Basic Static Assignment to Transportation Networks

Congested Networks

Introduction

Equilibrium assignment is generally expressed by fixed point models, that is, systems of nonlinear equations, or by variation inequalities. Hence only asymptotically converging algorithms are available.

We consider here the situation where O-D demands are fixed, but link performance measures and costs depend on link flows through the performance and cost functions. Conversely, link flows depend on link costs through the path choice probabilities, as described by the uncongested network assignment map. The user equilibrium approach to the study of the supply-demand interactions assumes that the state of the real-world system can be represented by a configuration of path flows that is consistent with the corresponding path costs. Equilibrium path flows and costs are defined by a system of nonlinear equations obtained by combining the supply model with the demand model:

$$g_{od}^{*} = \Delta_{od}^{T} c\left(\sum_{od} \Delta_{od} h_{od}^{*}\right) + g_{od}^{NA*} \qquad \forall od$$

$$V_{od}^{*} = -g_{od}^{*} \qquad \forall od$$

$$h_{od}^{*} = d_{od} p_{od}(V_{od}^{*}) \qquad \forall od$$

Or

$$g_{od}^{*} = \Delta_{od}^{T} c\left(\sum_{od} \Delta_{od} h_{od}^{*}\right) + g_{od}^{NA*} \quad \forall od$$

$$h_{od}^{*} = d_{od} p_{od}(-g_{od}^{*}) \quad \forall od$$

Equivalent equilibrium assignment models expressed in terms of link variables can be formulated by the system of nonlinear equations obtained by combining the uncongested network assignment map with the flow-dependent cost functions:

$$c^{*} = c(f^{*})$$

$$f^{*} = \sum_{od} d_{od} \Delta_{od} p_{od}\left(-\Delta_{od}^{T} c^{*} - g_{od}^{NA}\right)$$

Or

$$c^{*} = c(f^{*})$$

$$f^{*} = f_{UN}(c^{*}; d)$$

The above system of equations shows that, in congested networks, link flows may depend nonlinearly on on-demand flows (unlike uncongested network assignments). Thus, in this case, the effect of each O-D pair cannot be evaluated separately.

The circular dependence between flows and costs expressed by the equilibrium approach is depicted in Fig. 1. This figure particularizes the general framework for the fixed demand assumption role of the uncongested network assignment model in the equilibrium framework.

The formulation and analysis of the theoretical properties of equilibrium flows (and costs) depend on the type of model adopted to simulate path choices: probabilistic or deterministic. This selection defines, respectively, stochastic and deterministic equilibrium assignment models and corresponding solution algorithms, which are the subjects of the following sections.

