Mustansiriyah University
College of Engineering
Electrical Engineering Department
4th Class

Digital Signal Processing

(EE402**)**

-

2022-2023

Topics Covered
Introduction to Digital Signal Processing
Signal Sampling and Reconstructions: *Sampling of Continuous Signal, Signal Reconstruction, Aliasing Noise Level*
Digital Signals and Systems: *Classification of Systems, Linear System, Time-Invariant System, Causal System, Stability*
Digital Convolution: *Graphical Method, Table Lookup Method, Matrix by Vector Method, Linear Convolution and Circular Convolution, Deconvolution*
Frequency Response and Sinusoidal Steady State Response
Z-Transform (Review), Discrete Fourier Transform, Fast Fourier Transform
Fast Fourier Transform (FFT) Algorithms
Analog Filter Design: *Butterworth Filters , Chebyshev Filters.*
Digital Filter Design: *Infinite Impulse Response (IIR) filter , Finite Impulse Response (FIR) filter*
Realization of Digital Filters :*Realization of IIR Filters , Realization of FIR Filters*
Theoretical: 2 Hrs/Wk **Total hours (60 Theoretical)**

Suggested References:

1) "*Digital Signal Processing Principles, Algorithms, and Applications*", John G. Proakis, Dimitris G. Manolakis, Third Edition (1996).

2) "*Applied Digital Signal Processing Theory and Practice*", Dimitris G. Manolakis, Vinay K. Ingle, First Edition (2011).

Introduction to Digital Signal Processing

Signal (flow of information):

Signal is defined as any physical quantity that varies with Time, Space, or any other independent variables. For Example:

- Measured quantity that varies with time (or position).
- Electrical signal received from a transducer (Microphone, Thermometer, Accelerometer, Antenna, etc.)
- Electrical signal that controls a process.

Example:

- $s_1(t) = 5t$ or $s_1(t) = 5t^2$ → one variable
- $S(x,y) = 3x + 4xy + 6x^2$ → two variables x and y

A. Continuous-Time Signal or Analog Signal:

The analog signal is defined for every value of time and they take on values in the continuous interval as shown in Fig. 1.

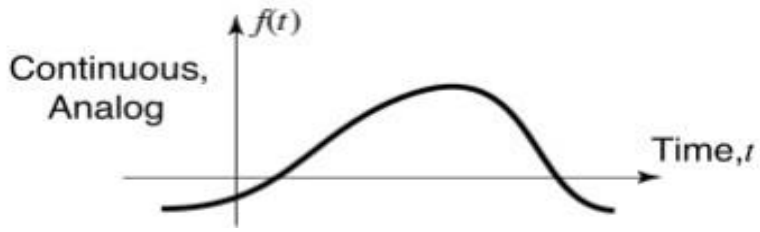


Fig. 1. Continuous or analog signal

- Continuous in time.
- Amplitude may take on any value in the continuous range of (-∞,∞).

❖ **Analog Processing**

- Differentiation, Integration, Filtering, Amplification.
- Differential Equations
- Implemented via passive or active electronic circuitry.

B. Discrete-Time signals:

Discrete signals are defined only at certain specific value of time as shown in Fig. 2.

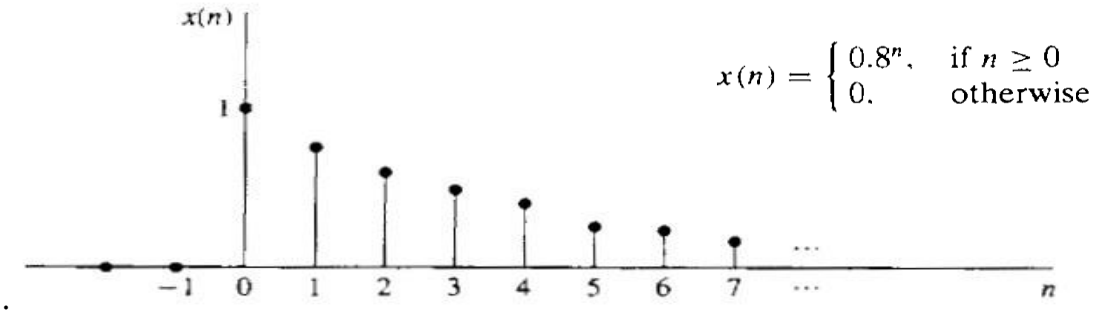


Fig. 2. Discrete signal

- Continuous in amplitude but discrete in time.
- Only defined for certain time instances.
- Can be obtained from analog signals via sampling.

C. Digital Signal:

Digital signal is the signal that takes on values from a finite set of possible values as shown in Fig. 3.

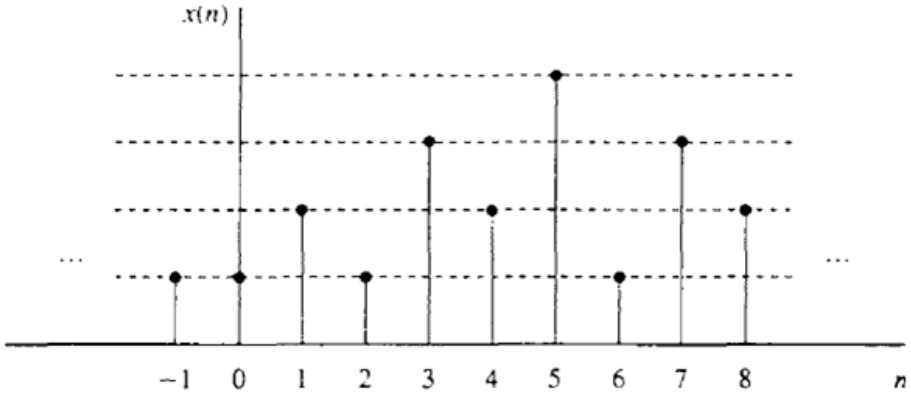


Fig. 3. Digital signal with four different amplitude values

- Discrete in amplitude & discrete in time.
- Can be obtained from discrete signals via quantization.

Finite and infinite length signal:

Finite length signal is nonzero over a finite interval $t_{min} < t < t_{max}$ as shown in Fig. 4.

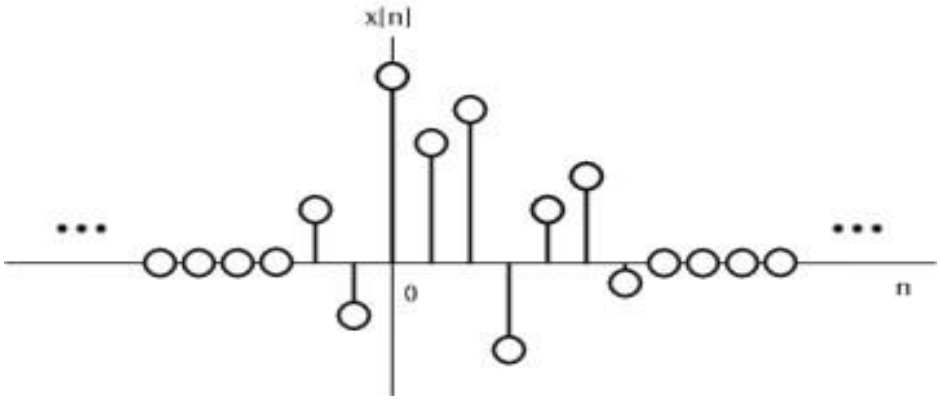


Fig. 4. Finite length signal

In contrast, the infinite length signal is nonzero over all real numbers.

What is signal processing?

Signals may have to be transformed in order to

- Amplify or filter out embedded information.
- Detect patterns.
- Prepare the signal to survive a transmission channel.
- Undo distortions contributed by a transmission channel.
- Compensate for sensor deficiencies.
- Find information encoded in a different domain.

Fig. 5 explains the main idea of the signal processor.

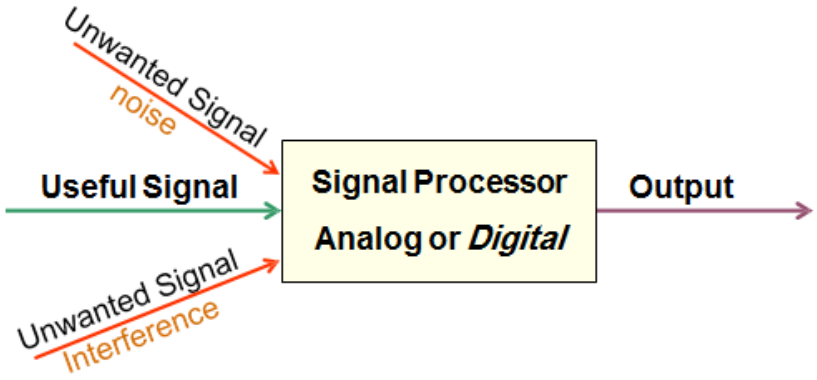


Fig. 5. Signal Processor

Analog signal processing:

Fig. 6 shows a basic block diagram of a typical analog signal processing system.

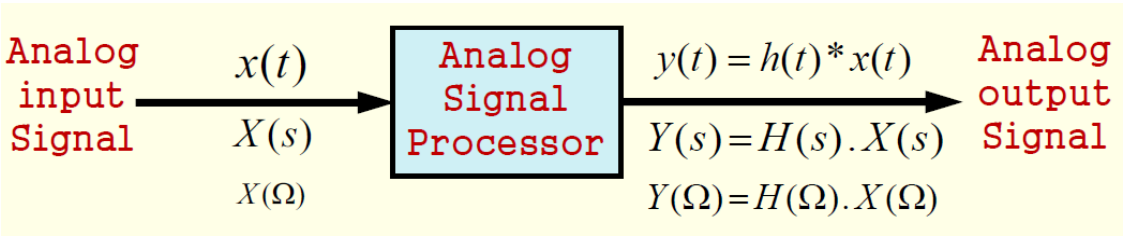


Fig. 6. A typical analog signal processing system

Where,

$h(t)$: The System Impulse Response
H(s): The System Transfer Function
H(Ω): The System Frequency Response

Analogue signal processing is achieved by using analogue components such as:

- Resistors.
- Capacitors.
- Inductors.

Limitations of analog signal processing:

- Accuracy limitations due to
  - Component tolerances
  - Undesired nonlinearities
- Limited repeatability due to
  - Tolerances
  - Changes in environmental conditions
    - Temperature
    - Vibration
- Sensitivity to electrical noise
- Limited dynamic range for voltage and currents
- Inflexibility to changes

- Difficulty of implementing certain operations
  - Nonlinear operations
  - Time-varying operations
- Difficulty of storing information

Digital signal processing (DSP) system:

Digital signal processing (DSP) is one of the most powerful technologies that will shape science and engineering in the twenty-first century. Revolutionary changes have already been made in aboard range of fields: communications, radar and sensor. DSP converts signals that naturally accrue in analog form (such as sound, video and information from sensors) to digital form and uses digital techniques to enhance and modify analog signal data for various applications. Fig. 7 shows a basic block diagram of a typical digital signal processing system.

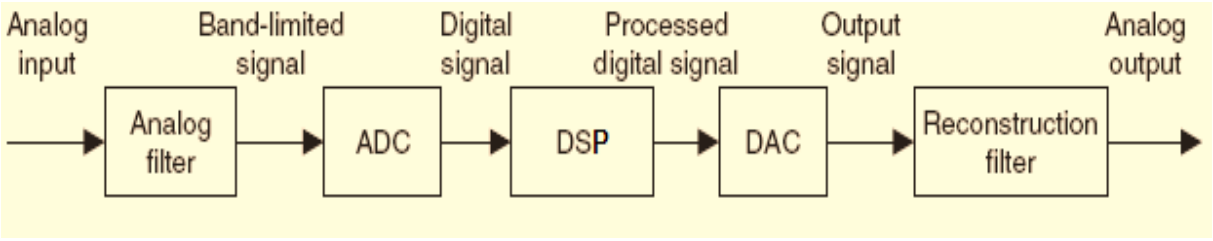


Fig. 7. A typical digital signal processing (DSP) system

The system consists of an analog filter, an analog-to-digital conversion (ADC) unit, a digital signal processor (DSP), a digital-to-analog conversion (DAC) unit, and a reconstruction (anti-image) filter.

As shown in the diagram, the **analog input signal**, which is continuous in time and amplitude, is generally encountered in our real life. Examples of such analog signals include current, voltage, temperature, pressure, and light intensity. Usually a transducer (sensor) is used to convert the non-electrical signal to the analog electrical signal (voltage). This analog signal is fed to an **analog filter**, which is applied to limit the frequency range of analog signals prior to the sampling process. The purpose of filtering is to significantly attenuate aliasing distortion.

The band-limited signal at the output of the analog filter is then sampled and converted via the **ADC** unit into the digital signal, which is discrete both in time and in amplitude.

The **DSP** then accepts the digital signal and processes the digital data according to DSP rules such as lowpass, highpass, and bandpass digital filtering, or other algorithms for different applications. Notice that the DSP unit is a special type of digital computer and can be a general-purpose digital computer, a microprocessor, or an advanced microcontroller; furthermore, DSP rules can be implemented using software in general. With the DSP and

corresponding software, a processed digital output signal is generated. This signal behaves in a manner according to the specific algorithm used.

The **DAC** unit converts the processed digital signal to an analog output signal. The signal is continuous in time and discrete in amplitude (usually a sample-and-hold signal).

The final stage in Fig. 7 is often another **analog filter** designated as a function to smooth the DAC output voltage levels back to the analog signal (i.e. to reconstruct the analog signal from the DAC output).

In contrast to the above, a direct analog processing of analog signals is much simpler since it involves only a signal processor. It is therefore natural to ask why we go to use the DSP systems. There are several good reasons:

1- Rapid advances in integrated circuit design and manufacture are producing more powerful DSP systems on a single chip at decreasing size and cost.

2- Digital processing is inherently stable and reliable.

3- Good processing techniques are available for digital signals, such as Data compression (or source coding), Error Correction (or channel coding), Equalization and Security.

4- Easy to mix signals and data using digital techniques known as Time Division Multiplexing (TDM).

5- It is easy to Change, Correct, or Update applications (software changes), such as-that needed in implementing adaptive circuits.

6- Sensitivity to electrical noise is minimal.

7- Digital information can be encrypted for security.

The list below by no means covers all DSP applications. Many more areas are increasingly being explored by engineers and scientists. Applications of DSP techniques will continue to have profound impacts and improve our lives.

1- *Digital audio and speech*: Digital audio coding such as CD players, digital crossover, digital audio equalizers, digital stereo and surround sound, noise reduction systems, speech coding, data compression and encryption, speech synthesis and speech recognition.

2- *Digital telephone*: Speech recognition, high-speed modems, echo cancellation, speech synthesizers, DTMF (dual-tone multi frequency) generation and detection, answering machines.

3- *Automobile industry*: GPS, Active Noise Cancellation, Cruise Control, Parking.

4- *Electronic communications*: Cellular phones, digital telecommunications, wireless LAN (local area networking), satellite communications.

5- *Medical imaging equipment*: ECG analyzers, cardiac monitoring, medical imaging and image recognition, digital x-rays, image processing, magnetic resonance, tomography and electrocardiogram.

6- *Multimedia*: Internet phones, audio, and video, hard disk drive electronics, digital pictures, digital cameras, DVD, JPEG, Movie special effects, video conferencing, text-to-voice and voice-to-text technologies.

7- *Military*: Radar, sonar, space photographs, remote sensing.

8- *Mechanical*: Motor control, process control, oil and mineral prospecting.

Signal Sampling and Reconstruction

Analog to digital (A/D) conversion:

The analog-to-digital conversion is basically a 2 step process:

- ❑ Sampling
  - • Converts continuous-time analog signal $x_a(t)$ to discrete-time continuous value signal $x(n)$.
  - • It is obtained by taking the "samples" of $x_a(t)$ at discrete-time intervals, T_s
- ❑ Quantization
  - • Converts discrete-time continuous valued signal to discrete time discrete valued signal.

These steps are shown in Fig. 8.

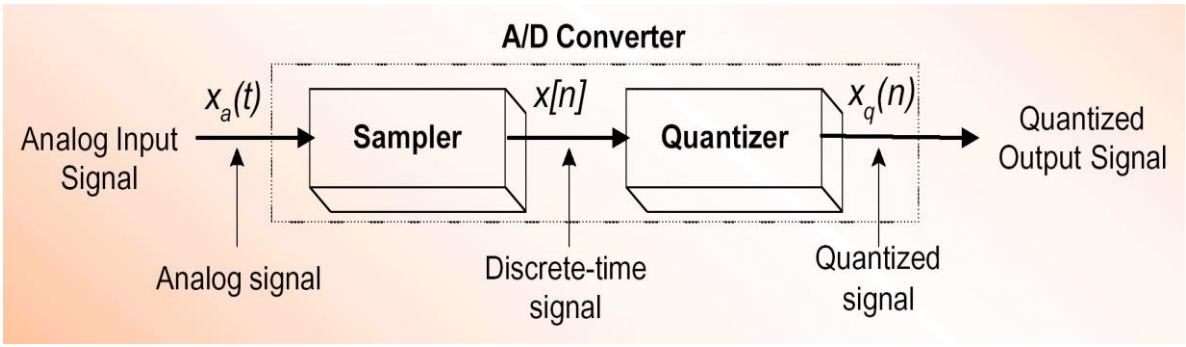


Fig. 8. Basic steps of ADC

Sampling of continuous signal

Sampling is the processes of converting continuous-time analog signal, $x_a(t)$, into a discrete-time signal by taking the "samples" at discrete-time intervals.

- ➢ Sampling analog signals makes them discrete in time but still continuous valued.
- ➢ If done properly (**Nyquist theorem** is satisfied), sampling does not introduce distortion.

Fig. 9 shows an analog (continuous-time) signal (solid line) defined at every point over the time axis and amplitude axis. Hence, the analog signal contains an infinite number of points.

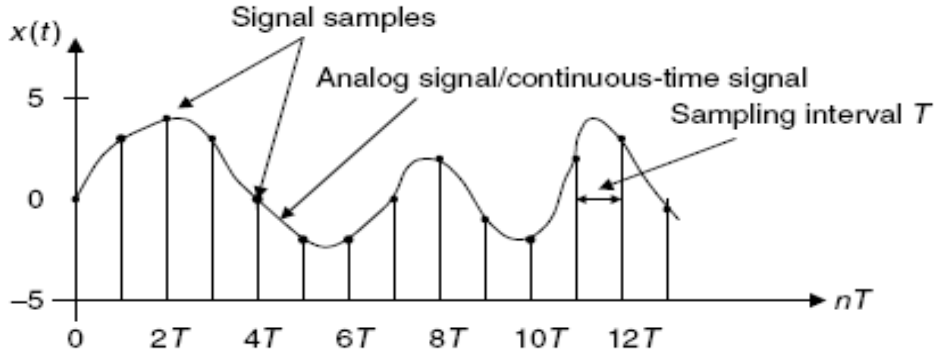


Fig. 9. Display of analog (continuous) signal and digital samples versus the sampling time instants

It is impossible to digitize an infinite number of points. Furthermore, the infinite points are not appropriate to be processed by the digital signal processor or computer, since they require an infinite amount of memory and infinite amount of processing power for computations. Sampling can solve such a problem by taking samples at the fixed time interval, as shown in Fig. 9 and Fig. 10, where the time ***T*** represents the **sampling interval** or **sampling period** in seconds. As shown in Fig. 10, each sample maintains its voltage level during the sampling interval *T* to give the ADC enough time to convert it. This process is called **sample and hold**.

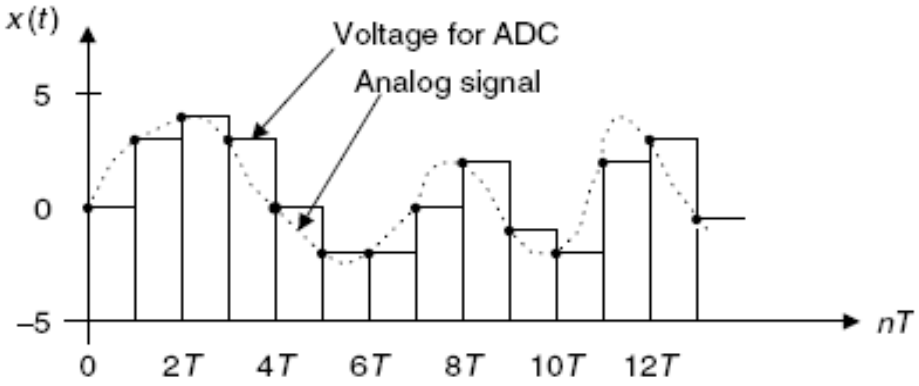


Fig. 10. Sample-and-hold analog voltage for ADC

For a given sampling interval *T*, which is defined as the time span between two sample points, the **sampling rate** or **sampling frequency** is the rate at which the signal is sampled, expressed as the number of samples per second (reciprocal of the sampling interval).

f_s=1/T Samples per second (Hz)

- If the signal is slowly varying, then fewer samples per second will be required than if the waveform is rapidly varying. So, the optimum sampling rate depends on the *maximum frequency* component present in the signal.

