Basic Static Assignment to Transportation Networks

Demand Models

Introduction

As stated earlier, it is assumed here that O-D demand flows are known and independent of cost variations; thus path choice – the way that paths flow themselves through the network – is the only choice dimension explicitly simulated. It is also assumed that the demand flows for different O-D pairs are expressed in consistent units. For private passenger modes such as cars, for example, they are typically measured in vehicles or drivers per unit of time, whereas for public (scheduled) transport modes they are usually expressed in terms of passengers per unit of time. Let:

$d_{od} \geq 0$ be the demand flow for O-D pair od, defined by the elements of the O-D matrix corresponding to the purpose, mode, and time band being analyzed

d the demand vector, whose components are the demand values d_{od} for each O-D pair od

Path choice behavior is simulated with random utility models, assuming that the relevant component of the systematic utility is equal to the negative of the generalized path cost:

$$V_{od} = -\beta g_{od} + V_{od}^{o} \; \forall \; od \qquad 1$$

Where:

β is a utility parameter, which is omitted in the following because it is assumed included in the scale parameter within the choice function, introduced below

V_{od} is a vector whose elements consist of the systematic path utilities V_k, $k \in K_{od}$, for users of O-D pair od

V_{od}^{o} is a vector whose elements are the parts of the systematic utility that depend on attributes other than path costs (such as users' socioeconomic attributes); with no loss of generality, from a mathematical point of view attributes in vector V_{od}^{o} may be considered within nonadditive path cost vector or vice versa, hence for simplicity this term is generally omitted in the following sections (clearly any change of the reference utility value does not modify the results of the model).

Thereafter, (path or link) costs are assumed measured in units commensurate with the utility by using appropriate coefficients (with the same meaning as β coefficients).

Path choice probabilities depend on the systematic utilities of the available paths through the path choice function. Let:

$p_{od,k} = p\left[\frac{k}{od}\right] \geq 0$ be the probability that a user on a trip from origin o to destination d will use path k, $k \in K_{od}$, with $\sum_{k\in K_{od}} p_{od,k} = 1$

$p_{od} \geq 0$ be the vector of path choice probabilities for users of O-D path od, whose elements are the probabilities $p_{od,k}, k \in K_{od}, with\ 1^T p_{od} = 1$

The random utility model used to simulate path choice is given by:

$$p_{od,k} = p\left[\frac{k}{od}\right] = prob[v_k - v_j \geq \varepsilon_j - \varepsilon_k \forall j \in K_{od}]$$

$$p_{od} = p_{od}(V_{od}) \forall od$$

Where ε_j denotes the random residual corresponding to the perceived utility of path j. If the random residuals are equal to zero (ε_j = 0), then the variance-covariance matrix of the random residuals is null (Σ = 0), and the resulting choice model is deterministic. On the other hand, if the variance-covariance matrix of the random residuals is non-null and nonsingular, |Σ| ≠ 0, then the model is probabilistic.

A relation between path choice probabilities and path costs for O-D pair od, known as the path choice map, is obtained by combining the path choice function with the systematic utility function:

$$p_{od,k} = p_{od,k}(V_{od}) = p_{od}(-g_{od}) \forall od, k$$

$$p_{od} = p_{od}(V_{od}) = p_{od}(-g_{od}) \forall od$$

The flow h_k on path k connecting O-D pair od, k ∈ K_{od}, is simply given by the product of the demand flow d_{od} and the probability of choosing path k:

$$h_k = d_{od} p_{od,k}$$

and is measured in demand units. Thus, for each O-D pair, the relationship between path flows, path choice probabilities and demand flows is given by:

$$h_{od} = d_{od} p_{od}(V_{od}) \quad \forall od \qquad 2$$

The whole demand model is defined by the relations (1) and (2) which, combined, describe the relationship between path flows and path costs:

$$h_{od} = d_{od} p_{od}(-g_{od}) \quad \forall od \qquad 3$$

The above equation (3) is a particular specification consistent with the assumptions introduced at the beginning of this section. It should be noted that the choice function p_{od} () may vary with the O-D pair.