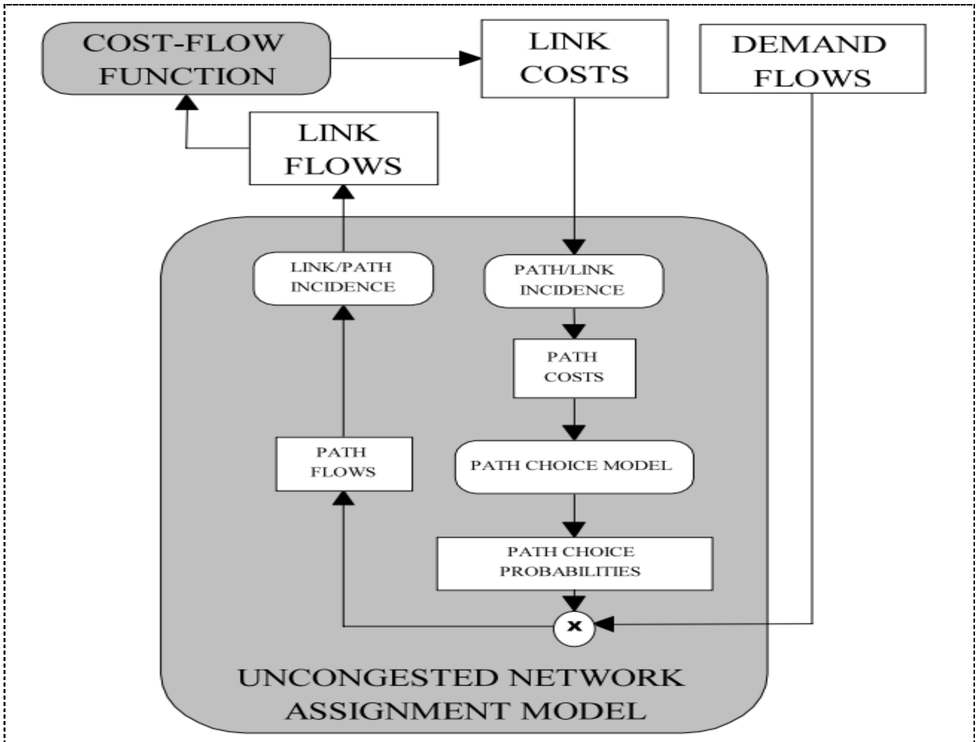


Fig. 1 Schematic representation of fixed demand equilibrium assignment models.

In general, algorithms for calculating equilibrium flows are based on recursive equations which, starting from an initial feasible link flow vector $f^0 \in S_f$, generate a sequence of feasible link flow vectors:

$$f^k = \emptyset(f^{k-1}) \in S_f$$

In each step, an assignment algorithm attempts to improve the solution estimate obtained in the preceding steps, but an exact equilibrium solution will not generally be found in a finite number of steps. However, if at any step k the equilibrium flow vector is generated, all subsequent elements of the sequence will remain equal to the equilibrium vector:

$$f^k = f^* \rightarrow f^j = f^* \qquad j > k$$

Furthermore, if link flow vectors in two successive steps are equal, they are the equilibrium vector:

$$f^k = f^{k-1} \rightarrow f^k = f^*$$

Under certain assumptions on the cost functions and the path choice model, it can be demonstrated that the sequence defined by the recursive equations converges to the equilibrium flow vector f^*, provided that it is unique:

$$\lim_{k\to\infty} f^k = f^*$$

Models for Stochastic User Equilibrium

Stochastic User Equilibrium (SUE) assignment is obtained by applying the equilibrium approach to congested networks under the assumption of probabilistic path choice behavior. The resulting path flows h^* corresponds to the condition in which, for each O-D pair, the perceived cost of the paths used at equilibrium is less than or equal to the perceived cost of every other path. Equilibrium path flows can be expressed as the solution of a fixed-point model defined on the feasible path flow set S_h and obtained by combining the supply model with the demand model:

$$h^*_{od} = d_{od} p_{od}(-\Delta^T_{od} c(\textstyle\sum_{od} \Delta_{od} h^*_{od}) - g^{NA}_{od} \quad \forall od \qquad 1$$

$$h^* = [h^*_{od}]_{od} \in S_h$$

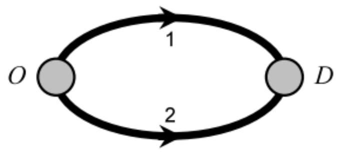
An equivalent fixed-point model using link flow variables f^* (and therefore defined on the feasible link flow set S_f) can be obtained by combining the stochastic uncongested network assignment function (disaggregated here by O-D pair to facilitate the analysis) with the flow-dependent cost functions:

$$f^* = f_{SUN}(c(f^*) \qquad or \quad f^* = \textstyle\sum_{od} d_{od} \Delta_{od} p_{od} (-\Delta^T_{od} c(f^*) - g^{NA}_{od}) \qquad 2$$

With:

$$f^* \in S_f$$

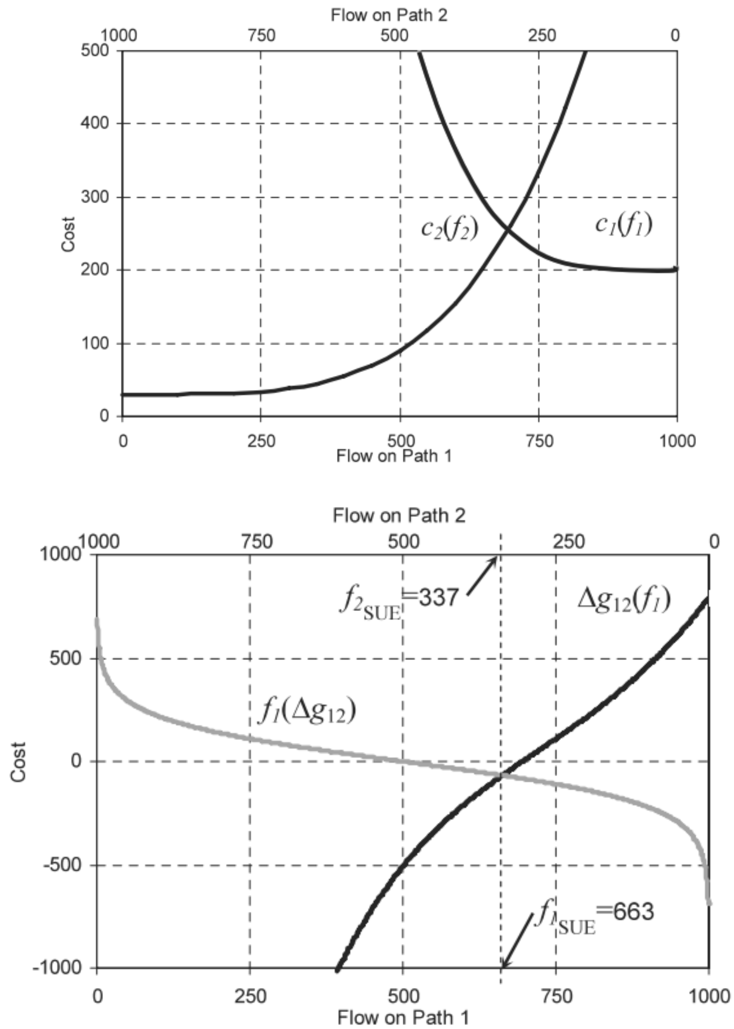
An example of stochastic equilibrium using a logit path choice model for a two-link/path network is given in Fig. 2. The stochastic equilibrium pattern is obtained at the intersection of the curves representing the supply and (inverse) demand equations. Note that the stochastic equilibrium configuration does not correspond to equal (systematic) costs on the two paths, which means that the intersection point of the two curves does not correspond to a zero value of the difference $g_1 - g_2$. In other words, at stochastic equilibrium, some travelers have higher (systematic) path costs than others. This result obviously depends on the assumptions made about path choice behavior. The perceived path cost is modeled as a random variable and therefore some users may choose higher (systematic) cost paths because they perceive them as the least cost.



$$c_1 = c_0\left(1 + a_1\frac{f_1}{Cap_1}\right)^{\gamma} = 30\left(1 + 2\left(\frac{f_1}{500}\right)^4\right);$$

$$c_2 = c_0\left(1 + a_2\frac{f_2}{Cap_2}\right)^{\gamma} = 350\left(1 + 2\left(\frac{f_2}{500}\right)^4\right);$$

$$h_1 + h_2 = d_{od} = 1000; \quad h_1 = f_1; \quad h_2 = f_2;$$



Supply equation $\quad \Delta g_{1,2}(f_1) = g_1(f_1) - g_2(f_2 = d - f_1)$

Demand equation $\quad f_1(\Delta g_{1,2}) = d_{o,d}\dfrac{1}{1 + \exp(\Delta g_{1,2}/\theta)}$

$f_2 = d_{od} - f_1$

Fig. 2 Example of Stochastic User Equilibrium (SUE; θ = 100).

Existence of Stochastic User Equilibrium Link Flows. The fixed-point model has at least one solution if the cost function c = c (f) and the path choice function $p_{od} = p_{od}(V_{od})$ (which defines the stochastic uncongested network assignment function $f = f_{SUN}(c; d)$) are both continuous.

The equilibrium solution f^* is a fixed point of the composite function $y = f_{SUN}(c(x))$ which, under the above assumptions (and for a connected network), is a continuous function defined over the nonempty, compact, and convex set S_f. Furthermore, the function $y = f_{SUN}(c(x))$ assumes values only in the feasible set S_f.