Nyquist sampling theorem or Nyquist criterion:

If an analog signal is not appropriately sampled, **aliasing** will occur, which causes unwanted signals in the desired frequency band (i.e. if the sampling is performed at a proper rate, no info is lost about the original signal and it can be properly reconstructed later).

"If a signal is sampled at a rate at least, but not exactly equal to twice the max frequency component of the waveform, then the waveform can be exactly reconstructed from the samples without any distortion". The condition is described as

$$f_s \geq 2 f_{\max}$$

Where, $f_{\max}$ is the maximum-frequency component of the analog signal to be sampled.

T_s is called the **Nyquist interval**: It is the longest time interval that can be used for sampling a band limited signal and still allow reconstruction of the signal at the receiver without distortion.

Example: Find the Nyquist frequency and Nyquist interval of the following signals:

a) speech signal containing frequencies up to 4 kHz
b) audio signal possessing frequencies up to 20 kHz

Solution:

a) to sample a speech signal containing frequencies up to 4 kHz, the Nyquist rate (minimum sampling rate fs) is chosen to be at least 8 kHz, or 8,000 samples per second (fs=2fm) and Nyquist interval (maximum time interval Ts) is 1/fs = 1/8 kHz = 0.125 ms.
b) to sample an audio signal possessing frequencies up to 20 kHz, at least 40,000 samples per second, or 40 kHz, of the audio signal are required and Nyquist interval (maximum time interval Ts) is 1/fs = 1/40 kHz = 25 μs.

Sampled signal spectrum:

Fig. 11 depicts the sampled signal $x_s(t)$ obtained by sampling the continuous signal $x(t)$ at a sampling rate of f_s samples per second. Mathematically, this process can be written as the product of the continuous signal and the sampling pulses (pulse train):

$$x_s(t) = x(t)\, p(t)$$

Where, $p(t)$ is the pulse train with a period $T = 1/f_s$.

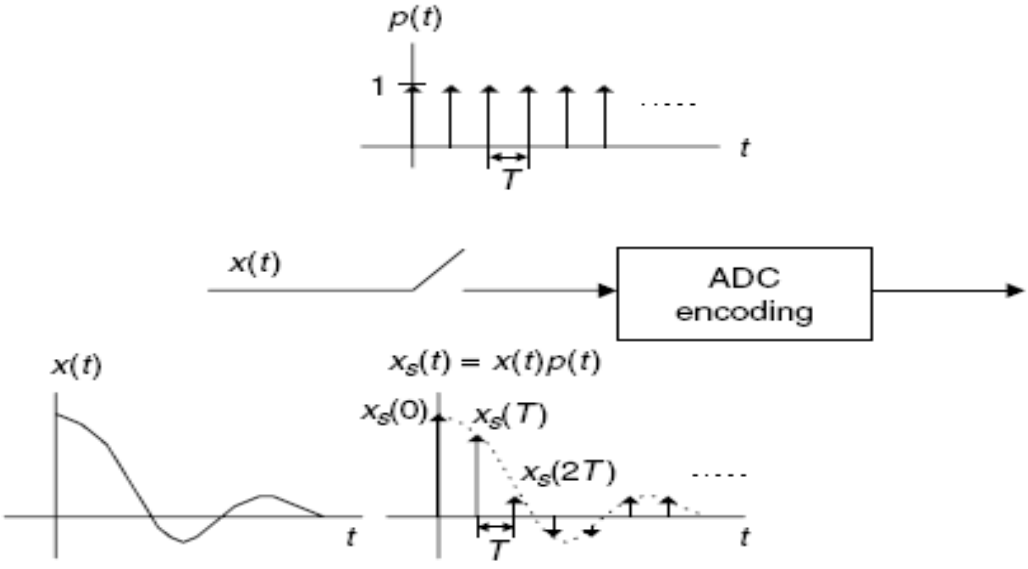


Fig. 11. The simplified sampling process

From the spectral analysis shown in Fig. 12, it is clear that the sampled signal spectrum

consists of the scaled baseband spectrum centered at the origin and its replicas centered at the frequencies of $\pm nf_s$ ($\pm n/T_s$) (multiples of the sampling rate) for each of n = 1,2,3, . . .

In Fig. 12, three possible sketches are classified. Given the original signal spectrum $X(f)$ plotted in Fig. 12(a), the sampled signal spectrum is plotted in Fig. 12(b), where, the replicas have separations between them. In Fig. 12(c), the baseband spectrum and its replicas are just connected. In Fig. 12(d), the original spectrum and its replicas are overlapped; that is, there are many overlapping portions in the sampled signal spectrum.

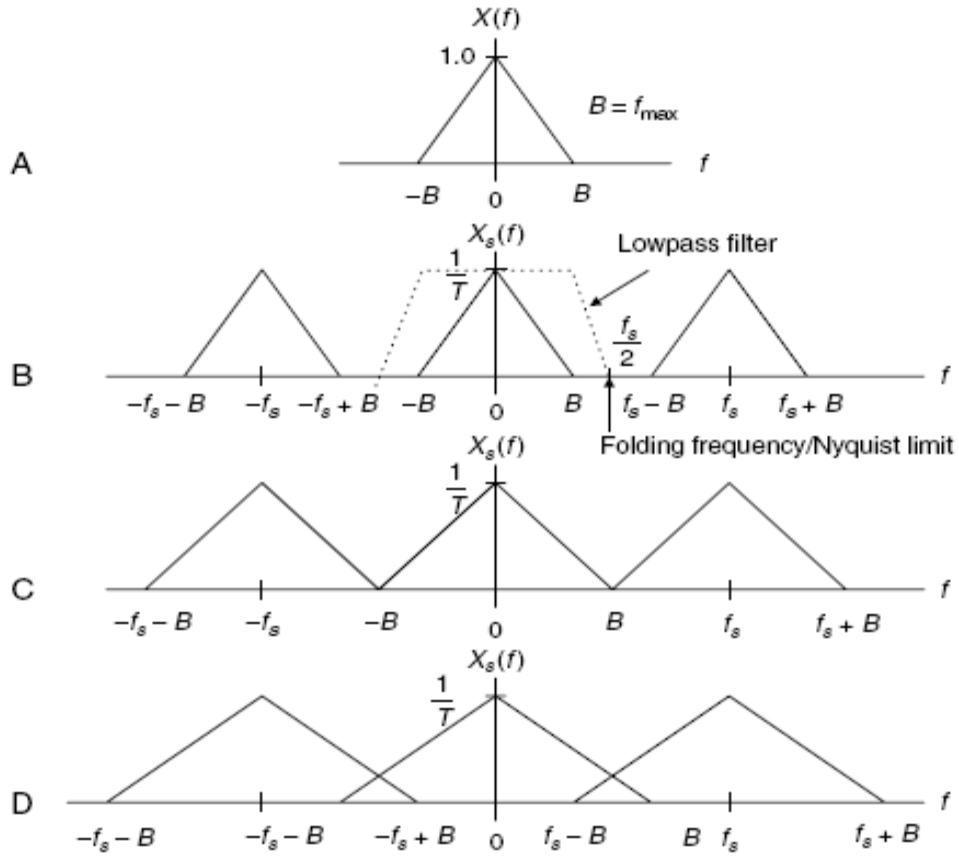


Fig. 12. Plots of the sampled signal spectrum

If applying a lowpass reconstruction filter to obtain exact reconstruction of the original signal spectrum,

- As long as $f_s > 2B$, no overlap of repeated replicas $X(f - n/T_s)$ will occur in $X_s(f)$. Hence, the signal at the output of the filter will be the original signal spectrum without distortion as shown in Fig. 13.
- If the waveform is undersampled (i.e. $f_s < 2B$), then there will be spectral overlap in the sampled signal. Hence, the signal at the output of the filter will be different from the original signal spectrum as shown in Fig. 14. [This is the outcome of aliasing].
- This implies that whenever the sampling condition is not met, an irreversible overlap of

the spectral replicas is produced.

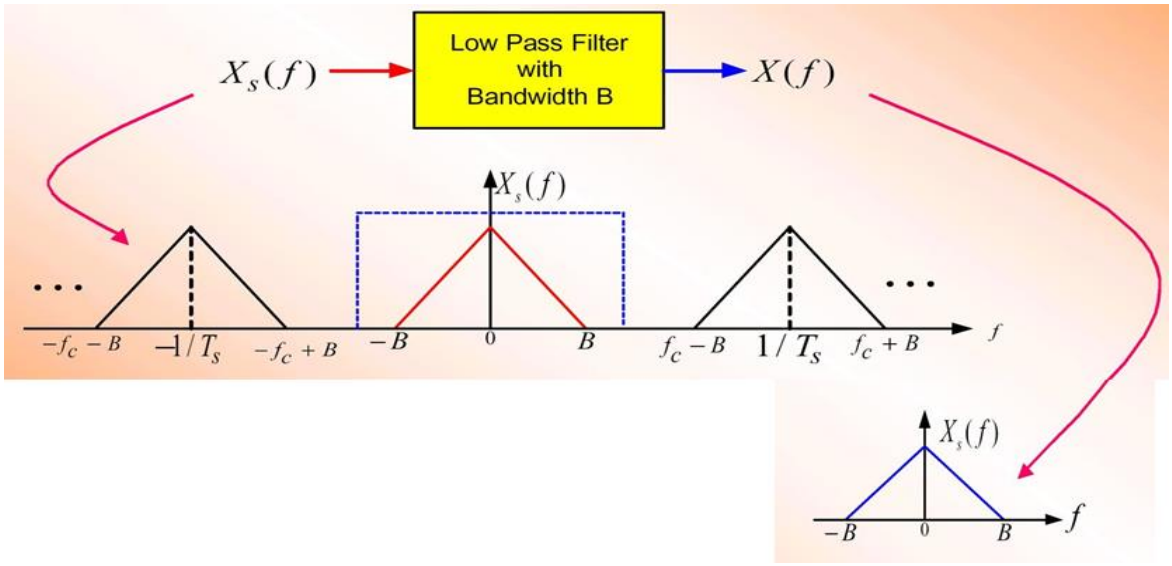


Fig. 13. Filter o/p in case of $f_s > 2B$

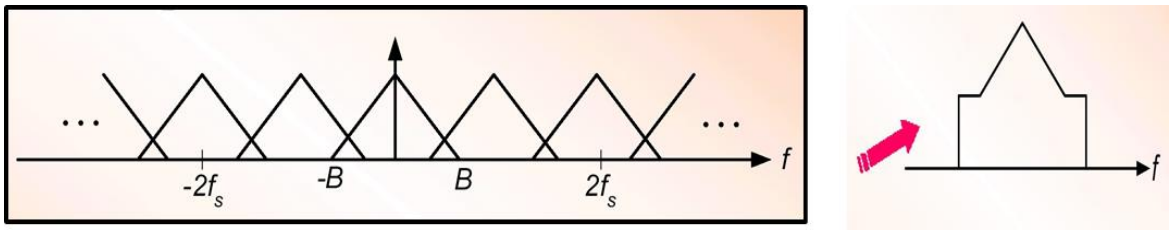


Fig. 14. Filter o/p in case of $f_s < 2B$

Example:

Suppose that an analog signal is given as

x(t) = 5 cos (2π.1000t), for t > 0, and is sampled at the rate of 8,000 Hz.

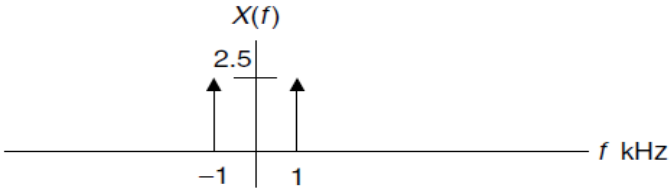
a. Sketch the spectrum for the original signal.

b. Sketch the spectrum for the sampled signal from 0 to 20 kHz.

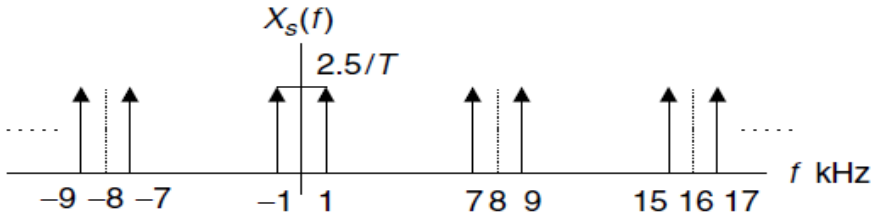
Sol.

a. Since the analog signal is sinusoid with a peak value of 5 and frequency of 1,000 Hz, we can write the sine wave using Euler's identity:

$$5\cos(2\pi \times 1000t) = 5 \cdot \left(\frac{e^{j2\pi\times 1000t} + e^{-j2\pi\times 1000t}}{2}\right) = 2.5e^{j2\pi\times 1000t} + 2.5e^{-j2\pi\times 1000t},$$



b. After the analog signal is sampled at the rate of 8,000 Hz, the sampled signal spectrum and its replicas centered at the frequencies $\pm nf_s$, each with the scaled amplitude being 2.5/T, are as shown in Figure below.



Notice that the spectrum of the sampled signal contains the images of the original spectrum; that the images repeat at multiples of the sampling frequency f_s (for our example, 8 kHz, 16 kHz, 24 kHz, . . .); and that all images must be removed, since they convey no additional information.

Signal reconstruction

Two simplified steps are involved, as described in Fig. 15. First, the digitally processed data $y(n)$ are converted to the ideal impulse train $y_s(t)$, in which each impulse has its amplitude proportional to digital output $y(n)$, and two consecutive impulses are separated by a sampling period of T; second, the analog reconstruction filter is applied to the ideally recovered sampled signal $y_s(t)$ to obtain the recovered analog signal.

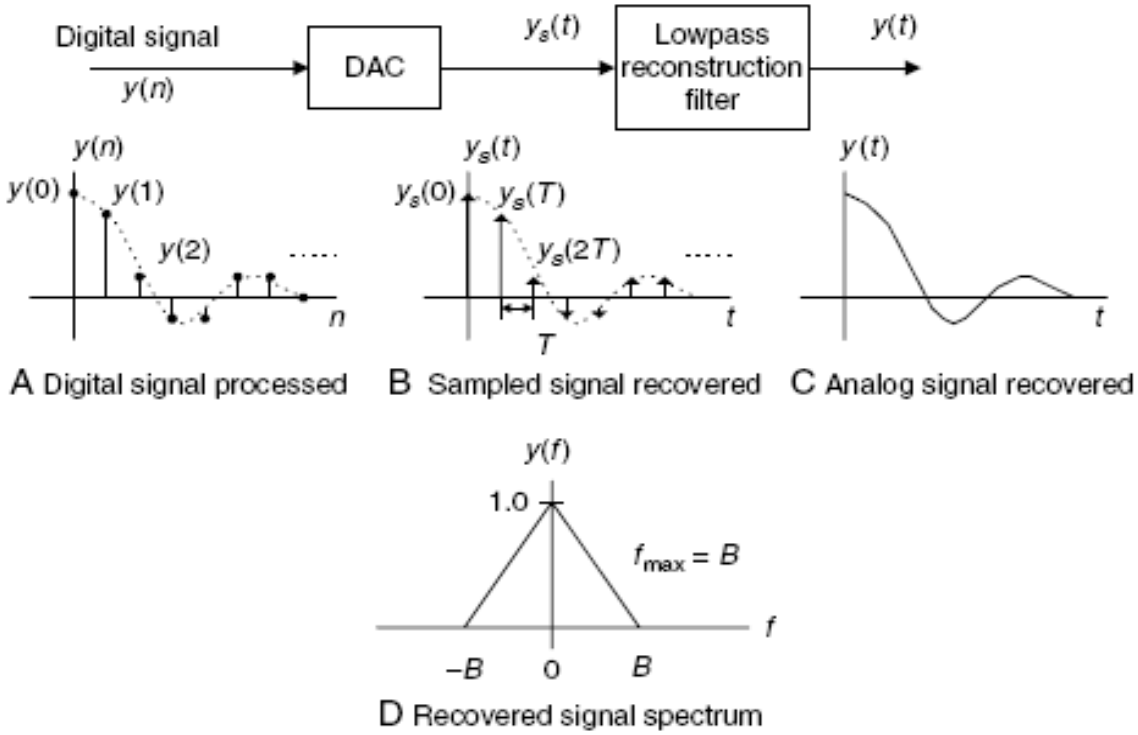


Fig. 15. Signal notations at reconstruction stage

The following three cases are listed for recovery of the original signal spectrum:

Case 1: $f_s = 2f_{max}$: Nyquist frequency is equal to the maximum frequency of the analog signal x(t), an ideal lowpass reconstruction filter is required to recover the analog signal spectrum. This is an impractical case.

Case 2: $f_s > 2f_{max}$: In this case, there is a separation between the highest frequency edge of the baseband spectrum and the lower edge of the first replica. Therefore, a practical lowpass reconstruction (anti-image) filter can be designed to reject all the images and achieve the original signal spectrum.

Case 3: $f_s < 2f_{max}$: This is aliasing, where the recovered baseband spectrum suffers spectral distortion, that is, contains an aliasing noise spectrum; in time domain, the recovered analog signal may consist of the aliasing noise frequency or frequencies. Hence, the recovered analog signal is incurably distorted.

Example: Assuming that an analog signal is given by

$x(t) = 5\cos(2\pi.2000t) + 3\cos(2\pi.3000t)$ for $t \geq 0$, and it is sampled at the rate of 8,000 Hz,

a. Sketch the spectrum of the sampled signal up to 20 kHz.

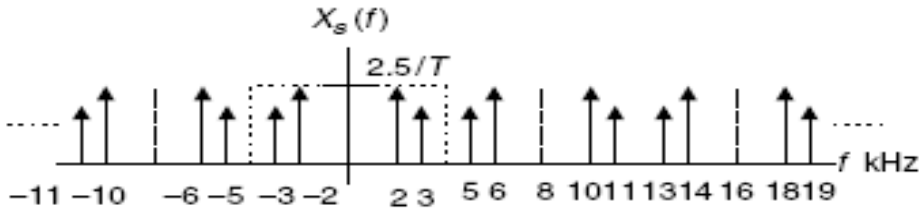
b. Sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to filter the sampled signal ($y(n)=x(n)$ in this case) to recover the original signal.

Sol.

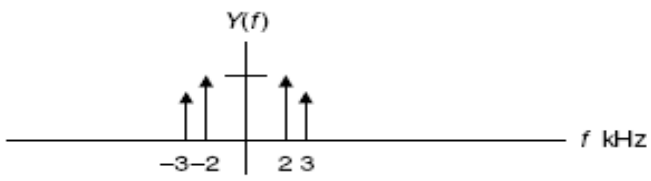
a. Using Euler's identity, we get

$$x(t) = \frac{3}{2}e^{-j2\pi\cdot 3000t} + \frac{5}{2}e^{-j2\pi\cdot 2000t} + \frac{5}{2}e^{j2\pi\cdot 2000t} + \frac{3}{2}e^{j2\pi\cdot 3000t}.$$

The two-sided amplitude spectrum for the sinusoids (sampled signal) is displayed in Fig.



b. Based on the spectrum in (a), the sampling theorem condition is satisfied; hence, we can recover the original spectrum using a reconstruction lowpass filter. The recovered spectrum is shown in the following Fig.



