Deflections

Deflection of Beams

Type of Deflections:

1. Unit Load Method

2. Moment Area Method

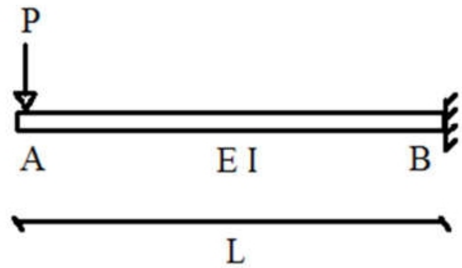
3. Conjugate Beam Method

Unit Load Method

Unit Load Method

Δ = Deflection = Displacement

θ = Rotation (Slope)



$$\Delta = \int_0^L \frac{M \,.\, m\Delta}{E \,.\, I} \,.\, dx$$

$$\theta = \int_0^L \frac{M \,.\, m\theta}{E \,.\, I} \,.\, dx$$

M = moment for applied load

m = moment for unit load

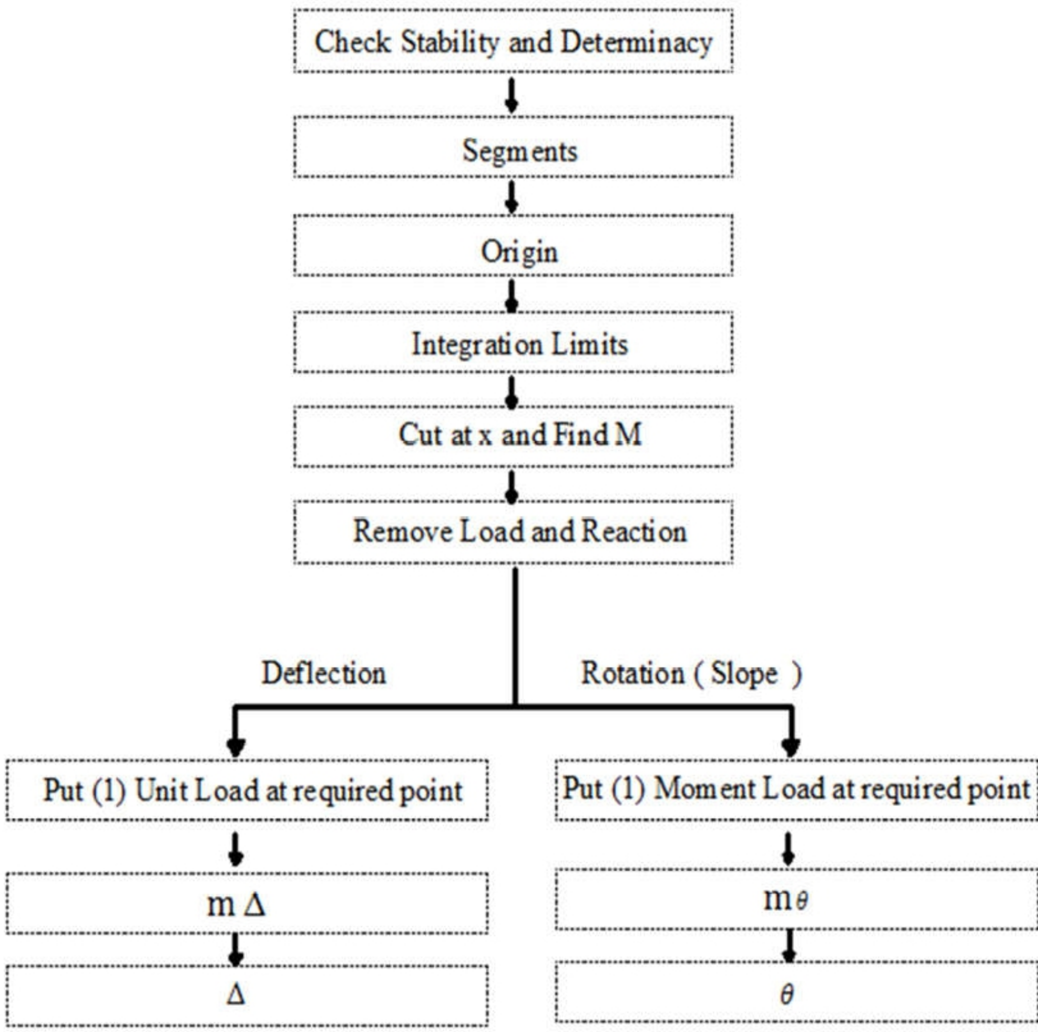
E = modulus of elasticity

I = moment of inertia

<u>Solution procedure</u>

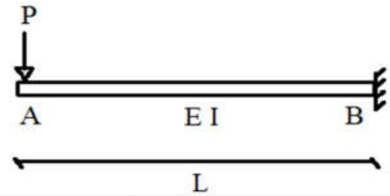
1. Check stability and determinacy of Beams.
2. Divided the beams in into segments.
3. Locate Origin for each point.
4. Specify The Integration Limits.
5. Cut at x distance and Determine M.
6. Remove all external Load and Reaction.
7. For Deflection Put (1) unit load at required point about x distance.
8. For Rotation (Slope) put (1) Moment unit at required point.

Deflections - Unit Load Method



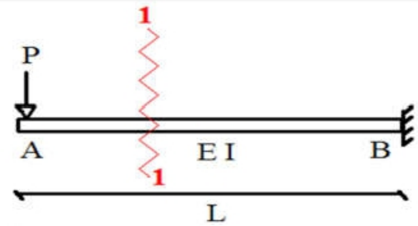
Example: Determine the Deflection at Point a.