All the above relations can be expressed using matrix notation (Fig1). Let:

$$g^T = [6 \quad 4 \quad 2 \quad 4 \quad 2 \quad 1]$$

$$p_{od,k} = \frac{\exp(-g_{od,k}/\theta)}{\sum_{j \in K_{od}} \exp(-g_{od,j}/\theta)};$$

$$\theta = 2$$

$$\boldsymbol{p}_{14} = \begin{bmatrix} 0.090 \\ 0.245 \\ 0.665 \end{bmatrix}; \qquad \boldsymbol{p}_{24} = \begin{bmatrix} 0.269 \\ 0.731 \end{bmatrix}; \qquad \boldsymbol{p}_{34} = [1.000]$$

P Matrix

Path \ O-D pair	1–4	2–4	3–4
1	0.090	0	0
2	0.245	0	0
3	0.665	0	0
4	0	0.269	0
5	0	0.731	0
6	0	0	1.000

$$\boldsymbol{h} = \boldsymbol{P} \cdot \boldsymbol{d}$$

$$\begin{bmatrix} 90 \\ 245 \\ 665 \\ 404 \\ 1097 \\ 800 \end{bmatrix} = \begin{bmatrix} 0.090 & 0 & 0 \\ 0.245 & 0 & 0 \\ 0.665 & 0 & 0 \\ 0 & 0.269 & 0 \\ 0 & 0.731 & 0 \\ 0 & 0 & 1.000 \end{bmatrix} \cdot \begin{bmatrix} 1000 \\ 1500 \\ 800 \end{bmatrix}$$

Fig. 1 Example of demand model with probabilistic path choice.

P be the path choice probability matrix, with a column for each O-D pair od, a row for each path k, and element (k, od) given by p[k/od] if path k connects the O-D pair, otherwise zero (P is a block diagonal matrix with blocks given by the vectors p_{od}).

The previous equations become:

P = P(V) = P(−g)

h = P(V)d

h = P(−g)d

Different probabilistic path choice models ($|\Sigma| \neq 0$) can be specified according to different assumptions on the joint probability density function of perceived utilities or random residuals. In any case a (one-to-one) function pod ( ) is obtained. An example is provided in Fig. 1. Some useful general requirements for the stochastic assignment are discussed below.

Continuity of the path choice model, $p_i = p_i(g_i)$, assures that small changes in path costs induce small changes in choice probabilities. If it is also continuously differentiable it has a continuous Jacobian, $Jac[p_i(g_i)]$. This feature, assured by commonly used joint probability density functions, guarantees the continuity of the resulting SNL function. Thus it is useful to state the existence of stochastic user equilibrium.

The monotonicity of the path choice model, $p_i = p_i\ (g_i)$, ensures that an increase in the cost of a path k induces a decrease in the corresponding choice probability. More generally, the path choice model, $p_i = p_i\ (g_i)$, should be no increasing monotone with respect to path costs. This feature guarantees the monotonicity of the resulting SNL function. Hence it is useful to state the uniqueness of solutions of stochastic user equilibrium. It is ensured if no other parameter of the perceived utility joint probability density functions depends on the mean say the systematic utility. The resulting choice function is called invariant.

Independence of linear transformations of utility ensures that no change in the scale of the utility affects the model results (as guaranteed by commonly used random residual joint probability density functions, such as Gumbel, or Normal distributions). For instance, it is not relevant whether travel time is measured in hours or minutes.

In addition to the above mathematical requirements, some modeling requirements presented below are useful to effectively simulate path choice behavior.

The similarity of perception of partially overlapping paths rules out counterintuitive results. Indeed two partially overlapping paths are likely not perceived as two totally separate paths. Introducing a positive covariance between any two overlapping paths can simulate similarity, as in the probit choice model, or a commonality factor as in the C-logit choice model.