Aliasing noise level

Given the DSP system shown in Fig. 16, where we can find the percentage of the aliasing noise level using the symmetry of the Butterworth magnitude function and its first replica. Then:

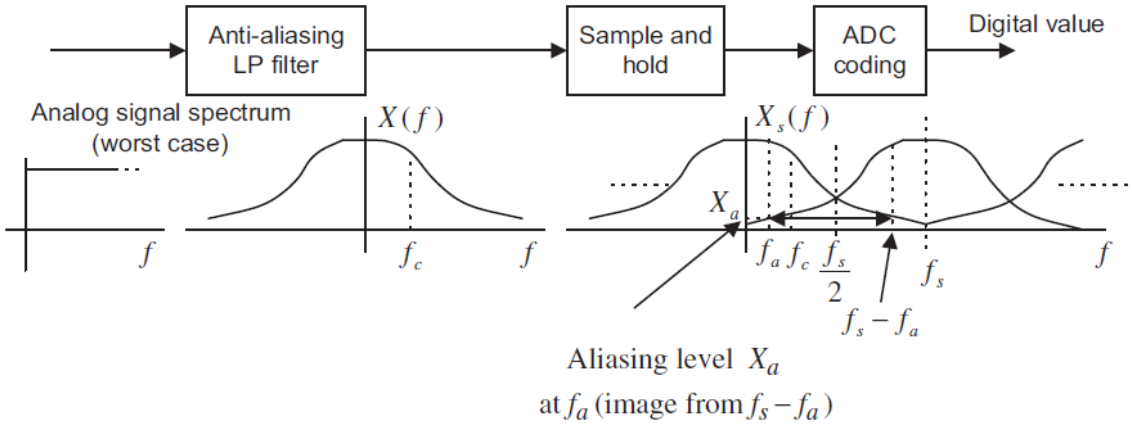


Fig. 16. DSP system with anti-aliasing filter

$$\text{Aliasing noise level \%} = \frac{\sqrt{1+\left(\frac{f_a}{f_c}\right)^{2n}}}{\sqrt{1+\left(\frac{f_s - f_a}{f_c}\right)^{2n}}} \quad \text{for } 0 \le f \le f_c$$

Where, n is the filter order, f_a is the aliasing frequency, f_c is the cutoff frequency, and f_s is the sampling frequency.

Example: In a DSP system with anti-aliasing filter, if a sampling rate of 8,000 Hz is used and the anti-aliasing filter is a second-order Butterworth lowpass filter with a cutoff frequency of 3.4 kHz,

a. Determine the percentage of aliasing level at the cutoff frequency.

b. Determine the percentage of aliasing level at the frequency of 1,000 Hz.

Sol.

$$f_s = 8000,\ f_c = 3400,\ \text{and } n = 2.$$

a. Since $f_a = f_c = 3400$ Hz, we compute

$$\text{aliasing noise level \%} = \frac{\sqrt{1+\left(\frac{3.4}{3.4}\right)^{2\times 2}}}{\sqrt{1+\left(\frac{8-3.4}{3.4}\right)^{2\times 2}}} = \frac{1.4142}{2.0858} = 67.8\%.$$

b. With $f_a = 1000$ Hz, we have

$$\text{aliasing noise level \%} = \frac{\sqrt{1+\left(\frac{1}{3.4}\right)^{2\times 2}}}{\sqrt{1+\left(\frac{8-1}{3.4}\right)^{2\times 2}}} = \frac{1.03007}{4.3551} = 23.05\%.$$

Digital Signal and Systems

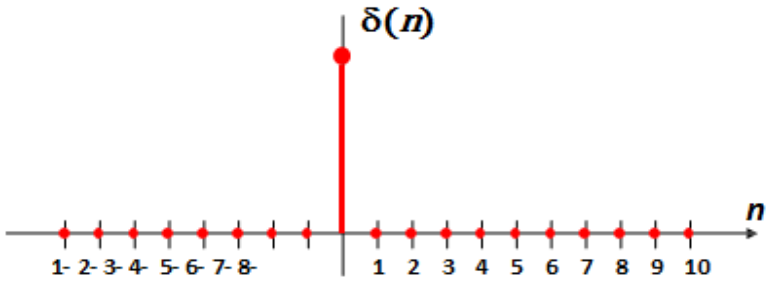
Discrete-Time Signals:

In digital signal processing, signals are represented as sequence of numbers called "samples". A sample value of a typical discrete-time signal or sequence is denoted as "$x[n]$" with the argument "n" being an integer in the range ($-\infty$ and ∞). It should be noted that $x[n]$ is defined only for integer values of "n" and undefined otherwise.

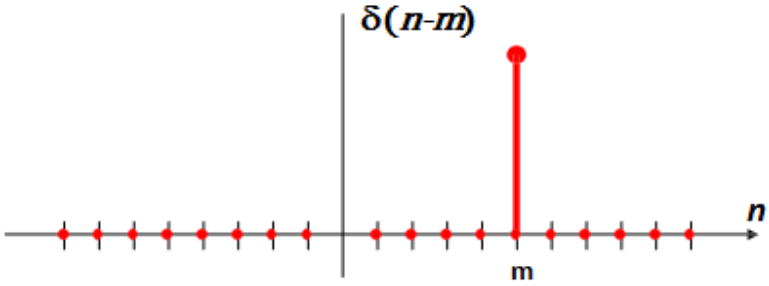
The most common basic sequences are described as follows:

- **delta function or unit-impulse (sample) sequence $\delta(n)$**

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

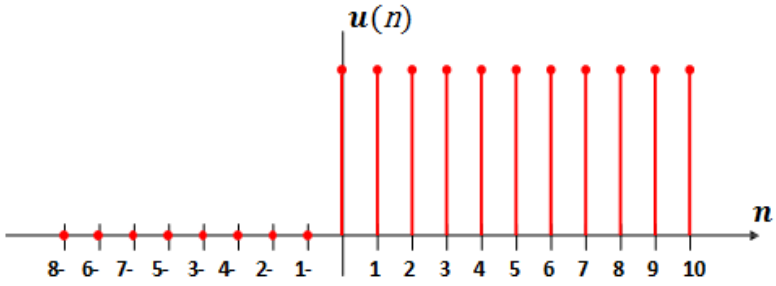


$$\delta(n-m) = \begin{cases} 1 & n = \mathrm{m} \\ 0 & n \neq \mathrm{m} \end{cases}$$

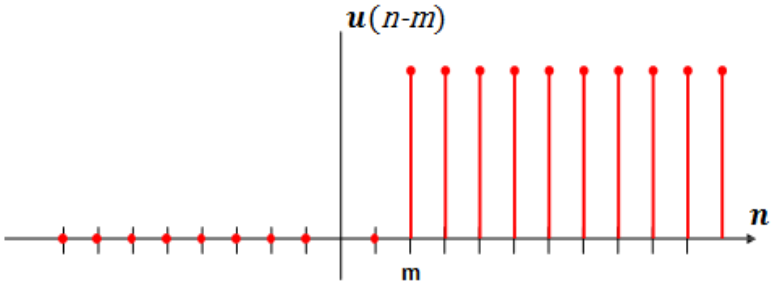


- **unit-step sequence $U(n)$**

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$u(n-m) = \begin{cases} 1 & n \geq \mathrm{m} \\ 0 & n < \mathrm{m} \end{cases}$$

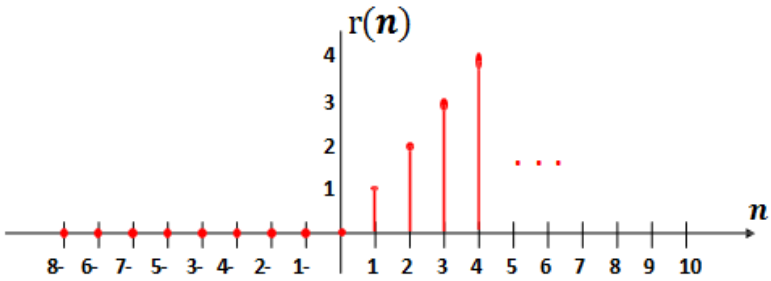


$$u(n)=\sum_{m=0}^{\infty}\delta(n-m)$$

$$\delta(n)=u(n)-u(n-1)$$

- **unit-ramp sequence *r(n)***

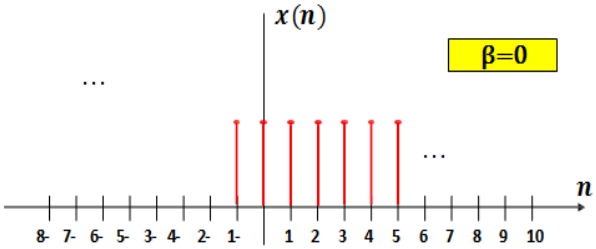
$$r(n)=nu(n)=\begin{cases} n & n\geq 0 \\ 0 & n<0 \end{cases}$$

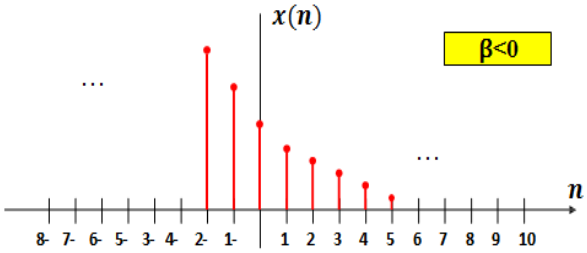


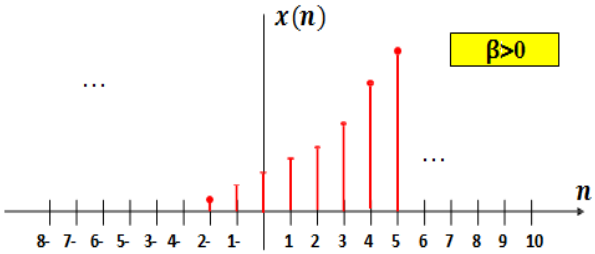
- **exponential sequence**

$$x(n)=Ae^{\beta n}$$

- If β=0, *x*(n)=*A*
- If β<0, *x*(n) is exponential decay.
- If β>0, *x*(n) is exponential growth.

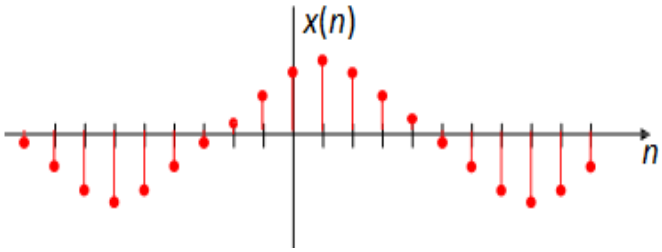






- **Sinusoidal sequence**

$$x(n)=A\cos(n\omega_0-\phi)$$



Note: $x(n)=x(t)\big|_{t=nT_s}$

For analog sine ⇨ $x(t)=\sin(wt)$, the discrete sine ⇨ $x(n)=\sin(wnT_s)$

$w_0=wT_s$ where $w=2\pi f_m$ and $T_s=1/f_s$

Example: Assuming a DSP system with a sampling time interval of 125 microseconds, convert each of the following analog signals $x(t)$ to the digital signal $x(n)$.

1. $10e^{-5000t}u(t)$
2. $10\sin(2000\pi t)u(t)$

sol.

1. $x(n)=x(nT_s)= 10e^{-5000\times0.000125n}u(nT_S)=10e^{-0.625n}u(n)$
2. $x(n)=x(nT_s)= 10\sin(2000\pi\times0.000125n)u(nT_S)=10\sin(0.25\pi n)u(n)$

Periodic Sequences:

A sequence $x(n)$ is defined to be periodic with period N if

$$x(n) = x(n+ N) \quad \text{for all } N$$

Where; N is an integer number.

Example: Is the sequence $x(n)=\cos(\pi n/4)$ periodic, find N.

sol.

Suppose it is periodic sequence with period N

$x(n)=x(n+N)$

$\cos(\pi n/4)=\cos(\pi(n+N)/4)$

$\pi n/4+\mathbf{2\pi k}= \pi n/4+\pi N/4 \Rightarrow N=2\pi k/(\pi/4)= 2\pi k/w_0=8k$ K: integer

for k=1, N=8

Example: Is the sequence $x(n)=\cos(3\pi n/8)$ periodic. If yes, find N.

sol.

Suppose it is periodic sequence with period N

$x(n)=x(n+N)$

$\cos(3\pi n/8)=\cos(3\pi(n+N)/8)$

$3\pi n/8+\mathbf{2\pi k}= 3\pi n/8+3\pi N/8 \Rightarrow N=2\pi k/w_0=2\pi k/(3\pi/8)$ K: integer

for k=3, N=16

Example: Is the sequence $x(n)=\cos(n)$ periodic. If yes, find N.

sol.

Suppose it is periodic sequence with period N

$x(n)=x(n+N)$

$\cos(n)=\cos(n+N)$

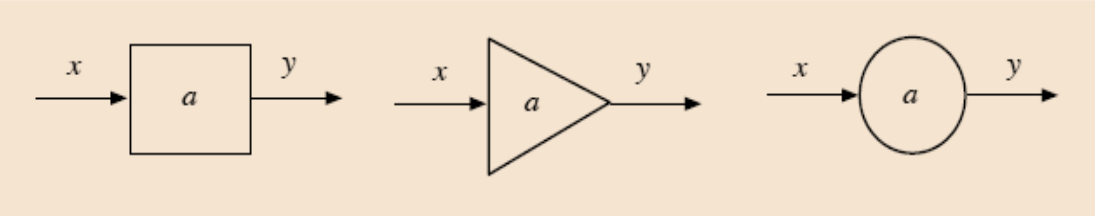
for $n+\mathbf{2\pi k}=n+N$, K: integer **There is no integer N $\Rightarrow$ Non-periodic sequence**

Operations on Sequences:

❑ **For input signal $x(n)$ and output signal $y(n)$**

(i) Scaling: $y(n)=\alpha x(n)$

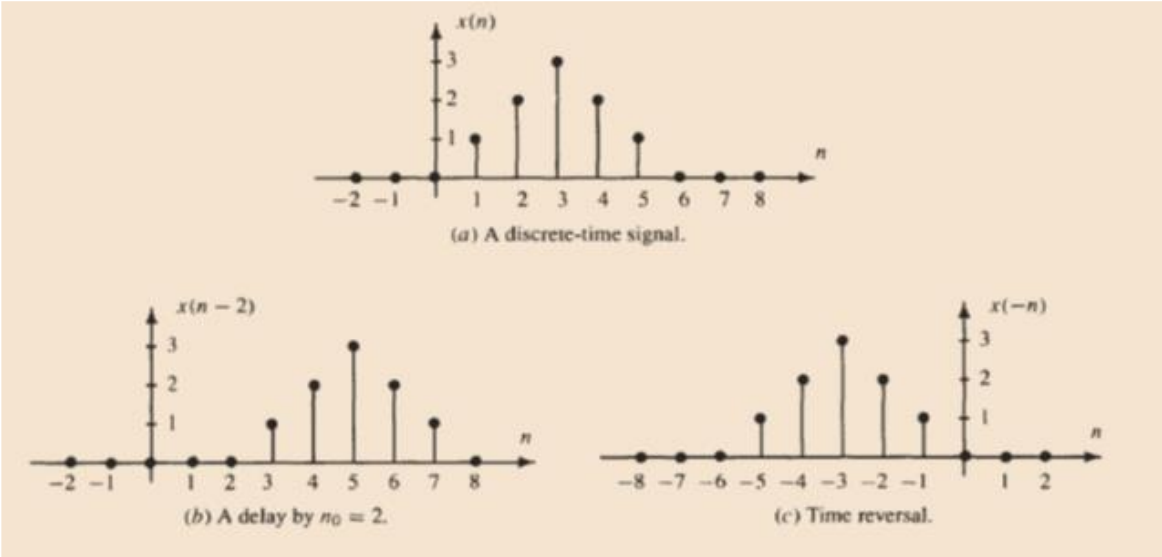
- α is called gain or scale factor.
- If |α|>1, called an amplification.
- If |α|<1, called an attenuating.
- If α <0, called inverting.
- Sometimes denoted by triangle or circle in block diagram:



(ii) Time shifting: $y(n) = x(n - n_0)$

- If n_0>0, called delay.
- If n_0<0, called predictor.

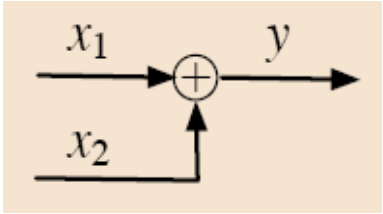
(iii) Reflection (Time reversal): $y(n) = x(-n)$



❑ **For multiple input signals $x_1(n)$, $x_2(n)$ and output signal $y(n)$**

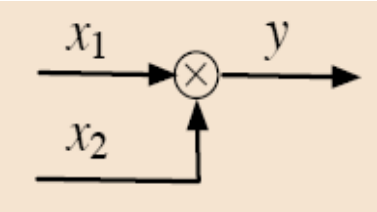
(i) Addition (summing):

$y(n)=x_1+x_2=x_1(n)+x_2(n)$



(ii)Product (multiplier or modulation):

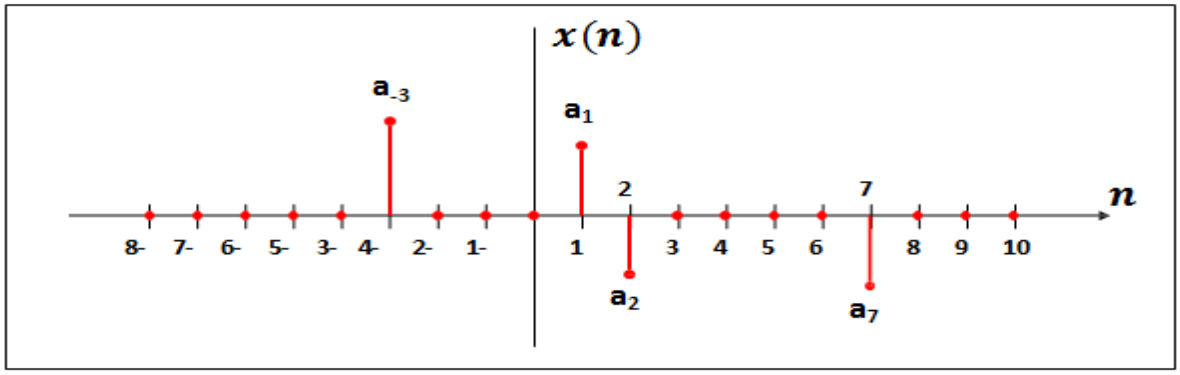
$y(n)=x_1.x_2=\{x_1(n)\ x_2(n)\}$



Sequence Representation Using Delay Unit:

Any arbitrary sequence $x(n)$ can be represented in terms of delayed and scaled impulse sequence $\delta[n]$ as shown in the figure

$$x(n)=\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$



$$x(n)=a_{-3}\delta(n+3)+a_1\delta(n-1)+a_2\delta(n-3)+a_7\delta(n-7)$$

Example: Represent the sequence $x[n]$ = {4, 2, -1, 1, 3, 2, 1, 5} as sum of shifted unit impulse.

sol.

Given $x[n]$ = {4, 2, -1, 1, 3, 2, 1, 5}; n = -3 -2 -1 0 1 2 3 4

$x[n] = x[-3]\delta[n+3] + x[-2]\ \delta[n+2] + x[-1]\ \delta[n+1] + x[0]\ \delta[n] + x[1]\ \delta[n-1] + x[2]\ \delta[n-2] + x[3]\ \delta[n-3] + x[4]\ \delta[n-4]$

$= 4\ \delta[n+3] + 2\ \delta[n+2] - \delta[n-1] + \delta[n] + 3\ \delta[n-1] + 2\ \delta[n-2] + \delta[n-3] + 5\ \delta[n-4]$

Example: Consider the following two sequences of length (5) defined for $0 \leq n \leq 4$:

$x[n]$ = {3.5, 41, 36, -9.5, 0}

$y[n]$ = {1.7, -0.5, 0, 0.8, 1}

Find:

a) $x[n].y[n]$
b) $x[n]+y[n]$
c) $7/2\ x[n]$

sol.

a) $x[n].y[n]$= {5.44, -20.5, 0, -7.6, 0}
b) $x[n]+y[n]$= {4.9, 40.5, 36, -8.7, 1}
c) $7/2\ x[n]$= {11.2, 143.5, 126, -33.25, 0}

Energy and Power of a Sequence:

Energy of a sequence is defined by

$$E = \sum_{n=-\infty}^{n=\infty} |x(n)|^2$$

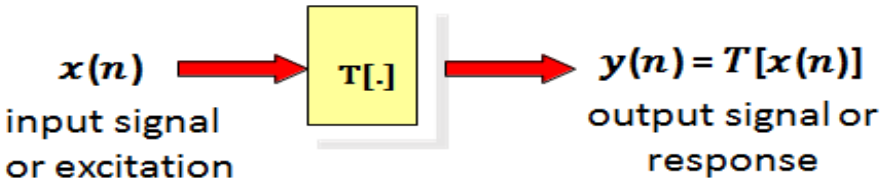
Power of a sequence is defined by

$$P = \frac{1}{N} \sum_{n=-\infty}^{n=\infty} |x(n)|^2$$

- A signal is called energy signal if $E < \infty$.
- A signal is called power signal if $0 < P < \infty$.
- A signal can be an energy signal, a power signal or neither type.
- An energy signal has zero power. $E < \infty$; $P = 0$
- A power signal has infinite energy. $P < \infty$; $E = \infty$

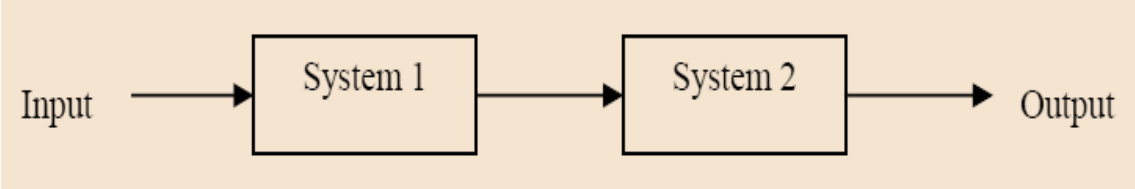
Discrete-Time Systems (Digital Processors):

A discrete-time system is a device or algorithm that operates on a discrete-time signal called the input or excitation (e.g. $x(n)$), according to some rule (e.g. T[.]) to produce another discrete-time signal called the output or response (e.g. $y(n)$). The transformation T[.], (also called operator or mapping) or processing performed by the system on $x(n)$ to produce $y(n)$.