Solution:



Segments	Origin	Integration Limits	M	m_Δ
$a \rightarrow b$	a	$0 \rightarrow L$	-p (X)	(- X)

Section 1-1

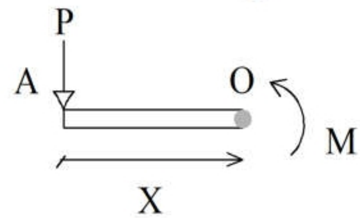


1.Determine M ,

- $\sum M_O = 0$ ⟲

  $M + P (x) = 0$

  $M = - P (x)$



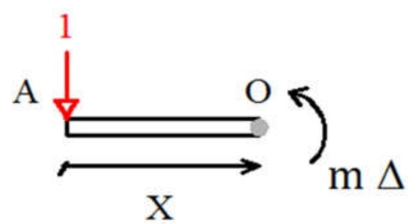
Section 2-2

2.Determine m_Δ

- $\sum M_O = 0$ ⟲

  $m_\Delta + 1 * x = 0$

  $m_\Delta = - x$



$$\Delta_a = \int_0^L \frac{M \,.\, m_\Delta}{E \,.\, I} \,.\, dx$$

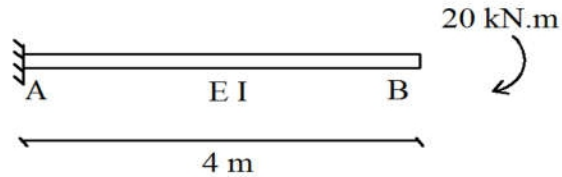
$$\Delta_a = \int_0^L \frac{(-P.\, x) \,.\, (-x)}{E \,.\, I} \,.\, dx$$

$$\Delta_a = \int_0^L \frac{P.\, x^2}{E \,.\, I} \,.\, dx$$

$$\Delta_a = \frac{P.\, x^3}{3\, E \,.\, I} \Big]_0^L$$

$$\boxed{\Delta_a = \frac{P.\, L^3}{3\, E \,.\, I}}$$

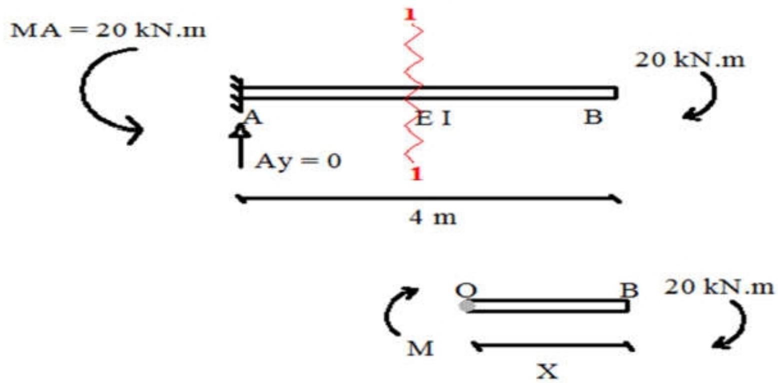
Example: Determine the Deflection and Rotation at Point B.



Solution:

Section 1-1

1. Determine M



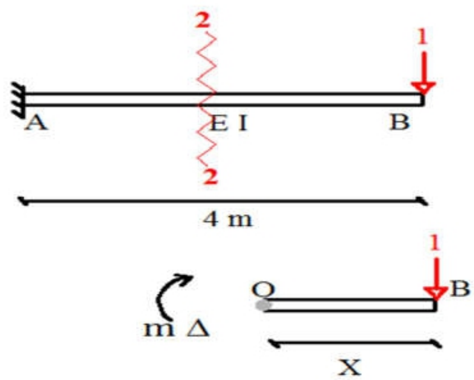
- $\sum M_O = 0$ ⟳

  $M + 20 = 0$

  $M = -20$

Section 2-2

2. Determine m_Δ



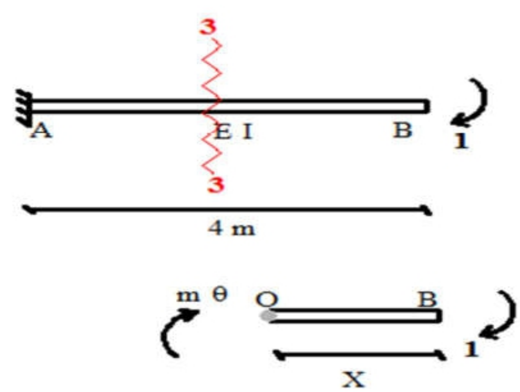
- $\sum M_O = 0$ ⟳

  $m_\Delta + 1 * x = 0$

  $m_\Delta = -x$

Section 3-3

3. Determine m_θ



- $\sum M_O = 0$ ⟳

  $m_\theta + 1 = 0$

  $m_\theta = -1$

Segments	Origin	Integration Limits	**M**	**m** $_\Delta$	**m** θ
AB	B	$0 \rightarrow 4$	-20	(- X)	-1

$$\Delta_B = \int_0^L \frac{M \, . \, m_\Delta}{E \, . \, I} \, . \, dx$$

$$\Delta_B = \int_0^4 \frac{(-20) . (-X)}{E \, . \, I} \, . \, dx$$

$$\Delta_B = \int_0^4 \frac{(-20) . (-X)}{E \, . \, I} \, . \, dx$$

$$\Delta_B = \frac{20 \, . \, x^2}{2 \, . \, E \, . \, I} \Big]_0^4$$

$$\Delta_B = \frac{160}{E \, . \, I}$$

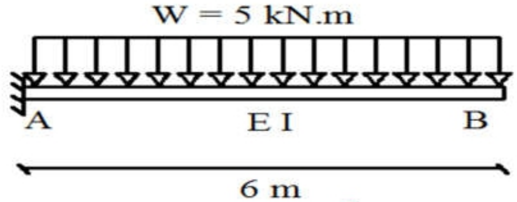
$$\theta_B = \int_0^L \frac{M \, . \, m\,\theta}{E \, . \, I} \, . \, dx$$

$$\theta_B = \int_0^4 \frac{(-20) \, . \, (-1)}{E \, . \, I} \, . \, dx$$

$$\theta_B = \frac{20 \, . \, x}{E \, . \, I} \Big]_0^4$$

$$\theta_B = \frac{80}{E \, . I}$$

Example: Determine the Deflection and Rotation at Point B.



