on the main road is. This is shown in Figure 8 for different functions. It shows that the inflow capacity decreases with the main road flow. However, the precise type of distribution matters less: the graphs for the different distribution functions are quite similar.

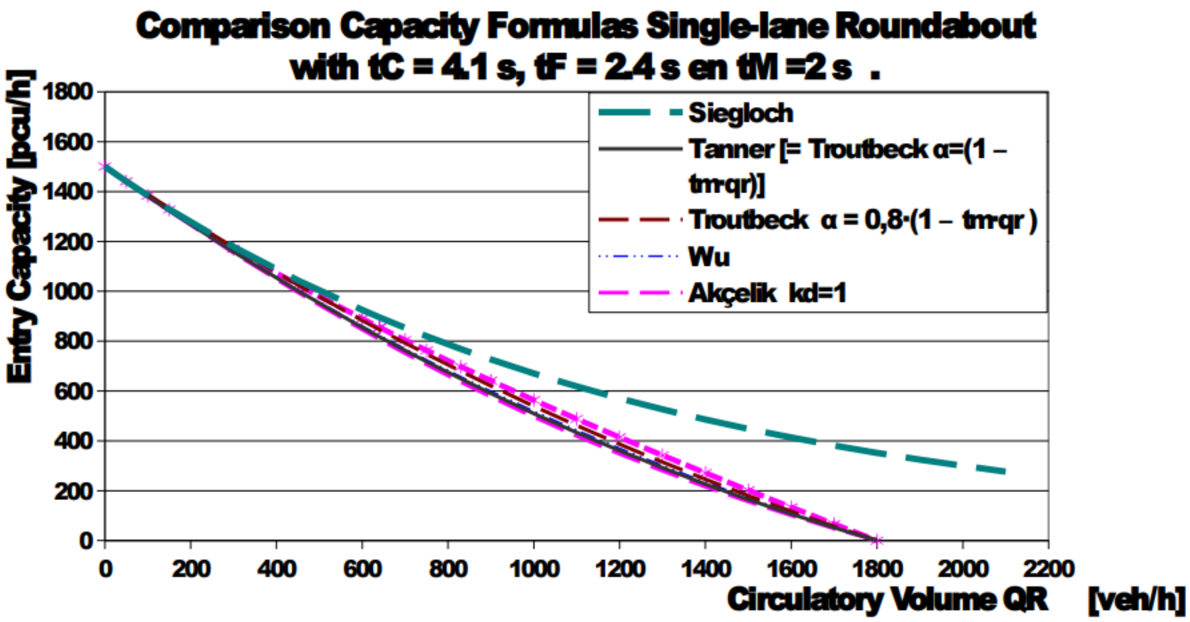

Figure 8: The inflow onto a road (or roundabout) as a function of the main road (roundabout) traffic – figure from Fortuijn and P. (2015)

Exercises

Q1/ To which distribution function for the number of arrivals per cycle (N) does this lead? Name this distribution.

The probability distribution function of X is given by:

$$p(X;\lambda) = P_r(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

With $e \sim 2.71828...$) and $k!$ is the factorial of k. The positive real number λ is equal to the expected value of X, which also equals the variance. Assume there is no traffic waiting when the traffic light turns red at the beginning of the cycle.

Answer: Poisson distribution

Q2/ Express the probability p that vehicles are remaining in the queue for direction 2 when the traffic light turns red at the end of the cycle in an equation (p = ...). Write your answer as a mathematical expression in which you specify the variables. Avoid infinite series. There is no need to calculate the final answer as a number

Answer:
The cycle time is 120 seconds. The expected number of vehicles in a cycle is
$120/60 * 800/60 = 160/6 = 26.7$
So $\lambda = 26.7$ (0.5) 120 seconds cycle time,
So 60 seconds per direction. 5 seconds are lost (2 s clearance time and 3 s startup loss),
So 55 seconds leading to the floor (55/2) =27 vehicles at maximum through a green phase.
The probability of an overflow queue is the probability that the number of cars arriving is 28 or larger,

$P(X >= 28)$; this can be calculated by:
$1 - P(X <= 27)$.

This is calculated as:

$$P = \sum_{k=0}^{27} \frac{26.7^k e^{26.7}}{k!}$$

Q3/ Argue whether this probability is higher or lower than if a uniform arrival process is assumed.

Answer:
The spread of a Poisson arrival process is larger, so the probability of having overflow queues is larger.