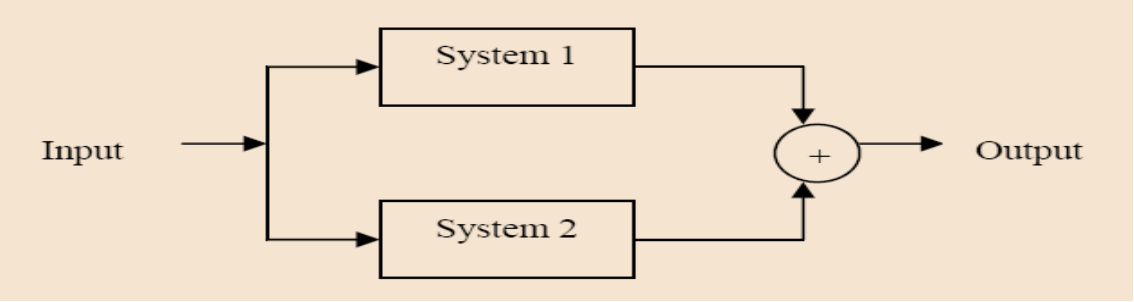


Interconnections of Systems:

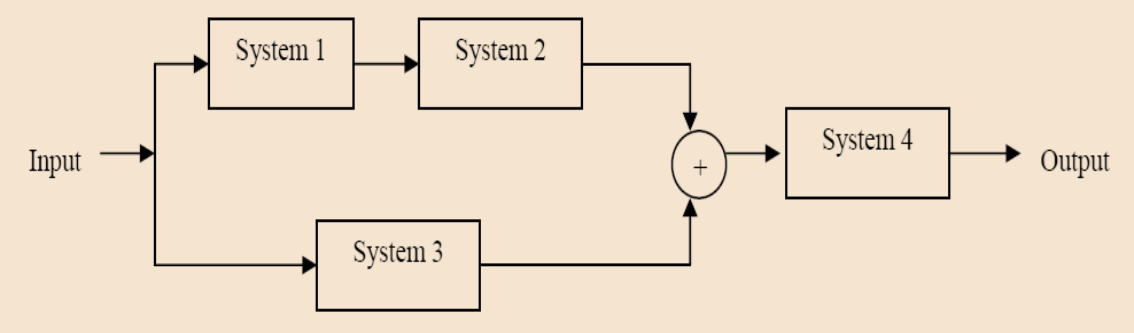
1. **Series or cascade interconnection**. The output of System 1 is the input to System 2.



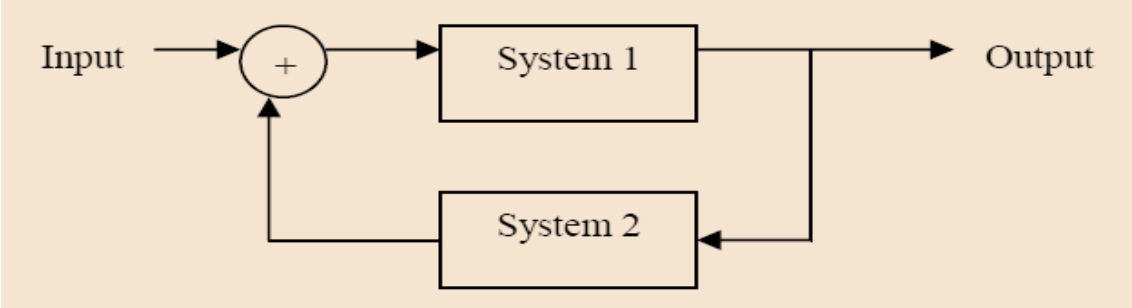
2. **Parallel interconnection.** The same input signal is applied to Systems 1 and 2.



3. **Combination of both cascade and parallel interconnection.**



4. **Feedback interconnection**. The output of System 2 is fed back and added to the external input to produce the actual input to System 1.



Classification of Discrete-Time Systems:

- Static (Memoryless) and Dynamic (Memory) Systems.
- Linear and Nonlinear Systems.
- Time-Invariant (TI) and Time-Varying Systems.
- Causal and Non-Causal Systems.
- Stable and Unstable Systems.

Static (Memoryless) and Dynamic (Memory) systems:

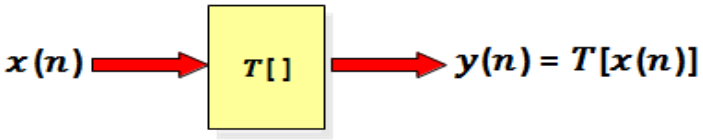
A discrete-time system is called **static** or **memoryless** if its output at any time instant n depends on the input sample at the same time, but not on the past or future samples of the input. For example $\boldsymbol{y(n) = \alpha x(n)}$, $\boldsymbol{y(n) = nx(n) + bx^3(n)}$.

In the other case, the system is said to be **dynamic** or to have **memory**, if the output of a system at time n depends not only on the value of input at the same instant n, but also on past or future values of the input. For example

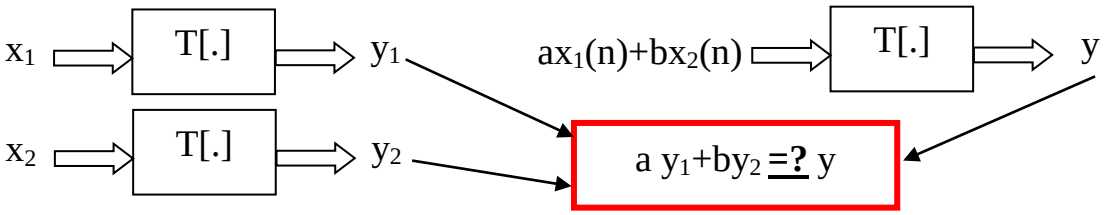
$$\boldsymbol{y[n] = \alpha x[n] + \beta x[n-1]},\ \boldsymbol{y(n) = \sum_{k=0}^{N} h(k)x(n-k)},\ \boldsymbol{y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)}.$$

Linear and Nonlinear Systems:

A discrete-time system is called **linear** if only if it satisfies the ***linear superposition principle***. In the other case, the system is called **non-linear**. If $y_1(n)$ and $y_2(n)$ are the responses to the inputs $x_1(n)$ and $x_2(n)$ respectively, then the input $\boldsymbol{x(n) = ax_1(n) + bx_2(n)}$ gives the output $\boldsymbol{y(n) = ay_1(n) + by_2(n)}$.



$$T[aX_1(n)+bX_2(n)] = aT[X_1(n)]+bT[X_2(n)]$$



Example: Test the linearity of the system

$$y(n) = 1/3(x(n+1)+x(n)+x(n-1))$$

sol.

The input x_1 then the output $y_1(n)= (1/3)a(x_1(n+1)+x_1(n)+x_1(n-1))$

The input x_2 then the output $y_2(n)= (1/3)b(x_2(n+1)+x_2(n)+x_2(n-1))$

$a\ y_1(n)+b\ y_2(n)= (1/3)a(x_1(n+1)+x_1(n)+x_1(n-1))+ (1/3)b(x_2(n+1)+x_2(n)+x_2(n-1))$

The input: $x(n)=ax_1(n)+bx_2(n)$, then the output

$y(n)=1/3(ax_1(n+1)+bx_2(n+1)+ax_1(n)+bx_2(n)+ax_1(n-1)+bx_2(n-1))$

$= (1/3)a(x_1(n+1)+x_1(n)+x_1(n-1))+(1/3)b(x_2(n+1)+x_2(n)+x_2(n-1))$

Then, $y(n)= a\ y_1(n)+b\ y_2(n)$ → The system is linear.

Example: Test the linearity of the accumulator system

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

sol.

The input x_1 then the output $y_1(n)= (1/3)a(x_1(n+1)+x_1(n)+x_1(n-1))$

The input x_2 then the output $y_2(n)= (1/3)b(x_2(n+1)+x_2(n)+x_2(n-1))$

$a\ y_1(n)+b\ y_2(n)= (1/3)a(x_1(n+1)+x_1(n)+x_1(n-1))+ (1/3)b(x_2(n+1)+x_2(n)+x_2(n-1))$

The input: $x(n)=ax_1(n)+bx_2(n)$, then the output

$y(n)=1/3(ax_1(n+1)+bx_2(n+1)+ax_1(n)+bx_2(n)+ax_1(n-1)+bx_2(n-1))$

$= (1/3)a(x_1(n+1)+x_1(n)+x_1(n-1))+(1/3)b(x_2(n+1)+x_2(n)+x_2(n-1))$

Then, $y(n)= a\ y_1(n)+b\ y_2(n)$ → The system is linear.

Let the input: $x(n)=ax_1(n)+bx_2(n)$, then the output

$$y(n) = \sum_{k=-\infty}^{n} ax_1(k) + bx_2(k)$$

$$= a \sum_{k=-\infty}^{n} x_1(k) + b \sum_{k=-\infty}^{n} x_2(k)$$

Then, $y(n)= a\, y_1(n)+b\, y_2(n)$ ⟹ The system is linear.

Example: Test the linearity of the system

$$\boldsymbol{y(n) = x^2(n)}$$

sol.

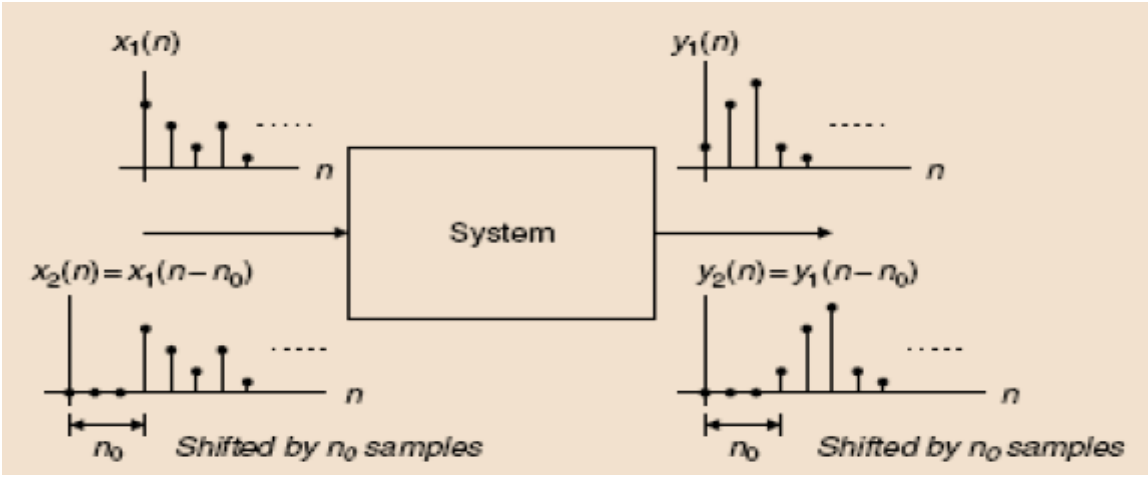
Let the input: $x(n)=ax_1(n)+bx_2(n)$, then the output $y(n)= [ax_1(n)+bx_2(n)]^2$

$$= a^2x_1^2(n) + 2ab x_1(n)x_2(n) + b^2x_2^2(n)$$

Then, $y(n)\neq a\, y_1(n)+b\, y_2(n)$ ⟹ The system is nonlinear.

Time-Invariant (TI) and Time-Varying Systems:

A **Time-Invariant (TI)** system is one in which if $y(n)$ is the output when the input $x(n)$ is applied, then $y(n-n_0)$ is the output when $x(n–n_0)$ is applied. In the other case, the system is called **time-variable**. Conceptually, a system is TI if the behavior and the input-output characteristics do not change with time. For example the system $\boldsymbol{y(n) = \alpha x(n)}$.



Example: Given the linear systems:

a. $\boldsymbol{y(n) = 2x(n-5)}$

b. $\boldsymbol{y(n) = 2x(3n)}$

c. $\boldsymbol{y(n) = nx(n)}$

Determine whether each of the following systems is time-invariant.

sol.

a. Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the system output is $y_1(n) = 2x_1(n-5)$. Again, let $x_2(n) = x_1(n-n_0)$ be the shifted input and $y_2(n)$ be the output due to the shifted input. We determine the system output using the shifted input as

$\boldsymbol{y_2(n) = 2x_2(n-5) = 2x_1(n-5-n_0)}$:

Meanwhile, shifting $y_1(n) = 2x_1(n-5)$ by n_0 samples leads to

$\boldsymbol{y_1(n-n_0) = 2x_1(n-n_0-5)}$

We can verify that $\boldsymbol{y_2(n) = y_1(n-n_0)}$. Thus the shifted input of n_0 samples causes the system output to be shifted by the same n_0 samples, thus, the system is ***time-invariant***.

b. Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the system output is $y_1(n) = 2x_1(3n)$. Again, let the input and output be $x_2(n)$ and $y_2(n)$, where $x_2(n) = x_1(n-n_0)$, a shifted version, and the corresponding output is $y_2(n)$. We get the output due to the shifted input $x_2(n) = x_1(n-n_0)$ and note that $x_2(3n) = x_1(3n-n_0)$:

$\boldsymbol{y_2(n) = 2x_2(3n) = 2x_1(3n-n_0)}$:

On the other hand, if we shift $y_1(n)$ by n_0 samples, which replaces n in

$y_1(n) = 2x_1(3n)$ by $n-n_0$, it yield

$\boldsymbol{y_1(n-n_0) = 2x_1(3(n-n_0)) = 2x_1(3n-3n_0)}$:

Clearly, we know that $\boldsymbol{y_2(n) \neq y_1(n-n_0)}$. Since the system output $y_2(n)$ using the input shifted by n_0 samples is not equal to the system output $y_1(n)$ shifted by the same n_0 samples, thus, the system is ***not time-invariant (time-varying system)***.

c. Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the output is $y_1(n) = n\,x_1(n)$. Again, let the input and output be $x_2(n)$ and $y_2(n)$, where $x_2(n) = x_1(n-n_0)$, a shifted version, and the corresponding output is $y_2(n)$. We get the output due to the shifted input $x_2(n) = x_1(n-n_0)$ and note that $x_2(n) = n\,x_1(n-n_0)$:

$\boldsymbol{y_2(n) = n\,x_2(n) = n\,x_1(n-n_0)}$:

On the other hand, if we shift $y_1(n)$ by n_0 samples, which replaces n in

$y_1(n) = n\,x_1(n)$ by $n-n_0$, it yield

$\boldsymbol{y_1(n-n_0) = (n-n_0)\,x_1(n-n_0)}$:

Clearly, we know that $\boldsymbol{y_2(n) \neq y_1(n-n_0)}$. Since the system output $y_2(n)$ using the input shifted by n_0 samples is not equal to the system output $y_1(n)$ shifted by the same n_0 samples, thus, the system is ***not time-invariant (time-varying system)***.

Note: Linear Time Invariant System (LTI) is the system that satisfies both the linearity and the time-invariance properties. Such systems are mathematically easy to analyze, and easy to design.

Causal and Non-Causal Systems:

A **causal** system is one in which the output $y(n)$ at time n depends only on the current input $x(n)$ at time n, its past input sample values such as $x(n-1)$, $x(n-2)$, . . . For example **y[n]**

= $\alpha x[n] + \beta x[n-1]$. Otherwise, if a system output depends on the future input values, such as $x(n + 1)$, $x(n + 2)$, . . . , the system is **noncausal**. For example **$y[n] = \alpha x[n] + \beta x[n+1]$**. The noncausal system cannot be realized in real time.

Example: Given the linear systems:

a. $y(n) = 0.5x(n) + 2.5x(n-2)$, for $n \geq 0$

b. $y(n) = 0.25x(n-1) + 0.5x(n+1) - 0.4y(n-1)$, for $n \geq 0$,

c. $$y(n) = \sum_{k=-2}^{2} h(k)\,x(n-k)$$

Determine whether each is causal.

sol.

a. Since for $n \geq 0$, the output $y(n)$ depends on the current input $x(n)$ and its past value $x(n-2)$, the system is **causal**.

b. Since for $n \geq 0$, the output $y(n)$ depends on the input's future value $x(n+1)$, the system is **noncausal**.

c. Since for $n \geq 0$, the output $y(n)$ depends on the input's future values $x(n+1)$ and $x(n+2)$, the system is **noncausal**.

Stable and Unstable Systems:

A system is said to be **bounded input-bounded output (BIBO) stable** if and only if every bounded input produces the bounded output. It means, that there exist some finite numbers say M_x and M_y, such that

$$|x(n)| \leq M_x < \infty \Rightarrow |y(n)| \leq M_y < \infty$$

For all n, If for some bounded input sequence $x(n)$, the output $y(n)$is unbounded (infinite), the system is classified as **unstable**.

Note: The system is stable, if its transfer function vanishes after a sufficiently long time. For a stable system:

$$S = \sum_{k=-\infty}^{\infty} |\, h(k) \,| < \infty$$

Where $h(k)$ = unit impulse response.

Example: Given the systems:

a. **$y[n] = (x[n])^2$**

b. Accumulator system $y[n]=\sum_{k=-\infty}^{n} x[k]$,

c. $y[n]=e^{x[n]}$

Determine whether each is stable.

sol.

a. If $|x[n]|\le B_x<\infty$ for all n, then $|y[n]|\le B_y=B_x^2<\infty$ for all n. Thus, the system is **stable**.

b. If $x[n]=u[n]=\begin{cases}0 & n<0\\ 1 & n\ge 0\end{cases}$: bounded

Then $y[n]=\sum_{k=-\infty}^{n}x[k]=\sum_{k=-\infty}^{n}u[k]=\begin{cases}0 & n<0\\ n+1 & n\ge 0\end{cases}$: not bounded

Thus, the accumulator system is **unstable**.

c. If $|x[n]|\le B_x<\infty$ for all n, then $|y[n]|\le B_y=e^{B_x}<\infty$ for all n. i.e., it is guaranteed that if the input is bounded by a positive number B_x, the output is bounded by a positive number e^{B_x}. Thus, the system is **stable**.

Note: $\sum_{k=-\infty}^{n}u[k]=\begin{cases}0 & n<0\\ n+1 & n\ge 0\end{cases}$

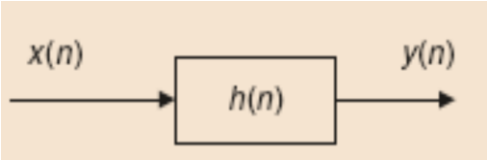
System Representation Using Its Impulse Response:

Any discrete-time can be characterized by one of the representations:

1) Difference Equation
2) Impulse Response $h(n)$
3) Transfer Function H(z)
4) Frequency Response H(W)

In this section, a Linear Time-Invariant (LTI) system will be represented by its impulse response (h(n)).

A LTI system can be completely described by its unit-impulse response, which is defined ***as the system response due to the impulse input δ(n) with zero initial conditions***, depicted in the following figure. Here $x(n)=\delta(n)$ and $y(n)=h(n)$.