The independence of link segmentation (within the network model) ensures that if a link is further divided into sub-links and link costs are redefined such that path costs are not affected, path perceived utility distribution is not affected either, nor are choice probabilities. This feature is clearly guaranteed for path-explicit formulations of the distribution of perceived utility (e.g., logit model). If the distribution of perceived utility is formulated from link distributions (e.g., some probit specifications). This feature is only guaranteed for distributions stable w.r.t. summation (e.g., Normal distribution).

The negativity of perceived utility ensures that no user perceives a positive utility to travel along any path. This feature is ensured by assuming lower bounded distributions (for instance, log-normal, or Gamma). According to this feature, a nonelementary path is always a worse choice than the elementary path within it, thus supporting the assumption of considering elementary paths alone. On the other hand, if this feature is not presented, a nonelementary path may be a better choice than the elementary path within it; hence, nonelementary paths should be included within the path choice set (which may no longer be finite), possibly leading to unrealistic situations (some algorithmic drawbacks may also arise). Several adopted distributions (Gumbel, MVN) fail to satisfy this requirement, even though this condition is not relevant in practice.

Deterministic path choice models ($\Sigma = 0$) usually result in a one-to-many map because, if there are several minimum cost paths between an O-D pair od, the choice probability vector $_{\text{pDET,od}}$, and therefore the path flow vector $h_{DET,od}$, are not uniquely defined. An example is given in Fig. 2.

$$g^T = [6 \quad 4 \quad 2 \quad 4 \quad 2 \quad 1]$$

$$p_{od,k} \begin{cases} \in [0, 1] & \text{if } g_{od,k} = \min_{j \in k_{od}} g_{od,j} \\ = 0 & \text{if } g_{od,k} > \min_{j \in k_{od}} g_{od,j} \end{cases}$$

$$\sum_{k \in k_{od}} p_{od,k} = 1$$

$$\boldsymbol{p}_{14} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \qquad \boldsymbol{p}_{24} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \qquad \boldsymbol{p}_{34} = [1]$$

P **Matrix**

Path \ O-D pair	1–4	2–4	3–4
1	0	0	0
2	0	0	0
3	1	0	0
4	0	0	0
5	0	1	0
6	0	0	1

$$\boldsymbol{h}_{\mathrm{DET}} \quad = \quad \boldsymbol{P} \quad \cdot \quad \boldsymbol{d}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1000 \\ 0 \\ 1500 \\ 800 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1000 \\ 1500 \\ 800 \end{bmatrix}$$

Fig. 2 Example of a demand model with deterministic path choice.

General requirements discussed above can be quite easily extended to a deterministic choice model.

It can be useful to reformulate the deterministic demand model (2) as a system of inequalities. This system is obtained by applying to each O-D pair condition on deterministic choice probabilities $p_{DET,od}$; it is repeated here for the convenience of the reader:

$$(V_{od})^T \left(p_{od} - p_{DET,od}\right) \leq 0 \ \forall p_{od} \colon p_{od} \geq 0 \,, 1^T p_{od} = 1 \ \forall od$$

Noting that $V_{od} = g_{od}$ and multiplying the above inequality by $d_{od} \geq 0$ ∀od yields:

$$g_{od}^T \left(h_{od} - h_{DET,od}\right) \geq 0 \ \ \forall h_{od} \colon h_{od} \geq 0, 1^T h_{od} = d_{od} \ \ \forall od \qquad \text{3b}$$

Condition (3b) underlies the deterministic assignment models described below. The deterministic demand model corresponds to a condition where, for each O-D pair, the cost of each path actually used is equal, and is less than or equal to the cost of any path not used:

$$h_{DET,k} > 0 \Rightarrow g_k = \min(g_{od}) \, k \in k_{od}$$

$g_k > \min(g_{od}) \Rightarrow h_{DET,k} = 0 \; k \in k_{od}$

In the literature, this condition is known as Wardrop's first principle.