Solution:

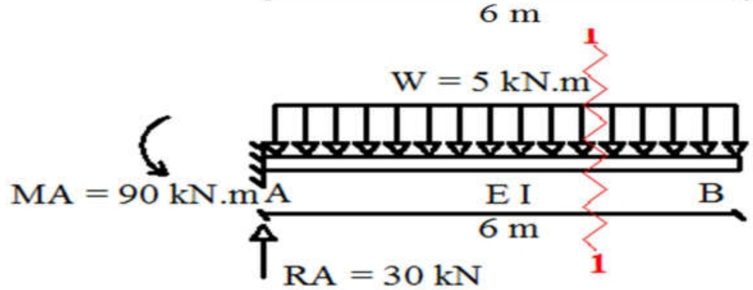
Section 1-1

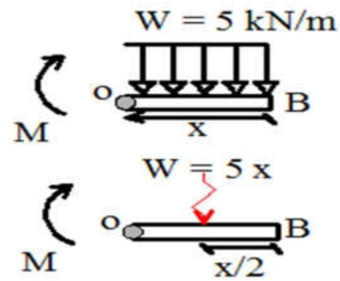
1. Determine M

○ $\sum M_O = 0$ ⟳

$M + 5x * \frac{x}{2} = 0$

$M = \frac{-5x^2}{2}$





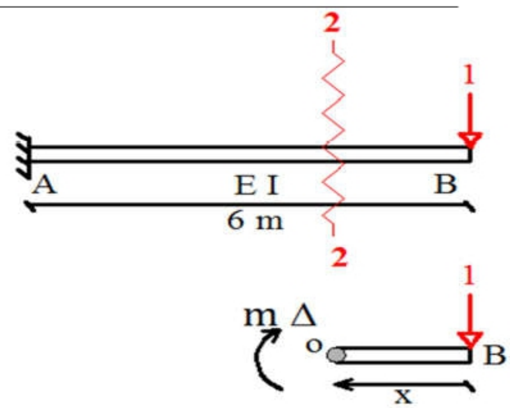
Section 2-2

2. Determine m_Δ

○ $\sum M_O = 0$ ⟳

$m_\Delta + 1 * x = 0$

$m_\Delta = -x$



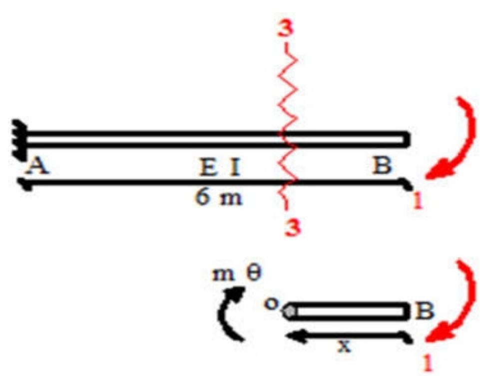
Section 3-3

3. Determine m_θ

○ $\sum M_O = 0$ ⟳

$m_\theta + 1 = 0$

$m_\theta = -1$



Segments	Origin	Integration Limits	M	m $_\Delta$	m $_\theta$
AB	B	0→ 6	$\frac{-5\,x^2}{2}$	(- X)	-1

$$\Delta_B = \int_0^L \frac{M\,.\,m_\Delta}{E\,.\,I}\,.\ dx$$

$$\Delta_B = \int_0^6 \frac{(\frac{-5\,x^2}{2}).(-X)}{E\,.\,I}\,.\ dx$$

$$\Delta_B = \frac{5\ X^4}{8\,E.I}\ \Big]\begin{matrix}6\\0\end{matrix}$$

$$\Delta_B = \frac{810}{E.I}$$

$$\theta_B = \int_0^L \frac{M\,.\,\mathrm{m}\,\theta}{E\,.\,I}\,.\ dx$$

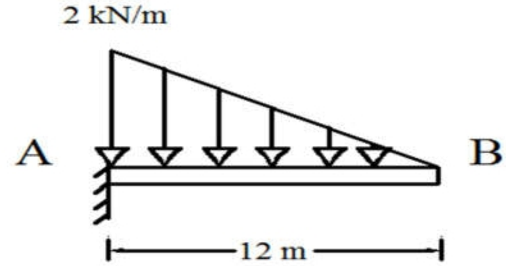
$$\theta_B = \int_0^6 \frac{\frac{-5\,x^2}{2}\,.\,(-1)}{E\,.\,I}\,.\ dx$$

$$\theta_B = \frac{5\ X^3}{6\,E.I}\ \Big]\begin{matrix}6\\0\end{matrix}$$

$$\theta_B = \frac{180}{E.I}$$

Example: Determine the Deflection and Rotation at Point B.