Note: The unit step function $u[n]$ is the running sum of the unit impulse $\delta[n]$, so the **step**

response *S*[*n*] of a LTI processor is the running sum of its impulse response. Therefore, if we denote the step response by *S*[*n*], we have

$$S[n] = y[n]\big|_{x[n]=u[n]} = \sum_{m=-\infty}^{n} h[m]$$

Alternatively, h[*n*] is the first order difference of *S*[*n*]

$$h[n] = y[n]\big|_{x[n]=\delta[n]} = S[n] - S[n-1]$$

Example: For a LTI system described by the following difference equation:

$$y(n) = 0.8y(n-1) + x(n)$$

a. Find and sketch the first four sample values of the impulse and step responses.

b. Determine the final value of the step response as $n \longrightarrow \infty$.

sol.

a. By setting $x(n)=\delta(n)$ in the system difference equation, then $y(n)=h(n)$ so,

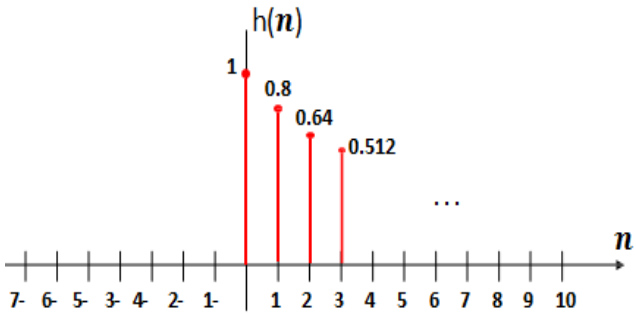
$$h(n)=0.8h(n-1)+\delta(n)$$

for $n=0$, $h(0)=(0.8)h(-1)+\delta(0)=1$

for $n=1$, $h(1)=(0.8)h(0)+\delta(1)=0.8$

for $n=2$, $h(2)=(0.8)h(1)+\delta(2)=(0.8)^2=0.64$

for $n=3$, $h(3)=(0.8)h(2)+\delta(3)=(0.8)^3=0.512$



b. By setting $x(n)=u(n)$ in the system difference equation, then $y(n)=S(n)$ so,

$$S(n)=0.8S(n-1)+u(n)$$

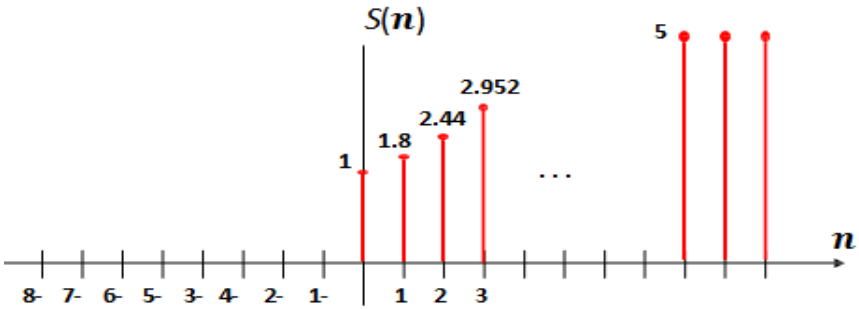
for $n=0$, $S(0)=(0.8)S(-1)+u(0)=1$

for $n=1$, $S(1)=(0.8)S(0)+u(1)=0.8+1=1.8$

for $n=2$, $S(2)=(0.8)S(1)+u(2)=0.8(0.8+1)+1=1+0.8+(0.8)^2=2.44$

for $n=3$, $S(3)=(0.8)S(2)+u(3)=0.8(1+0.8+(0.8)^2)+1=1+0.8+(0.8)^2+(0.8)^3=2.952$

Then, for $n=\infty$, $S(\infty)=1+0.8+(0.8)^2+(0.8)^3+(0.8)^4+\ldots+(0.8)^\infty=\sum_{n=0}^{\infty}(0.8)^n=\frac{1}{1-0.8}=5$



Note: $\sum_{n=0}^{\infty}(x)^n = \frac{1}{1-x}$ ***and*** $\sum_{n=0}^{N}(x)^n = \frac{1-x^{(N+1)}}{1-x}$

Example: Given the linear time-invariant system

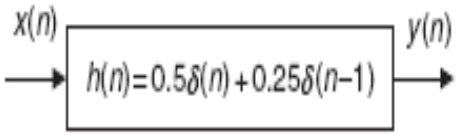
$y(n) = 0.5x(n) + 0.25x(n-1)$ with an initial condition $x(-1) = 0$

a. Determine the unit-impulse response $h(n)$.
b. Draw the system block diagram.
c. Write the output using the obtained impulse response.

sol.

a. $h(n) = 0.5\,\delta(n) + 0.25\,\delta(n-1)$, where $h(0)= 0.5$, $h(1) = 0.25$ and $h(n) = 0$ elsewhere.

b.



c. $y(n) = h(0)\,x(n) + h(1)\,x(n-1)$

From this result, it is noted that if the difference equation without the past output terms, $y(n-1), \ldots, y(n-N)$, that is, the corresponding coefficients $a_1, \ldots, a_N$, are zeros, the impulse response $h(n)$ has a finite number of terms. We call this a finite impulse response (FIR) system.

In general, we can express the output sequence of a LTI system from its impulse response and inputs as:

$$y(n) = \ldots + h(-1)\,x(n+1) + h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots$$

This equation called the ***digital convolution sum***.

Example: Given the difference equation

$y(n)= 0.25\,y(n-1) + x(n)$ **for** $n \geq 0$ **and** $y(-1) = 0$,

a. Determine the unit-impulse response $h(n)$.

b. Draw the system block diagram.

c. For a step input $x(n) = u(n)$, find the output responses for the first three samples using the difference equation.

sol.

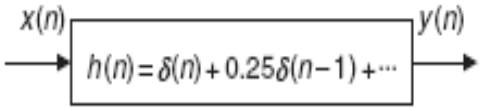
a. Let $x(n) = \delta(n)$, then $h(n) = 0.25\,h(n-1) + \delta(n)$
To solve for $h(n)$, we evaluate
$h(0) = 0.25\,h(-1) + \delta(0) = 0.25\,(0) + 1 = 1$
$h(1) = 0.25\,h(0) + \delta(1) = 0.25\,(1) + 0 = 0.25$
$h(2) = 0.25\,h(1) + \delta(2) = 0.25\,(0.5) + 0 = 0.0625$

…

With the calculated results, we can predict the impulse response as:

$h(n) = (0.25)^n\,u(n) = \delta(n) + 0.25\,\delta(n-1) + 0.0625\,\delta(n-2) + \ldots$

b. The system block diagram is given below



c. From the difference equation and using the zero-initial condition, we have

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0$$
$$n = 0, y(0) = 0.25y(-1) + x(0) = u(0) = 1$$
$$n = 1, y(1) = 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25$$
$$n = 2, y(2) = 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125$$
$$\ldots.$$

Notice that this impulse response h(n) contains an infinite number of terms in its duration due to the past output term y(n − 1). Such a system as described in the preceding example is called an infinite impulse response (IIR) system.

Digital Convolution

The Convolution Sum or Superposition Sum Representation of LTI Systems:

The **convolution** allows us to find the output signal from any LTI processor in response to any input signal. We can find the output signal $y(n)$ from an LTI processor by convolving its input signal $x(n)$ with a second function representing the impulse response h(n) of the processor. The convolution sum or superposition sum of the sequences $x(n)$ and h(n) can be represented by

$$\mathbf{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]}$$

N= N_1+N_2-1. Where N_1 = number of samples of $x(n)$, N_2= number of samples of h(n), and N= total number of samples. This operation is represented symbolically as ***x*(*n*)*h(*n*).**

Let nx is position of x sequence ,nh is position of h sequence ,then the position sequence of y (ny) can be found as

ny=[nx(0)+nh(0) … nx(N1-1)+nh(N2-1)]

Properties of Convolution:

1- *Commutatively*: Convolution is a commutative operation, meaning signals can be convolved in any order.

$$x(n) * h(n) = h(n) * x(n)$$

2- Associativity (Cascaded Connection)

Convolution is associative, meaning that convolution operations in series can be done in any order.

$$(x(n) * h(n)) * g(n) = x(n) * (h(n) * g(n))$$

3- Distributivity (Parallel Connection)

Convolution is distributive over addition

$$x(n) * [h(n) + g(n)] = x(n) * h(n) + x(n) * g(n)$$

The digital convolution can be performed by ***Direct method , graphical***, ***table lookup***, ***matrix by vector*** methods.

Graphical Method:

The convolution sum of two sequences can be found by using the following steps:

Step 1. Obtain the reversed sequence h(- k).

Step 2. Shift h(- k) by n samples to get h(n - k). If $n \geq 0$, h(- k) will be shifted to the right by n samples; but if $n < 0$, h(- k) will be shifted to the left by n samples.

Step 3. Perform the convolution sum that is the sum of the products of two sequences $x(k)$ and h(n - k) to get y(n).

Step 4. Repeat steps 1 to 3 for the next convolution value $y(n)$.

Example: Find the convolution of the two sequences $x[n]$ and h[n] given by $x[n]$ = [**3**, 1, 2] and h[n] = [**3**, 2, 1]. The bold number shows where n=0. Using:

a. Direct method.
b. Graphical method
c. Table Lookup Method

Sol: **a.** Using $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$x[n]$ = [**3**, 1, 2] and h[n] = [**3**, 2, 1] nx=[0 , 1 ,2] , nh=[0 , 1 , 2] ,

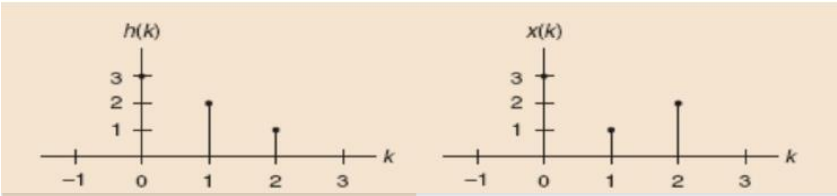
then ny=[0+0 … 2+2]=[0 … 4]=[0 1 2 3 4]

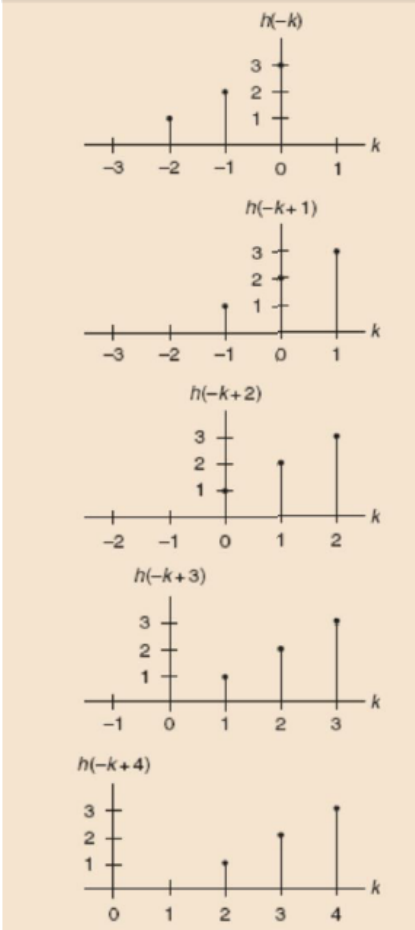
Total number of samples $N=N_1+N_2-1=3+3-1=5$ samples.

The values of k are equal to nx ,k =0,1,2

$$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9,$$
$$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9,$$
$$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11,$$
$$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5.$$
$$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2,$$
$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.$$

b. Graphical method





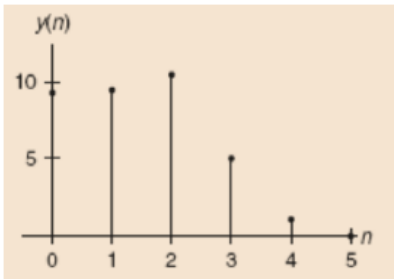
3x3=9

2x3+3x1=9

1x3+2x1+3x2=11

0x3+1x1+2x2+3x0=5

0x3+0x1+1x2+2x0=2



c. Table Lookup Method:

y(0) = 9
y(1) = 9
y(2) = 11
y(3) = 5
y(4) = 2

	y0)	y(1)	y(2)	
	3	2	1	
3	9	6	3	
1	3	2	1	y(3)
2	6	4	2	y(4)

Matrix by Vector Method:

Example : Find the **h*x** of the two sequences $x[n]$ and $h[n]$ given by $x[n]$ = [1 1 **2** 1 2 2 1 1] and $h[n]$ = [1 **2** -1 1] by using matrix be vector .

Solution:

nx=[-2 -1 0 1 2 3 4 5] , nh=[-1 0 1 2] → ny=[-2+(-1) . . . 5+2]=[-3 -2 -1 0 1 2 3 4 5 6 7]

Nx=8 , Nh=4 , N=8+4-1=11

Dimension of matrix become **N× Nx** =11×8

$$\boldsymbol{h} = [1,\ 2, -1,\ 1]$$
$$\boldsymbol{x} = [1,\ 1,\ 2,\ 1,\ 2,\ 2,\ 1,\ 1]$$
$$\boldsymbol{y} = H\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 5 \\ 3 \\ 7 \\ 4 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} \leftarrow$$

Circular Convolution

$$x_1(n) \otimes_N x_2(n) = \sum_{k=0}^{N-1} x_1((n-k) \mod N)\, x_2(k) = \sum_{k=0}^{N-1} x_1(k)\, x_2((n-k) \mod N)$$

The circular convolution can be performed by ***Direct method , Concentric Circle , graphical*** methods.

Note: **N** =maximum($\mathbf{N_1}$, $\mathbf{N_2}$)

Example: Use Direct, Concentric Circle and graphical methods to find circular convolution of $x_1(n)$=[**1** 2 2] and $x_2(n)$=[**0** 1 2 3].

Sol: **N** =maximum(**3 , 4)=4** **,**ny=[0 1 2 3] Applying the equation of circular convolution

1-Direct method : $y(n) = \sum_{k=0}^{3} x_1(k)x_2((n-k) \bmod 4) = \sum_{k=0}^{3} x_1(k)x_2((n-k) \blacksquare 4)$, ■=mod addition

n=0 , $y(0) = \sum_{k=0}^{3} x_1(k)x_2((-k) \bmod 4)$

$y(0) = x_1(0)x_2(-0 \blacksquare 4) + x_1(1)x_2(-1 \blacksquare 4) + x_1(2)x_2(-2 \blacksquare 4) + x_1(3)x_2(-3 \blacksquare 4)$

$y(0) = x_1(0)x_2(0) + x_1(1)x_2(3) + x_1(2)x_2(2) + x_1(3)x_2(1)$

$\mathbf{y(0)} = 1 \times 0 + 2 \times 3 + 2 \times 2 + 0 \times 1 = \mathbf{10}$

n=1 , $y(1) = \sum_{k=0}^{3} x_1(k)x_2((1-k) \bmod 4)$

$y(1) = x_1(0)x_2(1 \blacksquare 4) + x_1(1)x_2(0 \blacksquare 4) + x_1(2)x_2(-1 \blacksquare 4) + x_1(3)x_2(-2 \blacksquare 4)$

$y(1) = x_1(0)x_2(1) + x_1(1)x_2(0) + x_1(2)x_2(3) + x_1(3)x_2(2)$

$\mathbf{y(1)} = 1 \times 1 + 2 \times 0 + 2 \times 3 + 0 \times 2 = \mathbf{7}$

n=2 , $y(2) = \sum_{k=0}^{3} x_1(k)x_2((2-k) \bmod 4)$

$y(2) = x_1(0)x_2(2 \blacksquare 4) + x_1(1)x_2(1 \blacksquare 4) + x_1(2)x_2(0 \blacksquare 4) + x_1(3)x_2(-1 \blacksquare 4)$

$y(2) = x_1(0)x_2(2) + x_1(1)x_2(1) + x_1(2)x_2(0) + x_1(3)x_2(3)$

$\mathbf{y(2)} = 1 \times 2 + 2 \times 1 + 2 \times 0 + 0 \times 3 = \mathbf{4}$

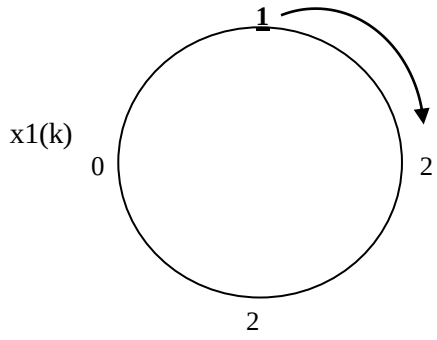
n=3 , $y(3) = \sum_{k=0}^{3} x_1(k)x_2((3-k) \bmod 4)$

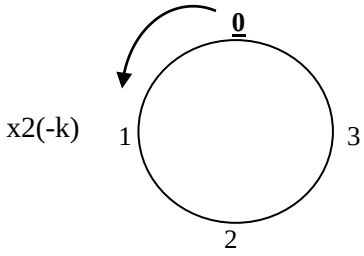
$y(3) = x_1(0)x_2(3 \blacksquare 4) + x_1(1)x_2(2 \blacksquare 4) + x_1(2)x_2(1 \blacksquare 4) + x_1(3)x_2(0 \blacksquare 4)$

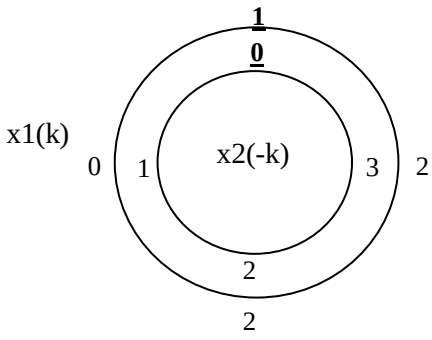
$y(3) = x_1(0)x_2(3) + x_1(1)x_2(2) + x_1(2)x_2(1) + x_1(3)x_2(0)$

$\mathbf{y(3)} = 1 \times 3 + 2 \times 2 + 2 \times 1 + 0 \times 0 = \mathbf{9}$

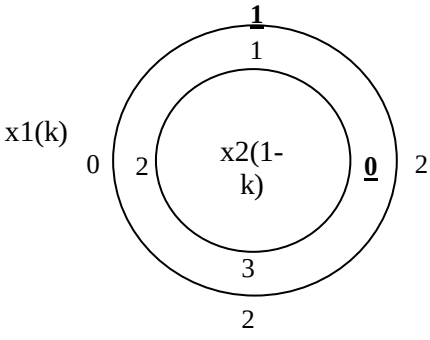
2- Concentric Circle



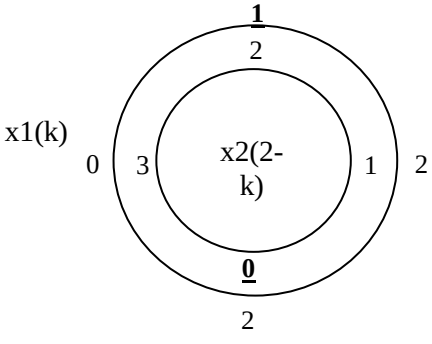




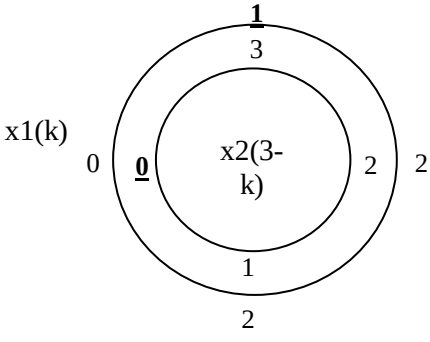
y(0)=1x0+2x3+2x2+0x1=10



y(1)=1x1+2x0+2x3+0x2=7

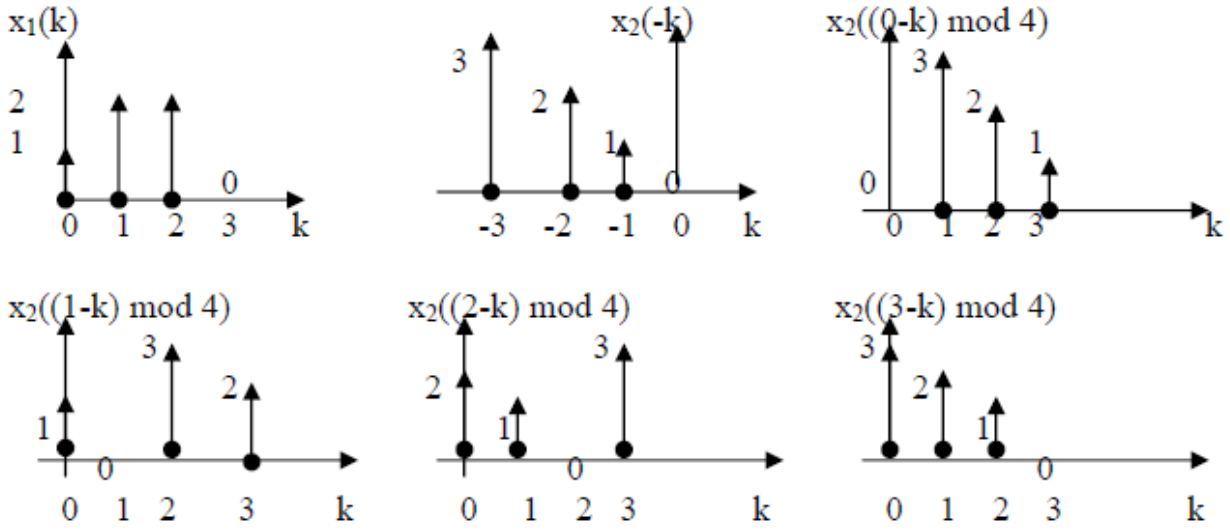


y(2)=1x2+2x1+2x0+0x3=7



y(3)=1x3+2x2+2x1+0x0=9

3- ***graphical method***



y=[10 7 7 9]