The above inequalities are equivalent to the definition of the deterministic path choice model. Thus the probability $p_{od,k}$ that a user of O-D pair *od* chooses path k is strictly positive only if the cost of path k is less than or equal to the cost of any other path that connects the O-D pair.

Feasible Path and Link Flow Sets

Vectors of path flow *h* are said to be feasible if they are compatible with the network topology and the O-D demand flows ***d***. The set S_h of feasible path flows contains nonnegative vectors $h \geq 0$ such that, for each O-D pair *od*, the sum of the elements of (sub)vector h_{od} is equal to the corresponding demand flow:

$$\sum_{k \in K_{od}} h_{od,k} = d_{od}$$

Or

$$1^T h_{od} = d_{od}$$

The above condition is definitely verified by any path flow vector h_{od} given by (3), due to features of the choice probability vector p_{od}, as well as it is no negativity.

The set S_h of feasible path flow vectors can therefore be expressed as:

$$S_h = \{h = [h_{od}]_{od}\}: \;\; h_{od} \geq 0, \;\; 1^T \, h_{od} = d_{od} \;\; \forall od \qquad 5$$

The set S_h is bounded because the path flow vector elements for each O-D pair *od* belong to the interval [0, d_{od}]; hence it is compact because it is also closed. It is also convex because it is defined by a system of linear equations and inequalities. Furthermore, it is nonempty if at least one path is available for each O-D pair. Moreover, regardless of the path cost vector $g = [g_{od}]_{od}$, the result of the demand (3) is by definition always a vector of feasible path flows:

$$h = [h_{od} = d_{od} p_{od}(-g_{od})]_{od} \in S_h \quad \forall g = [g_{od}]_{od}$$

In a similar way, a link flow vector is feasible if it is compatible with the network topology and the demand flows ***d***. Thus, a vector of link flows ***f*** is feasible if, according to the supply model, it corresponds to a feasible path flow as defined in the demand model. The set S_f of feasible link flows can be formally expressed as:

$$S_f(d) = \{f: \; f = \sum_{od} \Delta_{od} h_{od}, h_{od} \geq 0, \;\; 1^T \, h_{od} = d_{od} \;\; \forall od\} \qquad 6$$

That is,

$$S_f = \{f : f = \Delta h, \ \ \forall h \in S_h\}$$

Formulation (6) highlights the role of the demand flow vector ***d*** in the definition of the feasible link flow set S_f.

It should be noted that, in general, there are more paths than links in a transportation network; this means that the incidence matrix Δ has more columns than rows, and is therefore noninvertible. It follows that multiple feasible path flow vectors may lead to the same feasible link flow vector.

Network Performance Indicators

Each pattern of path and link costs and flows can be summarized by indicators that refer either to an O-D pair or to the system as a whole; these indicators are used in the following sections.

The total cost TC_{od} associated with an O-D pair *od* is given by the sum of the products of the corresponding path costs and flows:

$$TC_{od} = \sum_{k \in K_{od}} h_k g_k = (g_{od})^T h_{od} \ \ \ \forall od$$

The corresponding (weighted) average cost AC_{od} is obtained by dividing by the demand flow:

$$AC_{od} = \frac{TC_{od}}{d_{od}} = \frac{(g_{od})^T h_{od}}{d_{od}} \quad \forall od$$

The total network cost *TC* is given by the sum of the total O-D costs overall O-D pairs:

$$TC = \sum_{od} TC_{od} = \sum_{od} \sum_{k \in K_{od}} h_k g_k = \sum_k h_k g_k = g^T h$$

The network-level average cost *AC* is obtained by weighting the average costs of all the O-D pairs by the corresponding demand flows, that is, by weighting the path costs by the path flows:

$$AC = \frac{(\sum_{od} AC_{od} d_{od})}{(\sum_{od} d_{od})} = \frac{\left(\sum_{od} \sum_{k \in K_{od}} h_k g_k\right)}{\left(\sum_{od} \sum_{k \in K_{od}} h_k\right)} = \frac{(\sum_{od} TC_{od})}{(\sum_{od} d_{od})} = \frac{TC}{d_{..}} = \frac{g^T h}{1^T h} = g^T h / 1^T d$$

Where d.. = $\sum_{od} d_{od}$ = $\sum_{od} \sum_{k \in K_{od}} h_k$ = $1^T h$ = $1^T d$ denotes the total demand flow.