Monotonicity of the Stochastic Uncongested Network Assignment Function. If the path choice model is defined by a nondecreasing monotone function of the systematic utility, as in the case of the additive probabilistic model, the stochastic uncongested network assignment function is monotone nonincreasing with respect to link costs. Thus, if the cost of one or more of the links increases, the flow (or flows) on these links decreases, and vice versa. This property is formally expressed as:

$$(f_{SUN}(c) - \acute{f}_{SUN}(c' - c'') \leq 0 \qquad \forall c', c''$$

Given any two link cost vectors c ′ and c ″, consider the following notation.

$$g'_{od} = \Delta^T_{od} c' + g^{NA}_{od} \qquad V'_{od} = -g'_{od} \qquad P'_{od} = p_{od}(V'_{od})$$

$$h'_{od} = d_{od} p'_{od} \qquad f' = \sum_{od} \Delta_{od} h'_{od}$$

$$g''_{od} = \Delta^T_{od} c'' + g^{NA}_{od} \qquad V''_{od} = -g''_{od} \qquad p''_{od} = p_{od}(V''_{od})$$

$$h''_{od} = d_{od} p''_{od} \qquad f'' = \sum_{od} \Delta_{od} h''_{od}$$

From the monotonicity of the cost functions, with $f' = f_1^* \neq f'' = f_2^*$, it also follows that:

$$[c_1^* - c_2^*]^T (f_1^* - f_2^*) > 0$$

Models for Deterministic User Equilibrium

Deterministic User Equilibrium (DUE) assignment is obtained by applying the equilibrium approach for congested networks under the assumption of deterministic path choice behavior. Deterministic equilibrium link flows f^*, path flows h^*, and the corresponding costs c^* and g^* can be determined with a fixed-point model obtained by simultaneously applying the supply model and the demand model, as in the stochastic equilibrium case (an alternative is to utilize the deterministic uncongested network assignment map and flow-dependent cost functions). In this case, however, there are some mathematical complications arising from the fact that the deterministic demand model is expressed (such as the corresponding deterministic uncongested network assignment map) by a one-to-many map.

For this reason, the properties of deterministic equilibrium are usually studied through indirect formulations. The most general is the variation inequality formulation based on the specification of the deterministic demand model as the system of inequalities (7):

$$g(h^*)^T(h - h^*) \geq 0 \; \forall h \in S_h \qquad 3$$

By combining the demand model obtained by summing all O-D pairs with the supply model, expression (3) is obtained. In the case of congested networks, therefore, the resulting path (or link) flows correspond to the condition expressed by Wardrop's first principle.

Equivalent variation inequality models expressed in terms of link flows are obtained by combining the link cost functions with the inequality systems that represent deterministic uncongested network assignment:

$$c(f^*)^T(f - f^*) \geq 0 \; \forall f \in S_f \qquad 4$$

$$c(f^*)^T(f - f^*) + (g^{NA})^T(h - h^*) \geq 0 \; \forall f = \Delta h, \forall h \in S_h \qquad 5$$

Expressions (4) and (3) apply, respectively, to cases with zero and nonzero nonadditive path costs. Note that expressions (3)–(5) are different from those used for deterministic uncongested assignments in that the path and link costs depend on flows. In the presence of nonadditive path costs in terms of links, f^* and of the total nonadditive cost G^{NA*} at deterministic equilibrium:

$$c(f^*)^T(f - f^*) + (G^{NA} - G^{NA*}) \geq 0$$

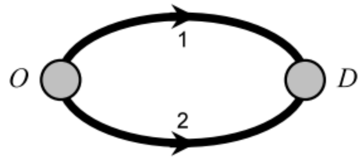
$$\forall f = \Delta h, G^{NA} = (g^{NA})^T h, \forall h \in S_h \qquad 6$$

An example of deterministic user equilibrium assignment for a two-link/path network is shown in Fig. 3. Note that the deterministic equilibrium flows correspond to the intersection point of the supply and demand curves (in this case, step curves) and the result in costs that are equal for the two paths since both are used.