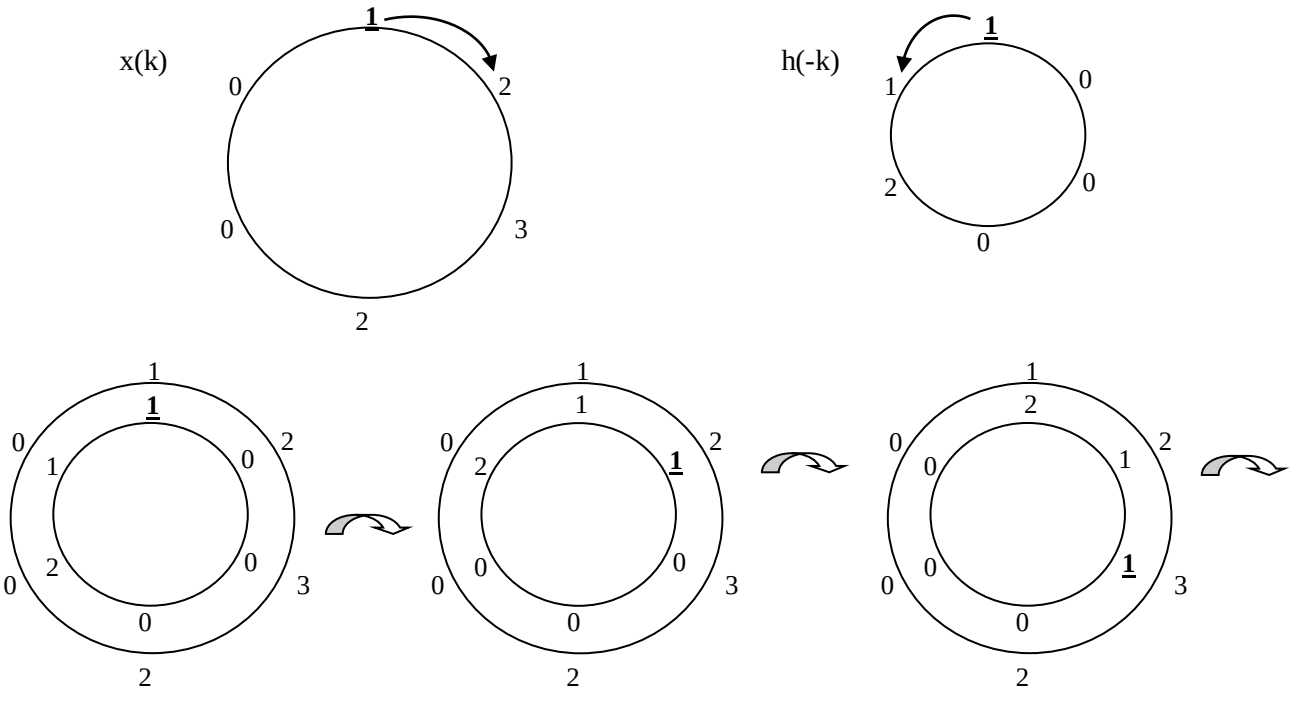
Example: If $x(n)$ = [1 2 3 2], and h(n) = [1 1 2]. Find $y(n)$ such that linear and circular convolution are the same.

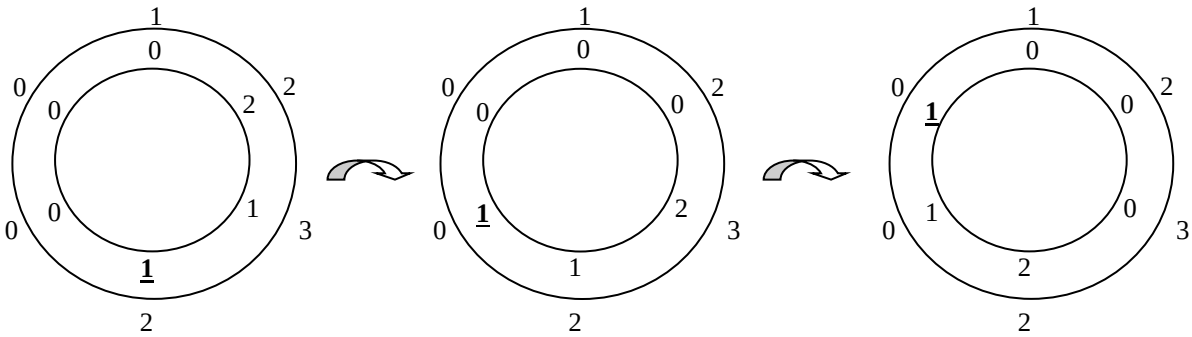
Sol: The circular and linear convolution are the same by zero padding

N = 4 + 3 – 1 = 6

Then $x(n)$ = [1 2 3 2 0 0] and h(n) = [1 1 2 0 0 0]

$x(n)$ is arranged in clockwise direction ,while h(n) is arranged in the opposite clockwise direction (bold numbers). Each time, only h(n) will be shifted with the ***clockwise direction*** to find $y(n)$.





Deconvolution:

The digital Deconvolution can be performed by ***Iterative Approach***, ***Polynomial Approach***, and ***Graphical Method***. In the following subsection, the polynomial approach will be explained.

Polynomial Approach:

A long division process is applied between two polynomials. For causal system, the remainder is always *zero*.

Example: If $y(n)$ = [15 -8 -5 2] and h(n) = [-3 1] find x(n) .

Solution: $y = 15 - 8x - 5x^2 + 2x^3$, and h = $-3 + x$. Applying long division, we obtain

$$
\begin{array}{r}
2x^2 + x - 5 \\
x-3 \;\overline{)\; 2x^3 - 5x^2 - 8x + 15} \\
\underline{2x^3 - 6x^2} \\
x^2 - 8x \\
\underline{x^2 - 3x} \\
-5x + 15 \\
\underline{-5x + 15} \\
\text{Remainder } 0
\end{array}
$$

result = $-5 + x + 2x^2$. Then $x(n)$ = [-5 1 2]

Example: If $y(n)$ = [12 10 14 6] and h(n) = [4 2] find x(n) .

Solution: $y = 12 + 10x + 14x^2 + 6x^3$, and h = $4 + 2x$. Applying long division, we obtain

result = $3 + x + 3x^2$. Then $x(n)$ = [3 1 3]

Frequency response and Sinusoidal Steady State Response

The Frequency domain representation:

In continuous linear time invariant (CLTI) system, it was important to know the frequency response of a system (H(jΩ)) , which could be used to find the steady-state response of the system. For discrete linear time invariant (DLTI) system, $H(e^{jW})$ will be used to find the frequency response of the system.

$W = \Omega T_s$	rad/sample	digital frequency.
$\Omega = 2 \pi f$	rad/sec.	analog frequency.
$T_s = 1 / f_s$	sec.	where, sampling rate = $1 / T_s$.

Response to a complex exponential sequence:

If $x(n) = e^{j\,nW}$ (4.1)

And $$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{\infty} h(k)\, e^{jW(n-k)} = e^{jnW} \sum_{k=-\infty}^{\infty} h(k)\, e^{-jWk} \quad (4.2)$$

Let $$H(e^{jW}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-jWk} \quad (4.3)$$

$$\therefore y(n) = e^{jnW}\, H(e^{jW}) \quad (4.4)$$

$$H(e^{jW}) = H_R(e^{jW}) + j\, H_I(e^{jW}) = |H(e^{jW})|\, \Phi(e^{jW}) \quad (4.5)$$

$$|H(e^{jW})| = [\, \{ H_R(e^{jW}) \}^2 + \{ H_I(e^{jW}) \}^2 \,]^{1/2} \quad (4.6.a)$$

$$\Phi(e^{jW}) = \text{argument}(H(e^{jW})) = \tan^{-1} [H_I(e^{jW}) / H_R(e^{jW})] \quad (4.6.b)$$

Response to a sinusoidal sequence:

If $x(n) = A \cos(W_o n + \theta) = \frac{A}{2}(e^{j\theta}\, e^{jW_o n} + e^{-j\theta}\, e^{-jW_o n})$ (4.7)

Substituting equation (4.7) into equation (4.4), and rearrange the terms, then:

$y(n) = 2 \text{ Re} [0.5 A H (e^{jW_o}) \; e^{jn W_o} \; e^{j\theta}]$

$y(n) = A\, |H(e^{jW_o n})|\, \cos [n W_o + \theta + \Phi(e^{jW_o})]$ (4.8)

Note: the output to a sinusoid is another sinusoid of the same frequency but with different phase and magnitude.

Example (1): A discrete time system has a unit sample response h(n)

$h(n) = 0.5\ \delta(n) + \delta(n-1) + 0.5\ \delta(n-2)$

a) Find the system frequency response. Plot magnitude and phase.

b) Find the steady-state response of the system to $x(n) = 5 \cos(\pi n /4)$.

c) Find the steady-state response of the system to $x(n) = 5 \cos(3 \pi n /4)$.

d) Find the total response to x(n) = u(n) assuming the system is initially at rest.

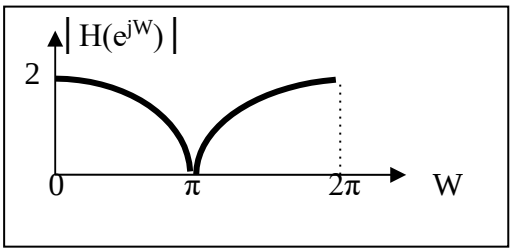
Solution:

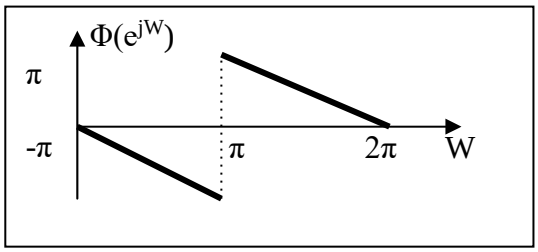
a) $H(e^{jW}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-jWn} = 0.5\, e^{-0} + e^{-jW} + 0.5\, e^{-j2W}$

$= e^{-jW} [\, 0.5\, e^{jW} + 1 + 0.5\, e^{-jW} \,] = e^{-jW} (\, 1 + \cos W \,)$

$|H(e^{jW})| = |\, e^{-jW} \,| \cdot |(\, 1 + \cos W \,)| = 1 + \cos W$

$\Phi(e^{jW}) = \tan^{-1}(e^{-jW}) + \arg(1 + \cos W) = -W$





b) Applying equation (4.8), where $W_0 = \pi / 4$

$|H(e^{jW_o})| = |H(e^{j\pi/4})| = 1 + \cos(\pi / 4) = 1.707$

$\Phi(e^{jW_o}) = -\pi / 4$

Then $y(n) = 5\,(1.707) \cos[(n\pi / 4) - (\pi / 4)] = 8.535 \cos[\pi (n - 1) / 4]$

C) $|H(e^{j 3\pi / 4})| = 1 + \cos(3\pi / 4) = 0.2928$

$\Phi(e^{jW_o}) = -3\pi / 4$

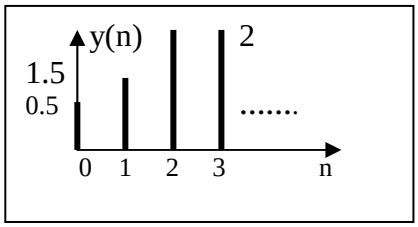
$y(n) = 5\,(0.2928) \cos[(n\pi/4) - (3\pi / 4)] = 1.4644 \cos[3\pi (n - 1) / 4]$

In part (b) the input signal is amplified, while in part (c) the input signal is attenuated.

d) $y(n) = x(n) \otimes h(n)$

$= 0.5\, x(n) + x(n-1) + 0.5\, x(n-2)$

$= 0.5\, u(n) + u(n-1) + 0.5\, u(n-2)$



Note: $\delta(t - t_o) \otimes f(t) = f(t - t_o)$

Properties of frequency response:

1- $H(e^{jW})$ is a continuous function in W.

2- $H(e^{jW})$ is periodic in W with period 2π.

3- $|H(e^{jW})|$ is an even function of W and symmetrical about π.

4- $\Phi(e^{jW})$ is an odd function of W and anti-symmetrical about π.

Example 2 :

Determine the magnitude and phase of $H(\omega)$ for the three-point moving average (MA) system

$$y(n) = \tfrac{1}{3}[x(n+1) + x(n) + x(n-1)]$$

and plot these two functions for $0 \le \omega \le \pi$.

Solution Since

$$h(n) = \{\tfrac{1}{3}, \underset{\uparrow}{\tfrac{1}{3}}, \tfrac{1}{3}\}$$

it follows that

$$H(\omega) = \tfrac{1}{3}(e^{j\omega} + 1 + e^{-j\omega}) = \tfrac{1}{3}(1 + 2\cos\omega)$$

Hence

$$|H(\omega)| = \tfrac{1}{3}|1 + 2\cos\omega| \qquad (4.4.16)$$

$$\Theta(\omega) = \begin{cases} 0, & 0 \le \omega \le 2\pi/3 \\ \pi, & 2\pi/3 \le \omega < \pi \end{cases}$$

Figure 4.37 illustrates the graphs of the magnitude and phase of $H(\omega)$. As indicated previously, $|H(\omega)|$ is an even function of frequency and $\Theta(\omega)$ is an odd function of

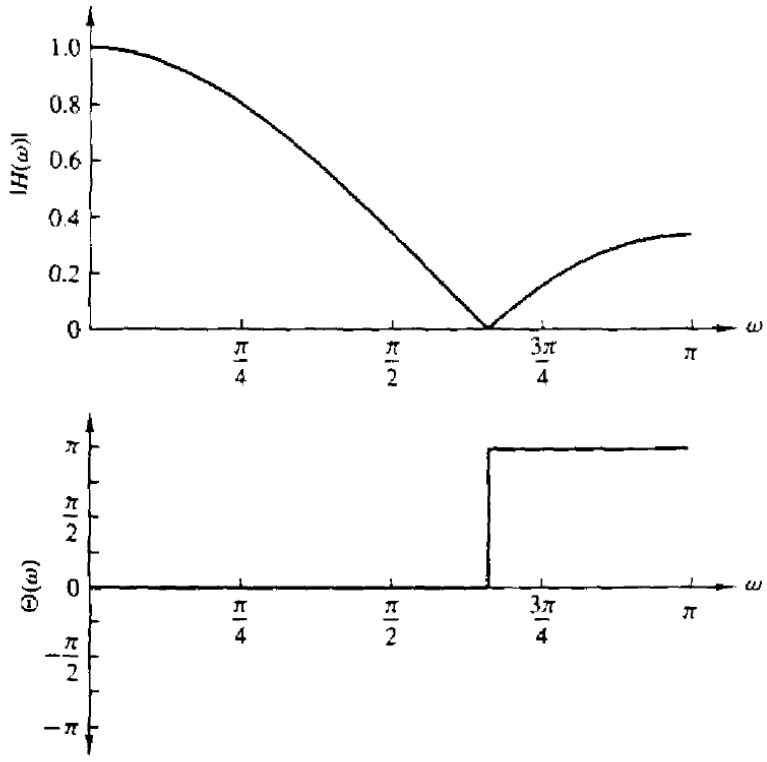


Figure 4.37 Magnitude and phase responses for the MA system in Example 4.4.2.

Example 4.4.1

Determine the output sequence of the system with impulse response

$$h(n) = (\tfrac{1}{2})^n u(n) \tag{4.4.6}$$

when the input is the complex exponential sequence

$$x(n) = Ae^{j\pi n/2} \qquad -\infty < n < \infty$$

Solution First we evaluate the Fourier transform of the impulse response $h(n)$, and then we use (4.4.5) to determine $y(n)$. From Example 4.2.3 we recall that

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \tag{4.4.7}$$

At $\omega = \pi/2$, (4.4.7) yields

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1 + j\frac{1}{2}} = \frac{2}{\sqrt{5}}\, e^{-j26.6^\circ}$$

and therefore the output is

$$\begin{aligned} y(n) &= A\left(\frac{2}{\sqrt{5}}e^{-j26.6^\circ}\right)e^{j\pi n/2} \\ y(n) &= \frac{2}{\sqrt{5}}Ae^{j(\pi n/2 - 26.6^\circ)} \qquad -\infty < n < \infty \end{aligned} \tag{4.4.8}$$

Example 3:

Determine the response of the system in Example 4.4.1 to the input signal

$$x(n) = 10 - 5\sin\frac{\pi}{2}n + 20\cos\pi n \qquad -\infty < n < \infty$$

Solution The frequency response of the system is given in (4.4.7) as

$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

The first term in the input signal is a fixed signal component corresponding to $\omega = 0$. Thus

$$H(0) = \frac{1}{1 - \frac{1}{2}} = 2$$

The second term in $x(n)$ has a frequency $\pi/2$. At this frequency the frequency response of the system is

$$H\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{5}}e^{-j26.6^\circ}$$

Finally, the third term in $x(n)$ has a frequency $\omega = \pi$. At this frequency

$$H(\pi) = \tfrac{2}{3}$$

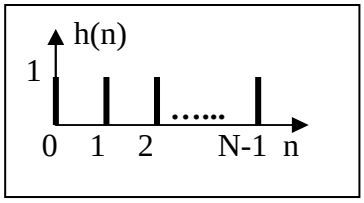
Hence the response of the system to $x(n)$ is

$$y(n) = 20 - \frac{10}{\sqrt{5}}\sin\left(\frac{\pi}{2}n - 26.6^\circ\right) + \frac{40}{3}\cos\pi n \qquad -\infty < n < \infty$$

Example 4: Find and plot the frequency response of a rectangular window filter if :

$h(n) = 1 \quad 0 \le n \le N-1$

$0 \quad$ elsewhere



Solution:

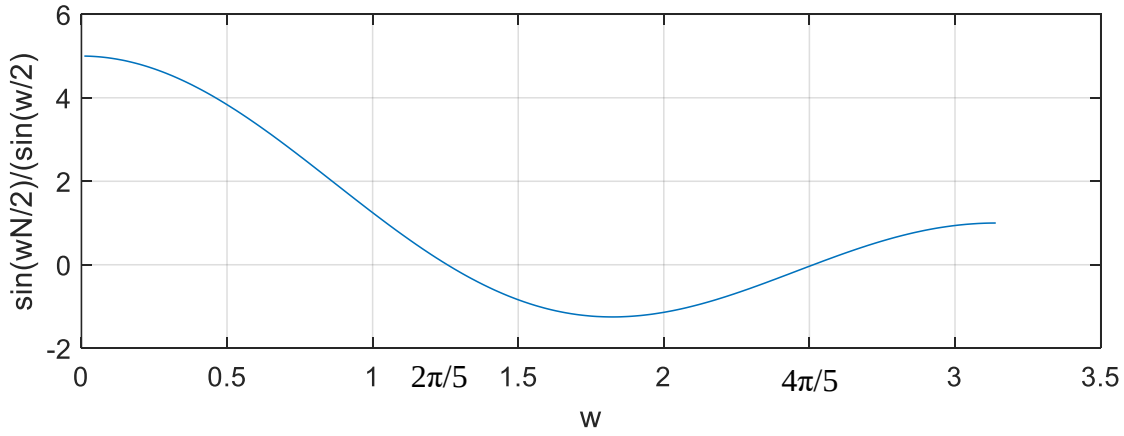
$$H(e^{jW}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-jWk} = \sum_{k=0}^{N-1} e^{-jWk} = \frac{1 - e^{-jWN}}{1 - e^{-jW}}$$