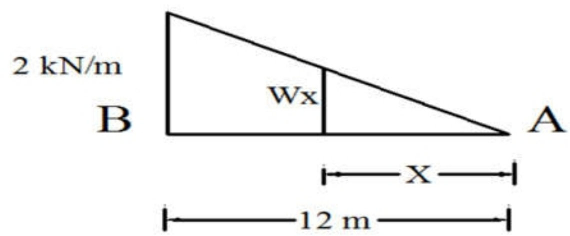
<u>Solution:</u>



$$\frac{wx}{x} = \frac{2}{12}$$

$$wx = \frac{2 * x}{12}$$

$$wx = \frac{x}{6}$$



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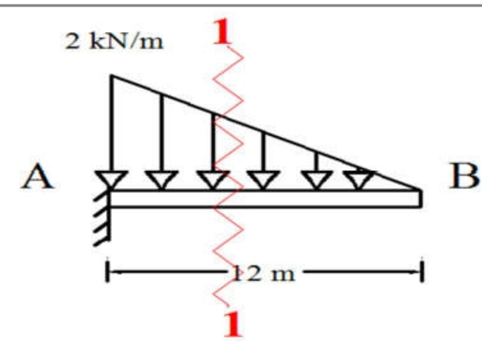
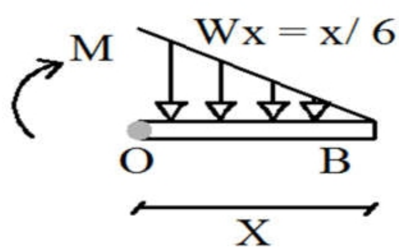
1. Determine M



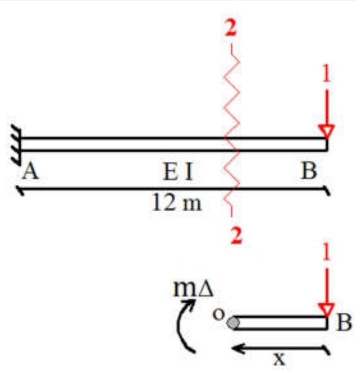
- $\sum M_O = 0$ ⟳

$$M + \left(\frac{1}{2} * \frac{x}{6} * x\right) * \frac{1}{3} * x = 0$$

$$M = -\frac{x^3}{36}$$



Determine m_Δ



- $\sum M_O = 0$ ⟳

$$m_\Delta + 1 * x = 0$$

$$m_\Delta = -x$$

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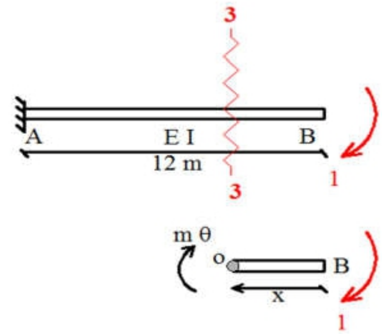
3. Determine m_{θ}



○ $\sum M_O = 0$ ⟳

$m_{\theta} + 1 = 0$

$m_{\theta} = -1$

Segments	Origin	Integration Limits	M	m_{Δ}	m_{θ}
AB	B	$0 \rightarrow 12$	$\frac{-x^3}{36}$	(- X)	-1

$$\Delta_B = \int_0^L \frac{M \cdot m_{\Delta}}{E \cdot I} \cdot dx$$

$$\Delta_B = \int_0^{12} \frac{(\frac{-x^3}{36}) \cdot (-X)}{E \cdot I} \cdot dx$$

$$\Delta_B = \frac{x^5}{180 * E \cdot I} \Big]_0^{12}$$

$$\Delta_B = \frac{1382.4}{E \cdot I}$$

$$\theta_B = \int_0^L \frac{M \cdot m\theta}{E \cdot I} \cdot dx$$

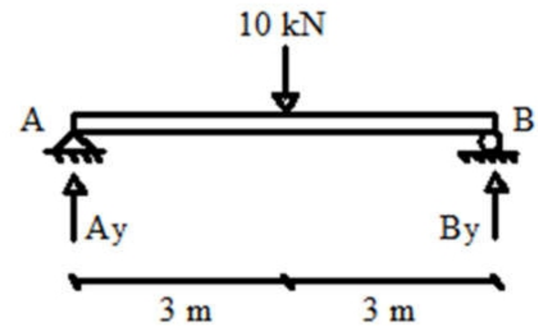
$$\theta_B = \int_0^{12} \frac{\frac{-x^3}{36} \cdot (-1)}{E \cdot I} \cdot dx$$

$$\theta_B = \frac{X^4}{144 * E.I} \Big]_0^{12}$$

$$\theta_B = \frac{144}{E.I}$$

Example: Determine the Maximum Deflection for Simply Support Beam AB by using Unit Load Method.

Solution:

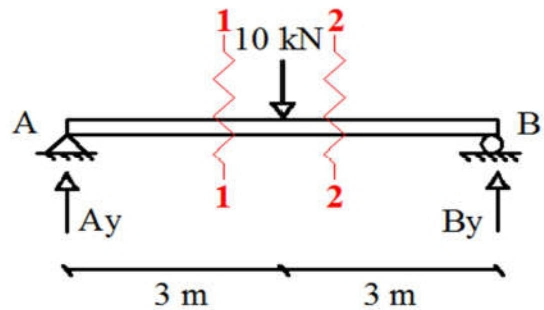


<u>Beam AB</u>

- $\sum M_B = 0$ $\circlearrowright$

  $Ay * 6 - 10 * 3 = 0$

  $Ay = 5\ kN \uparrow$

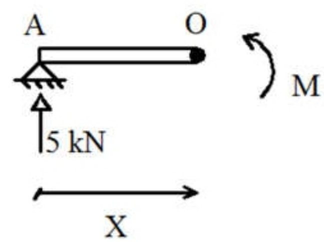


- $\sum Fy = 0 \uparrow$

  $Ay + By - 10 = 0$

  $By = 5\ kN \uparrow$

<u>Section 1-1</u>

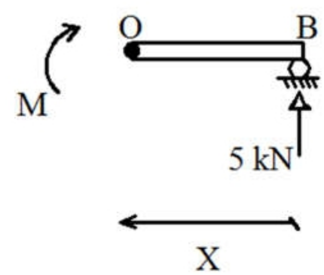


- $\sum M_O = 0$ $\circlearrowright$

  $5\ x - M = 0$

  $M = 5\ x$

<u>Section 2-2</u>

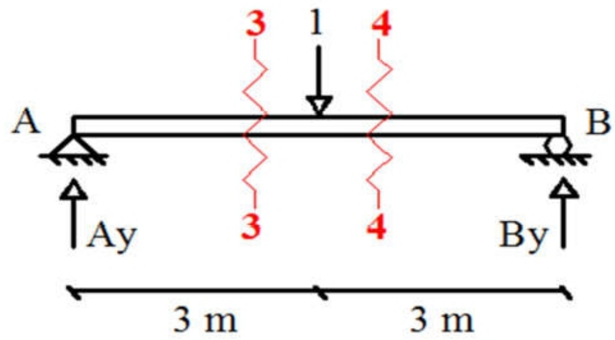


- $\sum M_O = 0$ $\circlearrowright$

  $M - 5\ x = 0$

  $M = 5\ x$

Beam AB



- $\sum M_B = 0$ ⟳

  Ay * 6 – 1 * 3 = 0

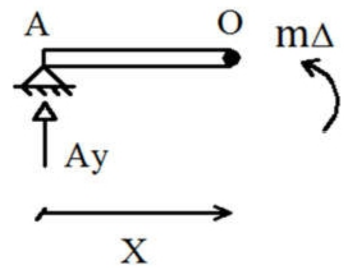
  Ay = 0.5 ↑

- $\sum Fy = 0$ ↑

  Ay + By – 1 = 0

  By = 0.5 ↑

Section 3-3

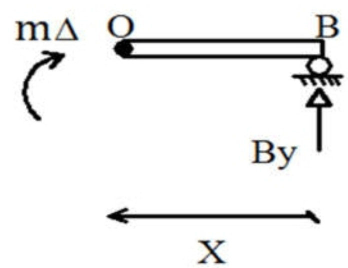


- $\sum M_O = 0$ ⟳

  0.5 x – m = 0

  $m_\Delta = 0.5\,x$

Section 4-4



- $\sum M_O = 0$ ⟳

  m – 0.5 x = 0

  $m_\Delta = 0.5\,x$

$$\text{Max } \Delta = \sum \int_0^L \frac{M\,.\,m\,\Delta}{E\,.\,I}\,.\,dx$$

$$\text{Max } \Delta = \int_0^3 \frac{5x\,.\,0.5x}{E\,.\,I}\,.\,dx + \int_0^3 \frac{5x\,.\,0.5x}{E\,.\,I}\,.\,dx$$

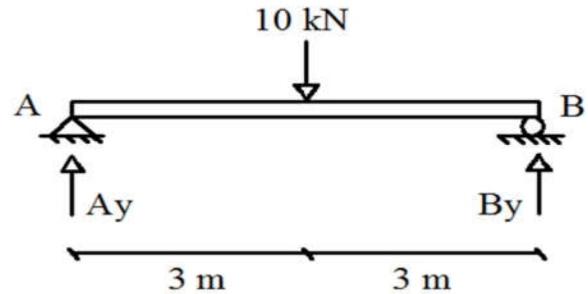
$$\text{Max } \Delta = 2 * \int_0^3 \frac{2.5x^2}{E\,.\,I}\,.\,dx$$

$$\text{Max } \Delta = 2 * \frac{2.5\ x^3}{3\,E\,.\,I}\Big]_0^3$$

$$\text{Max } \Delta = \frac{45}{E.I}$$

Example: Determine the Slope (Rotation) at Ends of Simply Support Beam AB by using Unit Load Method.

Solution:

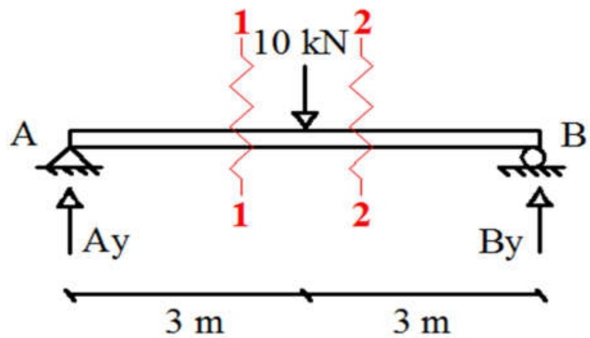


Beam AB

- $\sum M_B = 0$ ↻

  $Ay * 6 - 10 * 3 = 0$

  $Ay = 5$ kN ↑

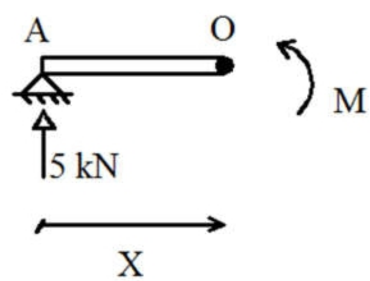


- $\sum Fy = 0$ ↑

  $Ay + By - 10 = 0$

  $By = 5$ kN ↑

Section 1-1

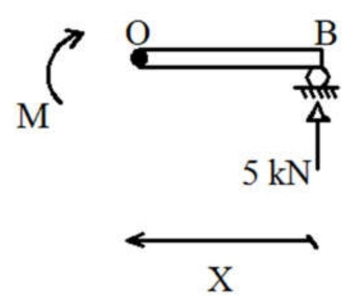


- $\sum M_O = 0$ ↻

  $5\,x - M = 0$

  $M = 5\,x$

Section 2-2



- $\sum M_O = 0$ ↻

  $M - 5\,x = 0$

  $M = 5\,x$

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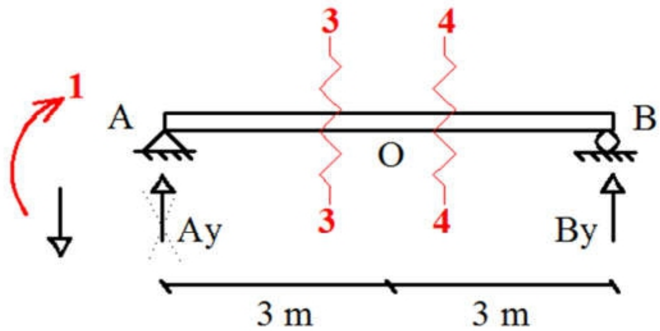
2. Determine m_θ