Conditions ensuring the existence and uniqueness of deterministic equilibrium link flows and costs are similar to those described for stochastic equilibrium. In particular, the continuity and monotonicity of the cost functions guarantee, respectively, the existence and uniqueness of the solution. It should be noted once again that these existence and uniqueness conditions are only sufficient; there may exist nonmonotone cost functions that give rise to a unique equilibrium vector.

Existence of Deterministic User Equilibrium Link Flows. The variation inequalities (3)–(5) have at least one solution if the cost functions are continuous functions defined on the nonempty, compact, and convex set of the feasible path flows S_h or link flows S_f.

The considerations regarding the continuity of cost functions discussed for SUE models apply also for DUE models. The existence of equilibrium link flows ensures the existence of the corresponding link costs $c^* = c(f^*)$, and of path costs and flows g^* and h^*.



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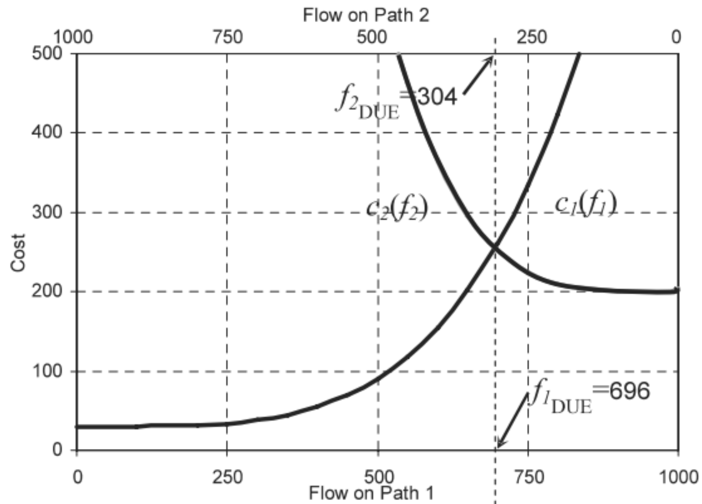
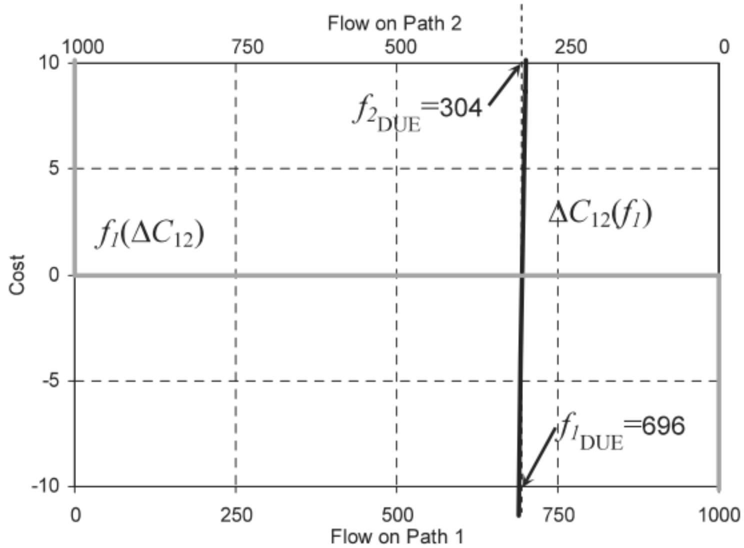
$$c_2 = c_{0,1}\left(1 + a_2 \frac{f_2}{Cap_2}\right)^{\gamma} = 200\left(1 + 2\left(\frac{f_2}{500}\right)^4\right);$$

$$F_1 + F_2 = d_{od} = 1000; \quad F_1 = f_1; \quad F_2 = f_2;$$





Supply equation $\quad \Delta C_{1,2}(f_1) = C_1(f_1) - C_2(f_2 = d - f_1)$

Demand equation $\quad f_1(\Delta C_{1,2}) = \begin{cases} 0 & \text{if } \Delta C_{1,2} > 0 \\ \in [0, d_{1,2}] & \text{if } \Delta C_{1,2} = 0 \\ d_{1,2} & \text{if } \Delta C_{1,2} < 0 \end{cases}$

$$f_2 = d_{od} - f_1$$

Fig. 3 Example of Deterministic User Equilibrium (DUE).