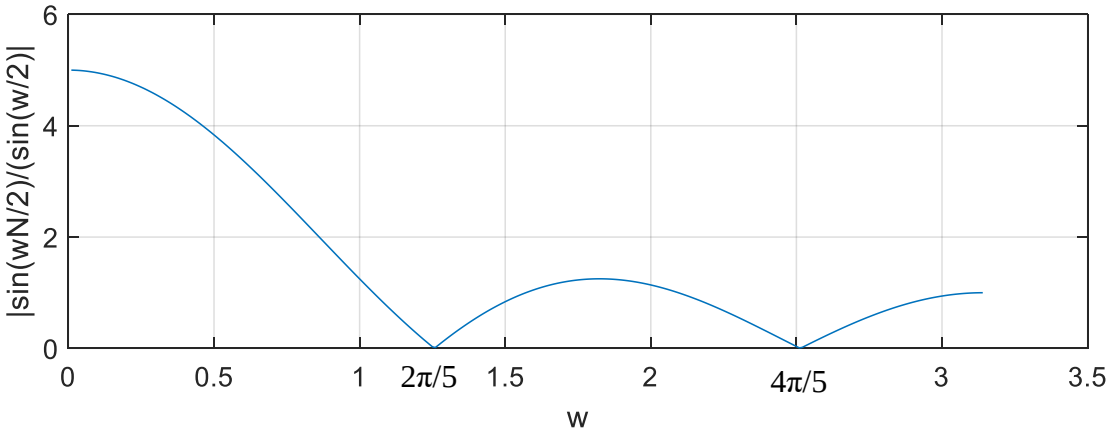
By using $\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}, \quad a \neq 1$

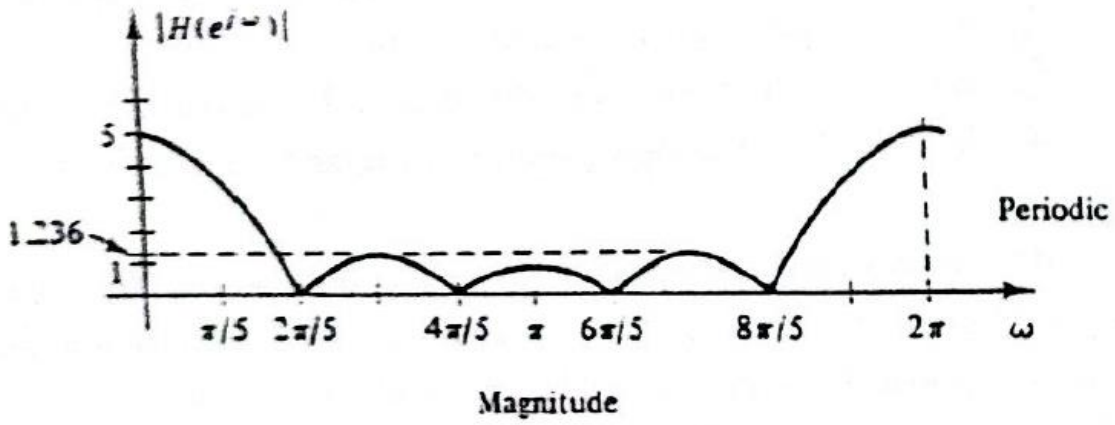
$$H(e^{jW}) \frac{e^{-jWN/2}(e^{jWN/2} - e^{-jWN/2})}{e^{-jW/2}(e^{jW/2} - e^{-jW/2})} = e^{-jW(N-1)/2} \frac{\sin(WN/2)}{\sin(W/2)}$$

$$\left| H(e^{jW}) \right| = \frac{\sin(WN/2)}{\sin(W/2)}$$

$$\Phi(e^{jW}) = -W(N-1)/2 + \arg\{ \frac{\sin(WN/2)}{\sin(W/2)} \}$$







Magnitude

$$\phi(w) = -w(N-1)/2 + Arg\left(\frac{\sin(wN/2)}{\sin(w/2)}\right)$$

$$Arg\left(\frac{\sin(wN/2)}{\sin(w/2)}\right) = \begin{cases} 0, 2\pi, \ldots & if \ \frac{\sin(wN/2)}{\sin(w/2)} \geq 0 \\ \pi, 3\pi, \ldots & if \ \frac{\sin(wN/2)}{\sin(w/2)} < 0 \end{cases}$$

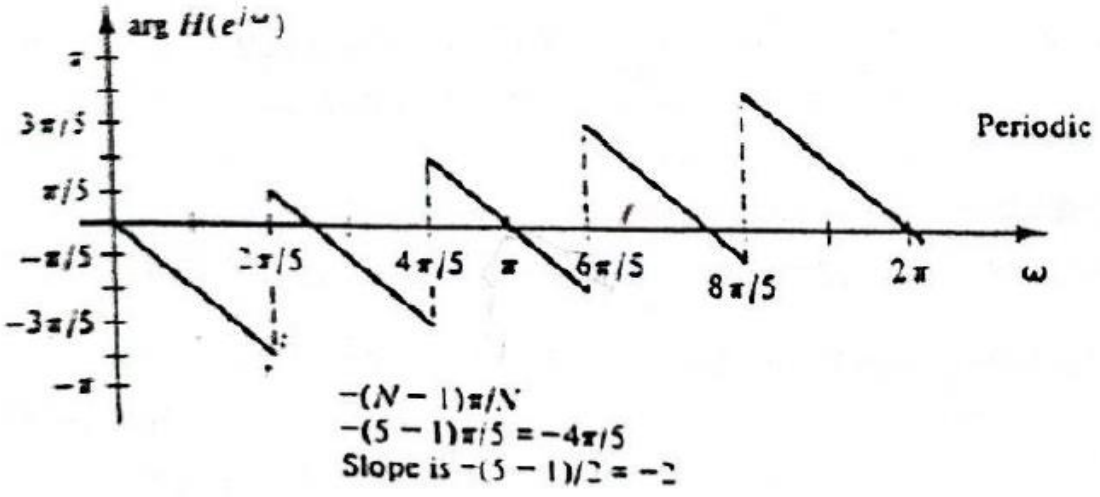
$$\frac{\sin(wN/2)}{\sin(w/2)} \geq 0 \Rightarrow \sin(wN/2) \geq 0 \Rightarrow 0 \leq wN/2 \leq \pi, \ 2\pi \leq wN/2 \leq 3\pi \ , \ldots$$
$$0 \leq w \leq 2\pi/N, \ 4\pi/N \leq w \leq 6\pi/N \ , \ldots$$

$$\frac{\sin(wN/2)}{\sin(w/2)} < 0 \Rightarrow \sin(wN/2) < 0 \Rightarrow \pi \leq wN/2 \leq 2\pi, \ 3\pi \leq wN/2 \leq 4\pi \ , \ldots$$
$$2\pi/N \leq w \leq 4\pi/N, \ 6\pi/N \leq w \leq 8\pi/N \ , \ldots$$

$$\therefore \phi(w) = -w(N-1)/2 + 0 \quad at \quad 0 \leq w \leq 2\pi/N$$
$$= -w(N-1)/2 + \pi \quad at \quad 2\pi/N \leq w \leq 4\pi/N$$
$$= -w(N-1)/2 + 2\pi \quad at \quad 4\pi/N \leq w \leq 6\pi/N$$
$$= -w(N-1)/2 + 3\pi \quad at \quad 6\pi/N \leq w \leq 8\pi/N$$

For $w \leq \pi$ rad, let N=5 samples

$$\therefore \phi(w) = -2w \quad at \quad 0 \leq w \leq 2\pi/5$$
$$= -2w + \pi \quad at \quad 2\pi/5 \leq w \leq 4\pi/5$$
$$= -2w + 2\pi \quad at \quad 4\pi/5 \leq w \leq 6\pi/5$$
$$= -2w + 3\pi \quad at \quad 6\pi/5 \leq w \leq 8\pi/5$$


arg $H(e^{j\omega})$
π
$3\pi/5$
$\pi/5$
$-\pi/5$
$-3\pi/5$
$-\pi$
$2\pi/5$
$4\pi/5$
π
$6\pi/5$
$8\pi/5$
2π
ω
Periodic
$-(N-1)\pi/N$
$-(5-1)\pi/5 = -4\pi/5$
Slope is $-(5-1)/2 = -2$

Phase

Z-Transform

5.1 Definition of Z.T

The z-transform is a very important tool in describing and analyzing digital systems. It also offers the techniques for digital filter design and frequency analysis of digital signals. The z-transform of a ***causal*** sequence $x(n)$, designated by $X(z)$ or $Z(x(n))$, is defined as:

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n)z^{-n}$$
$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \dots \qquad (5.1)$$

Where, z is the complex variable. Here, the summation taken from $n = 0$ to $n = \infty$ is according to the fact that for most situations, the digital signal $x(n)$ is the ***causal*** sequence, that is, $x(n) = 0$ for $n \leq 0$. For non-causal system, the summation starts at $n = -\infty$. Thus, the definition in Equation (5.1) is referred to as a **one-sided z-transform** or a **unilateral transform**. The region of convergence is defined based on the particular sequence $x(n)$ being applied. The z-transforms for common sequences are summarized below:

Line No.	$x(n)$, $n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\lvert z\rvert > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\lvert z\rvert > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
6	$a^nu(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\lvert z\rvert > e^{-a}$
8	$na^nu(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\lvert z\rvert > 1$
10	$\cos(an)u(n)$	$\frac{z[z - \cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\lvert z\rvert > 1$
11	$a^n\sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\lvert z\rvert > \lvert a\rvert$
12	$a^n\cos(bn)u(n)$	$\frac{z[z - a\cos(b)]}{z^2 - [2a\cos(b)]z + a^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z - e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$
15	$2\lvert A\rvert\lvert P\rvert^n\cos(n\theta + \phi)u(n)$ where P and A are complex constants defined by $P = \lvert P\rvert\angle\theta, A = \lvert A\rvert\angle\phi$	$\frac{Az}{z-P} + \frac{A^*z}{z-P^*}$	

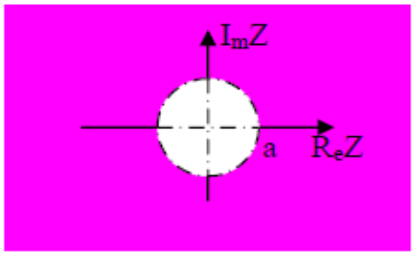
Example (1): Find Z.T including region of convergence of $x(n) = a^n u(n)$

Solution:

$$X(Z)=\sum_{n=0}^{\infty} a^n Z^{-n}=\sum_{n=0}^{\infty}(a Z^{-1})^n=\frac{1}{1-aZ^{-1}}=\frac{Z}{Z-a} \quad , \quad |aZ^{-1}|<1$$

Or $|Z| > |a|$

The region of convergence (ROC) is outside the unit circle only.



5.2 Properties of Z.T:

5.2.1 Linearity: The z-transform is a linear transformation, which implies

$$Z\,(a\,x_1(n) \pm b\,x_2(n)) = a\,X_1(Z) \pm b\,X_2(Z) \qquad (5.2)$$

Where; *a* and *b* are constants. **ROC = ROC1∩ ROC2**

5.2.2 Shift theorem (Delay) (without initial conditions): Given $X(z)$, the z-transform of a sequence $x(n)$, the z-transform of $x(n - m)$, the time-shifted sequence, is given by;

$$Z\{x(n-m)\}=Z^{-m}\,X(Z) \qquad (5.3)$$

5.2.3 Convolution: Given two sequences $x_1(n)$ and $x_2(n)$, their convolution can be determined as follows:

$$x(n) = x_1(n) \otimes x_2 = \sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k) = \sum_{k=-\infty}^{\infty} x_1(n-k)\,x_2(k) \qquad (5.4)$$

Where $\otimes$ designates the linear convolution. In z-transform domain, we have

$$X(Z) = X_1(Z)\,.\,X_2(Z) \qquad (5.5)$$

Basic Steps:

1. Compute z-Transform of each of the signals to convolve (time domain → z-domain):

$$X_1(z) = Z[x_1(n)] \quad , X_2(z) = Z[x_2(n)]$$

2. Multiply the two z-Transforms (in z-domain): $X(z) = X_1(z)\,X_2(z)$

3. Find the inverse z-Transform of the product (z-domain → time domain):

$$x(n) = z^{-1}[X(z)]$$

5.2.4 Multiplication by exponential:

$$Z\{a^n\ x(n)\} = X(Z)\ \Big|_{Z\to\frac{Z}{a}} \tag{5.6.a}$$

$$Z\{e^{\pm an}\ x(n)\} = X(Z)\ \Big|_{Z\to e^{\mp a}Z} \tag{5.6.b}$$

5.2.5 Initial and final value theorems:

$$\lim_{n\to 0} x(n) = \lim_{Z\to\infty} X(Z) = x(0) \quad \textit{initial value theorem} \tag{5.7.a}$$

$$\lim_{n\to\infty} x(n) = \lim_{Z\to 1} Z^{-1}\ (Z-1)\ \ X(Z) \quad \textit{final value theorem} \tag{5.7.b}$$

5.2.6 Multiplication by *n* (Differentiation of *X*(z)):

$$Z\{n\ x(n)\} = -\,Z\,\frac{d}{dZ}\,X(Z) \tag{5.8}$$

5.3 Inverse of Z.T

$$x(n) = Z^{-1}\ \{X(Z)\} \tag{5.9}$$

The inverse z-transform may be obtained by the following methods:

1. Using properties.
2. Partial fraction (P.F) expansion method.
3. Power series expansion (the solution is obtained by applying long division because the denominator can't be analyzed. It is not accurate method compared with the above three methods).

Example (2): Find *x*(*n*) using **partial fraction method**, if:

$$X(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}.$$

Solution:

Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1-z^{-1})(1-0.5z^{-1})}.$$

$$= \frac{z^2}{(z-1)(z-0.5)}$$

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}.$$

$$A = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \frac{z}{(z-0.5)}\Big|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\Big|_{z=0.5} = \frac{z}{(z-1)}\Big|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

Multiplying z on both sides gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

$$x(n) = 2u(n) - (0.5)^n u(n).$$

Example (3): Find the inverse transform of $X(z)$ using **partial fraction method**.

$$X(z) = \frac{z}{3z^2-4z+1}$$

Solution:

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{1}{3z^2-4z+1} = \frac{1}{3(z^2-\frac{4}{3}z+\frac{1}{3})} = \frac{1}{3(z-1)(z-\frac{1}{3})} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{3}}$$

$$A = \frac{X(z)}{z}(z-1)\,|_{\rightarrow}\quad z=1 \quad \rightarrow \quad \frac{1}{3(z-\frac{1}{3})}; \qquad A=1/2$$

$$B = \frac{X(z)}{z}(z-\frac{1}{3})\,|_{\rightarrow}\quad z=\frac{1}{3} \quad \rightarrow \frac{1}{3(z-1)}, \qquad B= -1/2$$

Therefore,

$$\frac{X(z)}{z} = \frac{1/2}{z-1} + \frac{-1/2}{z-\frac{1}{3}}$$

$$X(z) = \frac{\frac{1}{2}z}{z-1} + \frac{(-\frac{1}{2})z}{z-\frac{1}{3}}$$

$$\therefore \quad x(n) = \tfrac{1}{2}\,u(n) - \tfrac{1}{2}\,(\tfrac{1}{3})^n u(n)$$

Example (4): Find the inverse z-transform sequence of the following signal using **power series expansion (Long Division) method.**

$$X(z) = \frac{z^2 + z}{z^2 - 3z + 4}$$

Solution:

Represent the z-transform function $X(z)$ in terms of z^{-1} by dividing z^2 for both numerator and denominator.

$$X(z) = \frac{z^2+z}{z^2-3z+4} = \frac{1+z^{-1}}{1-3z^{-1}+4z^{-2}}$$

$$\begin{array}{l|l} & 1+4z^{-1}+8z^{-2}+8z^{-3} \\ 1-3z^{-1}+4z^{-2} & 1+z^{-1} \\ & \underline{1-3z^{-1}+4z^{-2}} \\ & \quad 4z^{-1}-4z^{-2} \\ & \quad \underline{4z^{-1}-12z^{-2}+16z^{-3}} \\ & \qquad 8z^{-2}-16z^{-3} \\ & \qquad \underline{8z^{-2}-24z^{-3}+32z^{-4}} \\ & \qquad\quad 8z^{-3}-32z^{-4} \end{array}$$

By examination, the sequence $x(n)$ is

$$x(n) = \delta(n) + 4\,\delta(n-1) + 8\,\delta(n-2) + 8\,\delta(n-3) + \ldots$$

$$x(0)=1,\ x(1)=4,\ x(2)=8,\ x(3)=8,\ \ldots$$

The long division procedure used in the example above can be carried out to any desired number of steps.

The disadvantage of this technique is that it does not give a closed form representation of the resulting sequence. In many applications, we need to obtain a closed-form result to infer general qualitative insights into the sequence $x(n)$. For most engineering investigation, the method of partial fraction expansion and a good z-transform table is often sufficient to generate the desired closed form solution.

5.4 Solution of linear constant coefficient difference equation using Z.T

$$Z\{x(n-m)\} = Z^{-m}\{X(Z) + \sum_{k=-m}^{-1} x(k)\,Z^{-k}\} \qquad (5.12)$$

Example (5): Solve $y(n) - (3/2)\,y(n-1) + (1/2)\,y(n-2) = (1/4)^n$, $y(-1) = 4$, $y(-2) = 10$ for $n \geq 0$

Solution:

$$Y(Z) - \frac{3}{2}\{Y(Z).Z^{-1} + y(-1)\} + \frac{1}{2}\{Z^{-2}\,Y(Z) + Z^{-1}\,y(-1) + y(-2)\} = \frac{Z}{Z-\frac{1}{4}}$$

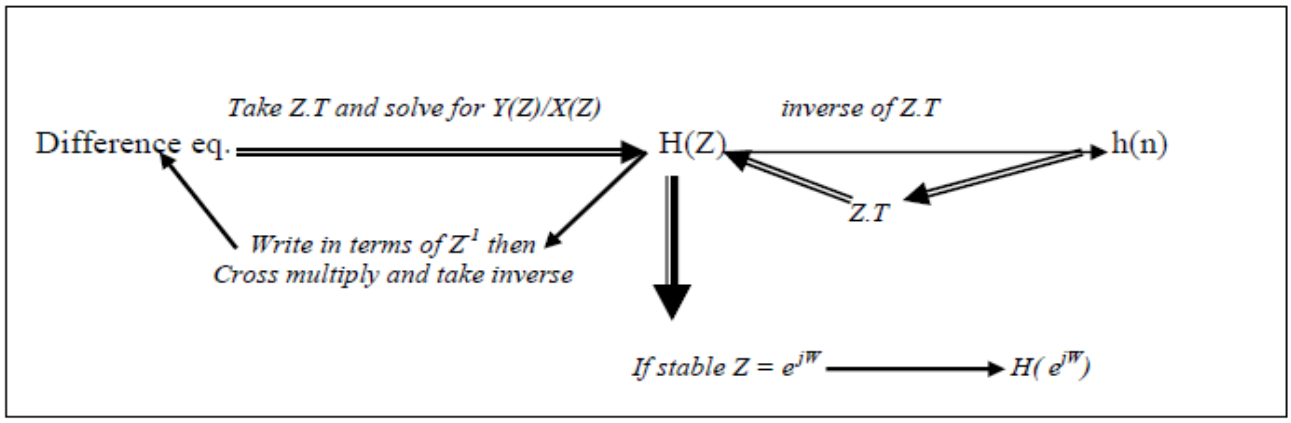
$$Y(Z)\{1 - \frac{3}{2}Z^{-1} + \frac{1}{2}Z^{-2}\} = \frac{Z}{Z-\frac{1}{4}} + 1 - 2Z^{-1}$$

$$Y(Z) = \frac{Z\,(2Z^2 - \frac{9}{4}Z + \frac{1}{2})}{(Z-\frac{1}{4})\,(Z-\frac{1}{2})\,(Z-1)}$$

$$Y(Z) = \frac{(1/3)\,Z}{(Z-\frac{1}{4})} + \frac{Z}{Z-\frac{1}{2}} + \frac{(2/3)\,Z}{Z-1}$$

$$y(n) = \{\frac{1}{3}(\frac{1}{4})^n + (\frac{1}{2})^n + \frac{2}{3}\}\,u(n)$$

5.5 Relations between system representations:



Continuous time system	Discrete time system
Differential equation	Difference equation
$H_a(S)$	$H(Z)$
$H(j\Omega)$	$H(e^{jW})$
$H_a(t) = L^{-1}\{H_a(S)\}$	$h(n) = Z^{-1}\{H(Z)\}$

Example (6): DSP system is described by the difference equation $y(n)=0.2y(n-1)+x(n)$, find the impulse response $h(n)$.