- $\sum M_B = 0$ ⟳

  $1 + Ay * 6 = 0$

  $Ay = -\ \frac{1}{6}$

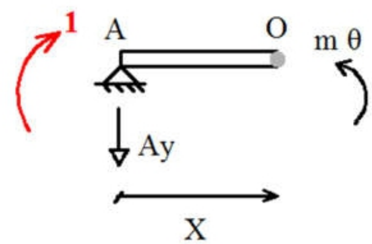
  $Ay = \frac{1}{6} \downarrow$

- $\sum Fy = 0 \uparrow$

  $By - \frac{1}{6} = 0$

  $By = \frac{1}{6} \uparrow$

<u>Section 3-3</u>



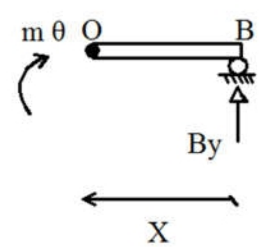
- $\sum M_O = 0$ ⟳

  $1 - \frac{1}{6} * x - m_\theta = 0$

  $m_\theta = 1 - \frac{1}{6} * x$

  $m_\theta = 1 - 0.1667\ x$

<u>Section 4-4</u>



- $\sum M_O = 0$ ⟳

  $m_\theta - \frac{1}{6} * x = 0$

  $m_\theta = 0.1667\ x$

$$\theta_A = \sum \int_0^L \frac{M \,.\, m\,\theta}{E \,.\, I} \,.\, dx$$

$$\theta_A = \int_0^3 \frac{5X * (1 - 0.1667\ x)}{E \,.\, I} \,.\, dx + \int_0^3 \frac{5X * 0.1667\ x}{E \,.\, I} \,.\, dx$$

$$\theta_A = \int_0^3 \frac{(5\ x - 0.8335\ x^2\ )}{E \,.\, I} \,.\, dx + \int_0^3 \frac{0.8335\ x^2}{E \,.\, I} \,.\, dx$$

$$\theta_A = \int_0^3 \frac{(5\ x\ )}{E \,.\, I} \,.\, dx - \int_0^3 \frac{(0.8335\ x^2\ )}{E \,.\, I} \,.\, dx + \int_0^3 \frac{0.8335\ x^2}{E \,.\, I} \,.\, dx$$

$$\theta_A = \frac{(5\ x^2)}{2\,E \,.\, I} \Big]_0^3 - \frac{(0.8335\ x^3\ )}{3\,E \,.\, I} \Big]_0^3 + \frac{(0.8335\ x^3\ )}{3\,E \,.\, I} \Big]_0^3$$

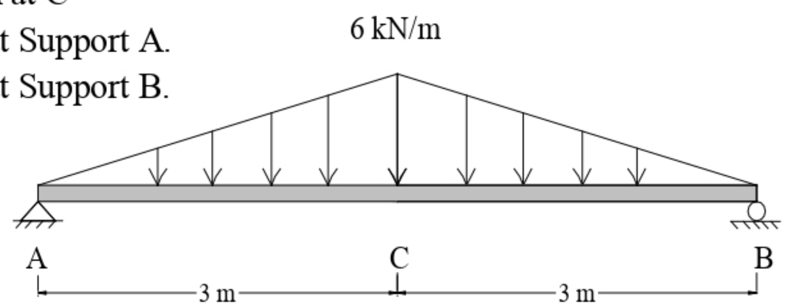
$$\theta_A = \frac{45}{2\,E \,.\, I} - \frac{22.5}{3\,E \,.\, I} + \frac{22.5}{3\,E \,.\, I}$$

$$\theta_A = \frac{22.5}{E \,.\, I} - \frac{7.5}{E \,.\, I} + \frac{7.5}{E \,.\, I}$$

$$\theta_A = \frac{22.5}{E \,.\, I}$$

Example: For Beam AB (EI Constant), use the Unit Load Method to Determine:

1. The Deflection at C
2. The Rotation at Support A.
3. The Rotation at Support B.



Solution

1. The Deflection at C

- $\sum M_A = 0$ ⟳

$$\left(\frac{1}{2} * 3 * 6\right) * \frac{2}{3} * 3 + \left(\frac{1}{2} * 3 * 6\right) * \left(3 + \frac{1}{3} * 3\right) - B_y * 6 = 0$$

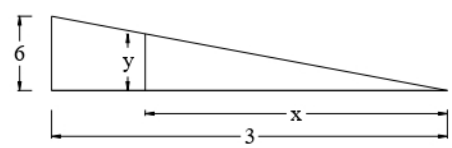
$B_y = 9$ kN ↑

- $\sum Fy = 0$ ↑

$$A_y + 9 - \left(\frac{1}{2} * 3 * 6\right) - \left(\frac{1}{2} * 3 * 6\right) = 0$$