With reference to additive and nonadditive path costs, the following also holds:

$$TC = (g^{ADD})^T h + (g^{NA})^T h = (\Delta^T c)^T h + (g^{NA})^T h = c^T f + (g^{NA})^T h$$

An expression that, when nonadditive path costs are zero (g^{NA} = 0), reduces to:

$$TC = c^T f = \sum_a f_a c_a \qquad 7$$

In other words, in the absence of nonadditive costs, the sum of the link costs multiplied by the corresponding flows coincides with the total network cost

An Expected Maximum Perceived Utility (or EMPU), s_{od}, can be associated with each O-D pair *od*; it depends on the path choice model. The EMPU is a function of the systematic utilities of the available paths (neglecting here the other attributes V^{o}_{od} for the sake of simplicity):

$$s_{od} = s_{od}(V_{od}) = s_{od}(-g_{od}) = s_{od}(-\Delta^{T}_{od}c - g^{NA}_{od}) \quad \forall od \tag{8}$$

The EMPU is greater than or equal to the maximum systematic utility and therefore to the average systematic utility as well:

$$s_{od} \geq \max(V_{od}) \geq (V_{od})^{T} p_{od} = \frac{(V_{od})^{T} h_{od}}{d_{od}} \qquad \forall od$$

The EMPU is, therefore, greater than or equal to the negative of the minimum cost over all the paths, which in turn is greater than or equal to the negative of the average cost:

$$s_{od} \geq -\min(g_{od}) \geq -(g_{od})^{T} h_{od}/d_{od} = -AC_{od} \qquad \forall od$$

The total EMPU, *TS*, is defined as the sum of each O-D pair's EMPU multiplied by the corresponding demand flow:

$$TS = \sum_{od} d_{od} s_{od}(V_{od}) = \sum_{od} d_{od} s_{od}(-g_{od}) = \sum_{od} d_{od} s_{od}(-\Delta^{T}_{od}c - g^{NA}_{od})$$

The corresponding average EMPU, *AS*, is obtained by dividing by the total demand flow:

$$AS = \sum_{od} d_{od} s_{od} / \sum_{od} d_{od} = \sum_{od} \frac{d_{od} s_{od}}{d} .. = \frac{TS}{d} ..$$

In conclusion, the total cost is an estimate, made without considering the effect of dispersion, of the disutility users receive when distributing themselves among paths according to path flows *h*, whereas the EMPU is the disutility users perceive when making path choices leading to path flows *h* including the effect of dispersion. From the preceding considerations, the following relations hold between the total and average values of EMPU and cost:

$$TS \geq -TC \qquad AS \geq -AC$$

Numerical examples of network indicators are presented in Fig. 3.

O-D pair	*Path*	*Cost*	*Flow*	*Total cost*	*Average cost*	$-\min(g)$	$\exp(-C/\theta)$	*Average EMPU* $s = \theta \times \ln(\sum \exp(-C/\theta))$	*Total EMPU*
1–4	1	6	90	540			0.00248		
	2	4	245	980			0.01832		
	3	**2**	665	1330			0.13534		
	Total		1000	2850			0.15613		−1857
					2.85	2.00		−1.85	
2–4	4	4	404	1616			0.01832		
	5	**2**	1096	2192			0.13534		
	Total		1500	3808			0.15365		2810
					2.54	2.00		−1.87	
3–4	6	**1**	800	800			0.36788		
	Total		800	800			0.36788		800
					1.00	1.00		−1.00	
Total network values			3300	7458					5467
Average network values					2.26	1.75		−1.66	

Fig. 3 Performance indicators for the network.