Solution:

Take the Z transform of the both sides

$$Z\{y(n) = 0.2y(n-1) + x(n)\}$$

$$Y(z) = 0.2z^{-1}Y(z) + X(z)$$

$$Y(z) - 0.2z^{-1}Y(z) = X(z)$$

$$Y(z)(1 - 0.2z^{-1}) = X(z)$$

Divide Y(z) by X(z) to get H(z)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.2z^{-1}} = \frac{z}{z-0.2}$$

Take inverse Z transform

$$h(n) = Z^{-1}\left\{\frac{z}{z-0.2}\right\} = (0.2)^n u(n)$$

Example (7): A relaxed (zero initial conditions) DSP system is described by the difference equation

$$y(n) + 0.1y(n-1) - 0.2y(n-2) = x(n) + x(n-1)$$

a. Determine system response due to the impulse input sequence (i.e. determine the impulse response $h(n)$).

b. Determine the system response due to the unit step input sequence (i.e. determine the step response $S(n)$).

Solution:

a. Applying the z-transform on both sides of the difference equation, we yield

$$Y(z) + 0.1\,Y(z)z^{-1} - 0.2\,Y(z)z^{-2} = X(z) + X(z)z^{-1}.$$

Factoring out $Y(z)$ on the left side and substituting $X(z) = Z(\delta(n))$ the right side in Equation (5.15) achieves

$$Y(z)(1 + 0.1z^{-1} - 0.2z^{-2}) = 1(1 + z^{-1}).$$

Then $Y(z)$ can be expressed as

$$Y(z) = \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}}.$$

To obtain the impulse response, which is the inverse z-transform of the transfer function, we multiply the numerator and denominator by z^2.

Thus

$$Y(z) = \frac{z^2 + z}{z^2 + 0.1z - 0.2} = \frac{z(z+1)}{(z-0.4)(z+0.5)}.$$

Using the partial fraction expansion method leads to

$$\frac{Y(z)}{z} = \frac{z+1}{(z-0.4)(z+0.5)} = \frac{A}{z-0.4} + \frac{B}{z+0.5},$$

$$\text{where } A = (z-0.4)\frac{Y(z)}{z}\bigg|_{z=0.4} = \frac{z+1}{z+0.5}\bigg|_{z=0.4} = \frac{0.4+1}{0.4+0.5} = 1.5556$$

$$B = (z+0.5)\frac{Y(z)}{z}\bigg|_{z=-0.5} = \frac{z+1}{z-0.4}\bigg|_{z=-0.5} = \frac{-0.5+1}{-0.5-0.4} = -0.5556.$$

Thus

$$Y(z) = \frac{1.5556z}{(z-0.4)} + \frac{-0.5556z}{(z+0.5)},$$

which gives the impulse response:

$$y(n) = 1.5556(0.4)^n u(n) - 0.5556(-0.5)^n u(n).$$

b. To obtain the response due to a unit step function, the input sequence is set to be

$$x(n) = u(n)$$

and the corresponding z-transform is given by

$$X(z) = \frac{z}{z-1},$$

Then the z-transform of the output sequence $y(n)$ can be yielded as

$$Y(z) = \left(\frac{z}{z-1}\right)\left(\frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}\right) = \frac{z^2(z+1)}{(z-1)(z-0.4)(z+0.5)}.$$

Using the partial fraction expansion method as before gives

$$Y(z) = \frac{2.2222z}{z-1} + \frac{-1.0370z}{z-0.4} + \frac{-0.1852z}{z+0.5},$$

and the system response is found by using Table 5.1:

$$y(n) = 2.2222u(n) - 1.0370(0.4)^n u(n) - 0.1852(-0.5)^n u(n).$$

Discrete Fourier Transform (DFT)

6.1 Definition of DFT

In time domain, representation of digital signals describes the signal amplitude versus the sampling time instant or the sample number. However, in some applications, signal frequency content is very useful than as digital signal samples.

The algorithm transforming the time domain signal samples to the frequency domain components is known as the discrete Fourier transform, or DFT. The DFT also establishes a relationship between the time domain representation and the frequency domain representation. Therefore, we can apply the DFT to perform frequency analysis of a time domain sequence. In addition, the DFT is widely used in many other areas, including spectral analysis, acoustics, imaging/ video, audio, instrumentation, and communications systems.

6.2 Fourier Series Coefficients of Periodic Digital Signals

To estimate the spectrum of a periodic digital signal $x(n)$, sampled at a rate of f_s Hz with the fundamental period $T_0=NT$, where there are N samples within the duration of the fundamental period and $T=T_s= 1/f_s$ is the sampling period. Fig. 6.1 shows periodic digital signal.

Fourier series expansion of a periodic signal x(t) in a complex form is:

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt, \quad -\infty \le k \le \infty \qquad (6.1)$$

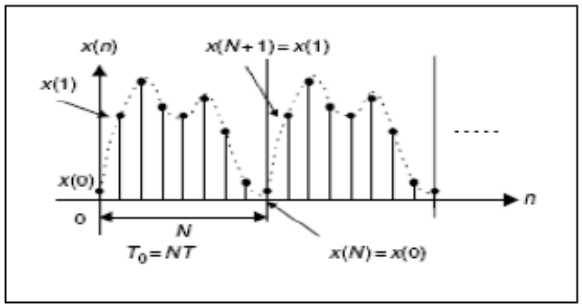


Fig. 6.1 periodic digital signal

Where, k is the number of harmonics corresponding to the harmonic frequency of kf_0 and $W_0=2\pi/T_0$ and $f_0=1/T_0$ are the fundamental frequency in radians per second and the fundamental frequency in Hz, respectively. To apply Equation (6.1), we substitute $T_0=NT$, $W_0=2\pi/T_0$ and approximate the integration over one period using a summation by substituting $dt=T$ and $t=nT$. We obtain:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}, \quad -\infty \le k \le \infty \qquad (6.2)$$

Since the coefficients c_k are obtained from the Fourier series expansion in the complex form, the resultant spectrum c_k will have two sides. Therefore, the two-sided line amplitude spectrum $|c_k|$ is periodic, as shown in Fig. 6.2.

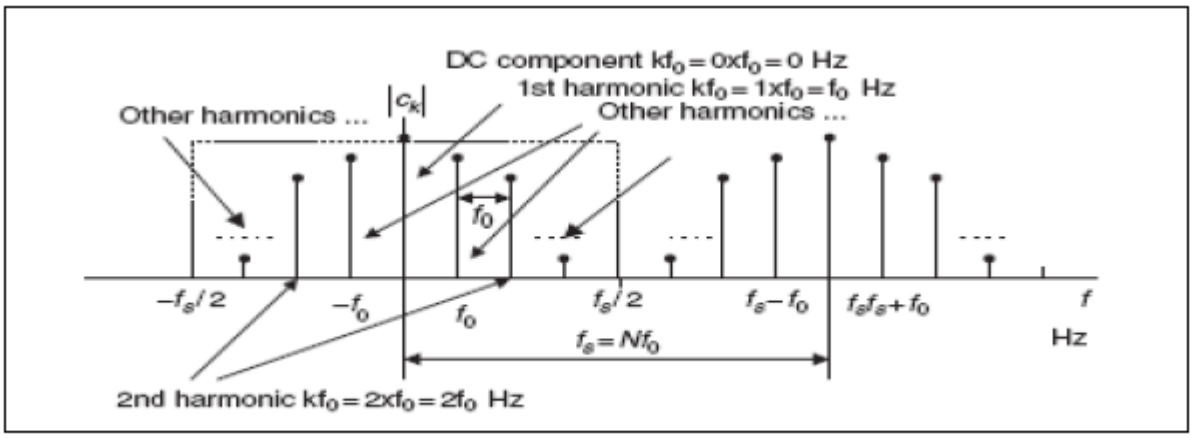


Fig. 6.2 Amplitude Spectrum of periodic Digital signal

As displayed in Figure 6.2 we note the following points:

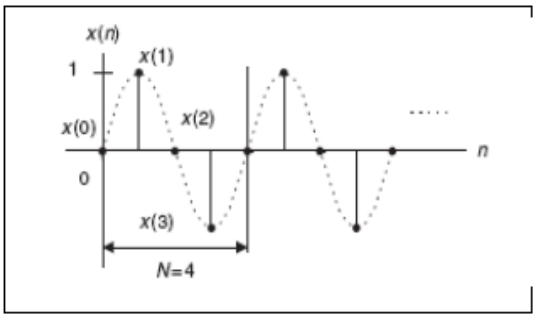
a. Only the line spectral portion between the frequency $-f_s/2$ and frequency $f_s/2$ (folding frequency) represents the frequency information of the periodic signal.
b. The spectrum is periodic for every Nf_0 Hz.
c. For the k_{th} harmonic, the frequency is $f = kf_0$ Hz. f_0 is called the frequency resolution.

Example (1): The periodic signal $x(t) = \sin(2\pi t)$ is sampled using the rate $f_s = 4$ Hz.

a. Compute the spectrum c_k using the samples in one period.
b. Plot the two-sided amplitude spectrum $|c_k|$ over the range from −2 to 2 Hz

Solution:

The fundamental frequency $W_0 = 2\pi$ radians per second and $f_0 = 1$, and the fundamental period $T_0 = 1$ second. Since using the sampling interval $T = 1/f_s = 0.25$ second. The sampled signal is $x(n) = \sin(0.5\,\pi\,n)$, and the first eight samples of it are plotted as shown



a. Choosing one period, $N = 4$, we have $x(0) = 0$; $x(1) = 1$; $x(2) = 0$; and $x(3) = -1$. Using Eq. (6.2),

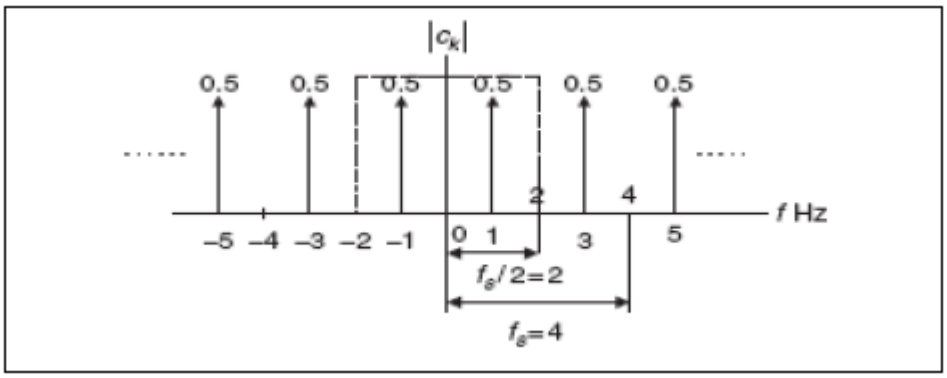
$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x(n) = \frac{1}{4}(x(0) + x(1) + x(2) + x(3)) = \frac{1}{4}(0 + 1 + 0 - 1) = 0$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi \times 1n/4} = \frac{1}{4}\left(x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}\right)$$

$$= \frac{1}{4}(x(0) - jx(1) - x(2) + jx(3)) = \frac{1}{4}(0 - j(1) - 0 + j(-1)) = -j0.5$$

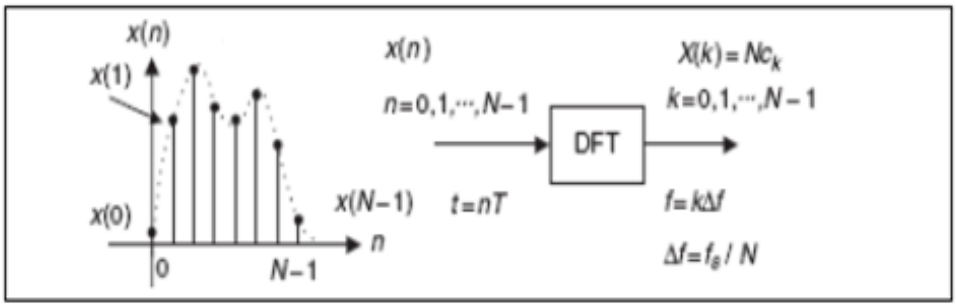
Similarly, $c_2 = 0$ and $c_3 = j0.5$. Using periodicity, it follows that $c_{-1} = c_1 = -j0.5$, and $c_{-2} = c_2 = 0$.

b. The amplitude spectrum for the digital signal is sketched below:



6.3 Discrete Fourier Transform Formulas

Given a sequence $x(n)$, $0 \le n \le N-1$, its DFT is defined as:



$$X(k)=\sum_{n=0}^{N-1}x(n)\,e^{-j\frac{2\pi kn}{N}}=\sum_{n=0}^{N-1}x(n)\,W_N^{kn},\ for\ k=0,1,..N-1 \tag{6.3}$$

Where the factor W_N (called the twiddle factor in some textbooks) is defined as

$$W_N=e^{-j\frac{2\pi}{N}}=\cos\left(\frac{2\pi}{N}\right)-j\sin\left(\frac{2\pi}{N}\right) \tag{6.4}$$

The **inverse DFT** is given by:

$$x(n)=\frac{1}{N}\sum_{n=0}^{N-1}X(k)\,e^{j\frac{2\pi kn}{N}}=\frac{1}{N}\sum_{n=0}^{N-1}X(k)\,W_N^{-kn},\ for\ n=0,1,..N-1 \tag{6.5}$$

Example (2): Given a sequence $x(n)$ for $0 \le n \le 3$, where $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$. Evaluate its DFT $X(k)$.

Solution:

Since $N=4$, $W_4=e^{-j\pi/2}$, then using:

$$X(k)=\sum_{n=0}^{3}x(n)\,W_4^{kn}=\sum_{n=0}^{3}x(n)e^{-j\frac{\pi kn}{2}}.$$

Thus, for $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n)e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0}$$
$$= x(0) + x(1) + x(2) + x(3)$$
$$= 1 + 2 + 3 + 4 = 10$$

for $k = 1$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$$
$$= x(0) - jx(1) - x(2) + jx(3)$$
$$= 1 - j2 - 3 + j4 = -2 + j2$$

for $k = 2$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$
$$= x(0) - x(1) + x(2) - x(3)$$
$$= 1 - 2 + 3 - 4 = -2$$

and for $k = 3$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j\frac{3\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}$$
$$= x(0) + jx(1) - x(2) - jx(3)$$
$$= 1 + j2 - 3 - j4 = -2 - j2$$

Example (3): Find the inverse DFT for $X(k)$ in Example 2 to determine the time domain sequence $x(n)$.

Solution:

a. Since $N = 4$ and $W_4^{-1} = e^{j\frac{\pi}{2}}$, using Equation (4.10) we achieve a simplified formula,

$$x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)W_4^{-nk} = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi kn}{2}}.$$

Then for $n = 0$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j0} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0}\right)$$
$$= \frac{1}{4}(10 + (-2 + j2) - 2 + (-2 - j2)) = 1$$

for $n = 1$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}\right)$$
$$= \frac{1}{4}(X(0) + jX(1) - X(2) - jX(3))$$
$$= \frac{1}{4}(10 + j(-2 + j2) - (-2) - j(-2 - j2)) = 2$$

for $n = 2$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{jk\pi} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right)$$
$$= \frac{1}{4}(X(0) - X(1) + X(2) - X(3))$$
$$= \frac{1}{4}(10 - (-2 + j2) + (-2) - (-2 - j2)) = 3$$

and for $n = 3$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi 3}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}}\right)$$
$$= \frac{1}{4}(X(0) - jX(1) - X(2) + jX(3))$$
$$= \frac{1}{4}(10 - j(-2 + j2) - (-2) + j(-2 - j2)) = 4$$

Mapping the frequency bin k to its corresponding frequency is as follows:

$$\omega = \frac{k\omega_s}{N} \text{ (radians per second)}, \tag{6.6}$$

Since $w_s = 2\,\pi\,f_s$, then:

$$f = \frac{kf_s}{N} \text{ (Hz)}, \tag{6.7}$$

We can define the frequency resolution as the frequency step between two consecutive DFT coefficients to measure how fine the frequency domain presentation is and achieve.

$$\Delta\omega = \frac{\omega_s}{N} \text{ (radians per second)}, \tag{6.8}$$

$$\Delta f = \frac{f_s}{N} \text{ (Hz)}. \tag{6.9}$$

Example (4): In example (2), if the sampling rate is 10 Hz,

a. Determine the sampling period, time index, and sampling time instant for a digital sample $x(3)$ in time domain.
b. Determine the frequency resolution, frequency bin number, and mapped frequency for each of the DFT coefficients $X(1)$ and $X(3)$ in frequency domain.

Solution:

a. In time domain, we have the sampling period calculated as

$$T = 1/f_s = 1/10 = 0.1 \text{ second.}$$

For data $x(3)$, the time index is $n = 3$ and the sampling time instant is determined by

$$t = nT = 3 \cdot 0.1 = 0.3 \text{ second.}$$

b. In frequency domain, since the total number of DFT coefficients is four, the frequency resolution is determined by

$$\Delta f = \frac{f_s}{N} = \frac{10}{4} = 2.5\,\text{Hz.}$$

The frequency bin number for $X(1)$ should be $k = 1$ and its corresponding frequency is determined by

$$f = \frac{kf_s}{N} = \frac{1 \times 10}{4} = 2.5\,\text{Hz.}$$

Similarly, for $X(3)$ and $k = 3$,

$$f = \frac{kf_s}{N} = \frac{3 \times 10}{4} = 7.5\,\text{Hz.}$$

6.4 The DFT as a Linear Transformation

The formulas for the DFT and IDFT may be expressed as:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

$$W_N = e^{-j2\pi/N}$$

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1\cdots & 1 & \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

With these definitions, the N-point DFT may be expressed in matrix form as

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \tag{5.1.24}$$

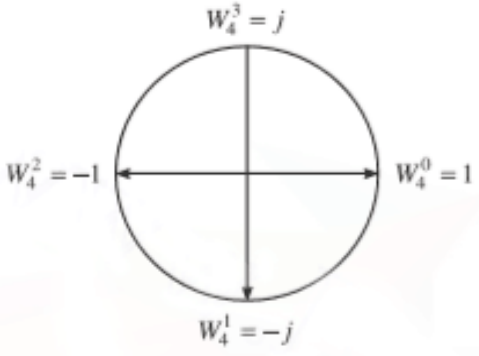
In fact, the IDFT as given by (5.1.21), can be expressed in matrix form as

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N \tag{5.1.26}$$

Cyclic property of twiddle factors

For a 4-point DFT

$W_4^0 = 1$ $W_4^1 = -j$ $W_4^2 = -1$ $W_4^3 = j$

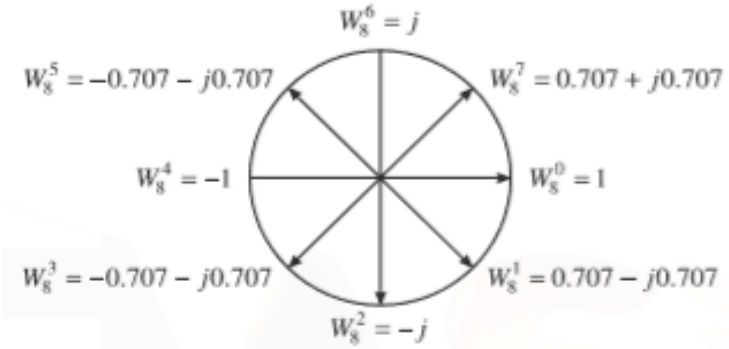


(a) 4-point DFT

For an 8-point DFT

$W_8^0 = 1$ $W_8^1 = 0.707-0.707j$ $W_8^2 = -j$ $W_8^3 = -0.707-0.707j$ $W_8^4 = -1$

$W_8^5 = -0.707+0.707j$ $W_8^6 = j$ $W_8^7 = 0.707+0.707j$



(b) 8-point DFT

Example (5): Compute the DFT of the four-point sequence

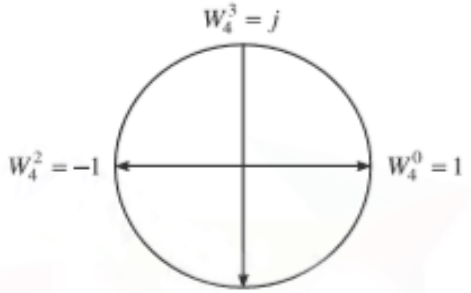
$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

Solution The first step is to determine the matrix $\mathbf{W}_4$. By exploiting the periodicity property of $\mathbf{W}_4$ and the symmetry property

$$W_N^{k+N/2} = -W_N^k$$

the matrix $\mathbf{W}_4$ may be expressed as

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$



(a) 4-point DFT

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\mathbf{X}_4 = \mathbf{W}_4\mathbf{x}_4 = \begin{bmatrix} 6 \\ -2 + 2j \\ -2 \\ -2 - 2j \end{bmatrix}$$