$A_y = 9$ kN ↑

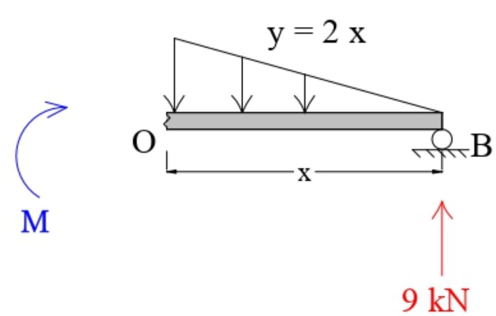
- <u>Form B to C</u> , $0 \leq x \leq 3$



$$\frac{6}{3} = \frac{y}{x}$$

y = 2 x

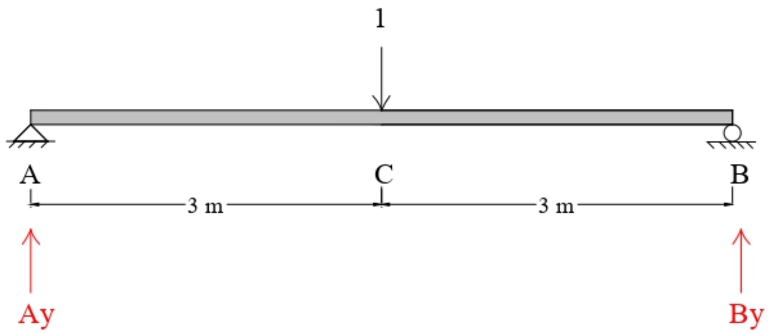
- $\sum M_O = 0$ ⟳



$$M + \left(\frac{1}{2} * x * 2x\right) * \frac{x}{3} - 9 * x = 0$$

$$M = -\frac{x^3}{3} + 9x$$

✓ m_Δ



- $\sum M_A = 0$ ⟳

  $1*3 - B_y * 6 = 0$

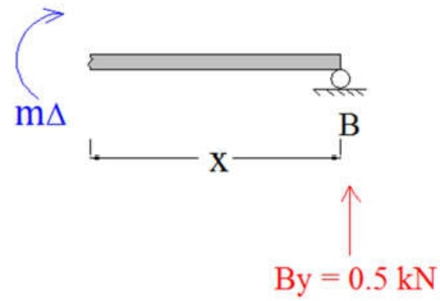
  $B_y = 0.5$ ↑

- $\sum Fy = 0$ ↑

  $A_y + 0.5 - 1 = 0$

  $A_y = 0.5$ ↑

<u>Sec 1-1</u>



- $\sum M_O = 0$ ⟳

  $m - 0.5\,x = 0$

  $m_\Delta = 0.5\,x$

$$\Delta_C = \sum \int_0^L \frac{M \,.\, m\Delta}{E \,.\, I} \,.\, dx$$

$$\Delta_C = 2 * \frac{1}{E \,.\, I} * \{ \int_0^3 \left(-\frac{x^3}{3} + 9\,x\right) * 0.5x \,.\, dx \}$$

$$\Delta_C = 2 * \frac{1}{E \,.\, I} * \{ \int_0^3 \left(-\frac{x^4}{6} + 4.5\,x^2\right) . \, dx\}$$

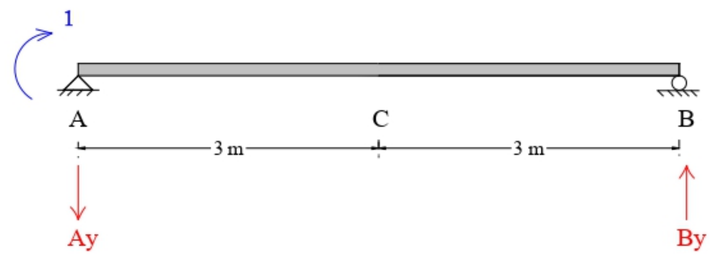
$$\Delta_C = 2 * \frac{1}{E \,.\, I} * \{ \int_0^3 \left(-\frac{x^4}{6}\right) . \, dx + \int_0^3 (4.5\,x^2) \,.\, dx\}$$

$$\Delta_C = 2 * \frac{1}{E \,.\, I} * \{ \left(-\frac{x^5}{30}\right)]_0^3 + (1.5\,x^3)]_0^3 \}$$

$$\Delta_C = 2 * \frac{1}{E \,.\, I} \{ -8.1 + 40.5\}$$

$$\Delta C = \frac{64.8}{E \,.\, I}$$

2. The Rotation at Support A.



- $\sum M_A = 0$ ↻

  $1 - B_y * 6 = 0$

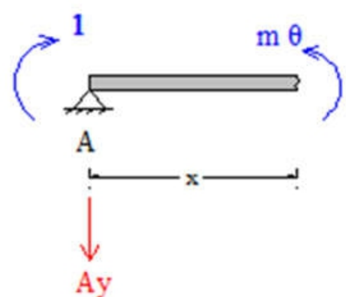
  $B_y = \frac{1}{6}$ ↑

- $\sum F_y = 0$ ↑

  $\frac{1}{6} - A_y = 0$

  $A_y = \frac{1}{6}$ ↓

<u>Sec 3-3</u>

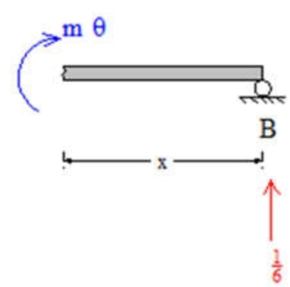


- $\sum M_O = 0$ ↻

  $1 - m_\theta - \frac{1}{6} * x = 0$

  $m_\theta = 1 - \frac{x}{6}$

<u>Sec 4-4</u>



- $\sum M_O = 0$ ↻

  $m_\theta - \frac{1}{6} * x = 0$

  $m_\theta = \frac{x}{6}$

$$\theta_{\mathrm{A}} = \sum \int_0^L \frac{M \,.\, m\,\theta}{E \,.\, I} \,.\; dx$$

$$\theta_{\mathrm{A}} = \int_0^3 \frac{\left(-\frac{x^3}{3} + 9\,\mathrm{x}\right).\,(1 - \frac{x}{6})}{E \,.\, I} \,.\; dx + \int_0^3 \frac{\left(-\frac{x^3}{3} + 9\,\mathrm{x}\right).\,(\frac{x}{6})}{E \,.\, I} \,.\; dx$$

$$\theta_{\mathrm{A}} = \int_0^3 \frac{-\frac{x^3}{3} + \frac{x^4}{18} + 9x - 1.5\,\mathrm{x}^2}{E \,.\, I} \,.\; dx + \int_0^3 \frac{\left(-\frac{x^4}{18} + 1.5\,\mathrm{x}^2\right)}{E \,.\, I} \,.\; dx$$

$$\theta_{\mathrm{A}} = \frac{-\frac{x^4}{12} + \frac{x^5}{90} + 4.5\,x^2 - 0.5\,\mathrm{x}^3}{E \,.\, I} \Big]_0^3 + \frac{\left(-\frac{x^5}{90} + 0.5\,\mathrm{x}^3\right)}{E \,.\, I} \Big]_0^3$$

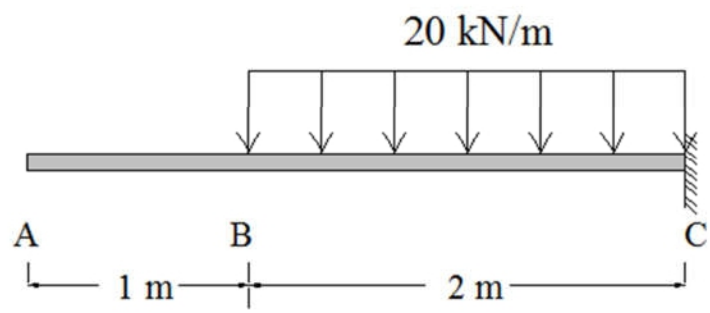
$$\theta_{\mathrm{A}} = \frac{-\frac{x^4}{12} + \frac{x^5}{90} + 4.5\,x^2 - 0.5\,\mathrm{x}^3}{E \,.\, I} + \frac{\left(-\frac{x^5}{90} + 0.5\,\mathrm{x}^3\right)}{E \,.\, I}$$

$$\theta_{\mathrm{A}} = \frac{22.95}{E \,.\, I} + \frac{(10.8)}{E \,.\, I}$$

$$\theta_{\mathrm{A}} = \frac{33.75}{E \,.\, I}$$

Example: Determine the Deflection and Slope at point B by using Unit Load Method.

Solution

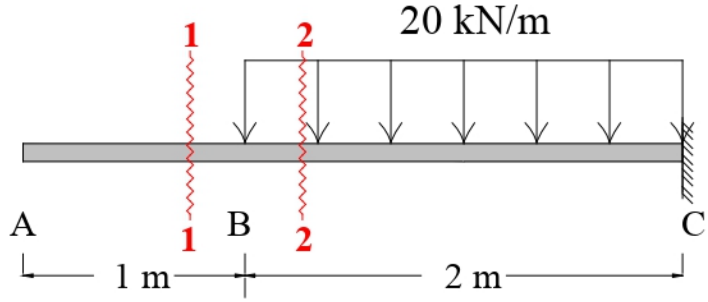


Section 1-1

$M = 0$

$m_\theta = 0$

$m_\Delta = 0$

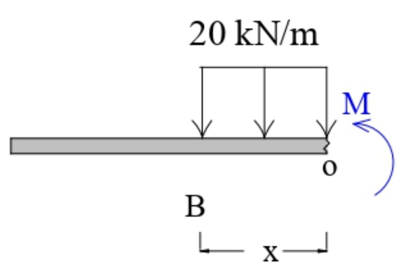


Section 2-2

- $\sum M_O = 0$ ⟳

  $-M - 20 * x * \frac{x}{2} = 0$

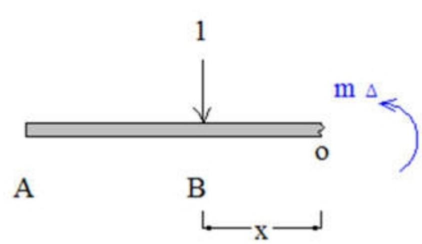
  $M = -10x^2$



- $\sum M_O = 0$ ⟳

  $-m_\Delta - 1 * (x) = 0$

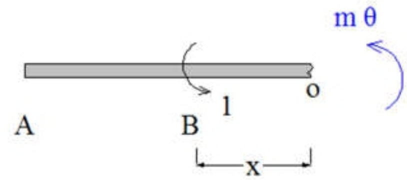
  $m_\Delta = -x$



- $\sum M_O = 0$ ⟳

  -1- $m_\theta = 0$

  $m_\theta = -1$



Segments	Origin	Integration Limits	M	m_Δ	m_θ
AB	A	0→ 1	0	0	0
BC	B	0→ 2	-10 x^2	-x	-1

$$\Delta_B = \int_0^2 \frac{-10\,x^2 * \;-x}{E\,.\,I}\,.\;dx$$

$$\Delta_B = \frac{10\,x^4}{4\,E\,.\,I}\Big]_0^2$$

$$\Delta_B = \frac{40}{E\,.\,I}$$

$$\theta_B = \int_0^2 \frac{-10\,x^2 * -1}{E\,.\,I}\,.\;dx$$

$$\theta_B = \frac{10\,x^3}{3\,E\,.\,I}\Big]_0^2$$

$$\theta_B = \frac{26.667}{E\,.\,I}$$