Review of Matrix Theory

$$A = [a_{ij}]_{m \times n} = \begin{array}{l} \text{Column} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{array}{l} \text{Row} \\ \\ \\ \to \text{diagonal} \end{array}$$

if $m = n \Rightarrow$ square matrix

* identity matrix has the diagonal '1' else '0'.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ldots$$

Notes:

* $A = B$ if $a_{ij} = b_{ij}$ $\forall$ i and j.
* if $A = B$ and $B = C \Rightarrow A = C$.
* $A + B = B + A$
* $(A + B) + C = A + (B + C)$.
* $A + 0 = A$, $A - A = A + (-A) = 0$
* $\alpha(A + B) = \alpha A + \alpha B$.
* $(\alpha + \beta) A = \alpha A + \beta A$. $\quad$ α and β are const.
* $(\alpha\beta) A = \alpha(\beta A) = \beta(\alpha A)$.

- Multiplication:

$$\boxed{A_{m\times n} \cdot B_{n\times p} = C_{m\times p}}$$

where $A = [a_{ij}]_{m\times \underline{n}}$ and $B = [b_{ij}]_{\underline{n}\times p}$

then $C = [c_{ij}]_{m\times p} = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]$

H.W: $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, Find $AB = ?$

<u>Notes:</u>

1. $A.B \neq B.A$ Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

2. $A.0 = 0.A = 0$

3. $A.I = I.A = A$

4. $(A+B).C = AC + BC$
   $A.(B+C) = AB + AC$.

5. $(AB).C = A.(BC) = ABC$

6. $\alpha(AB) = (\alpha A)B = A.(\alpha B)$ α is cnst.

7. if $A.B = 0$ does not mean $A = 0$ or $B = 0$

<u>Ex:</u> $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$.

Transpose and Inverse

let $A = [a_{ij}]_{m \times n} \Rightarrow A^T = [a_{ji}]_{n \times m}$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$.

Notes:

1. if $A^T = A$, then A is said Symmetric.
   if $A^T = -A$, then A is said Skew-Symmetric

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 5 \end{bmatrix}$ is symmetric matrix

$B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ is Skew-Symm. matrix

all are zero.

2. $(A^T)^T = A$
3. $(A+B)^T = A^T + B^T$
4. $(\alpha A)^T = \alpha . A^T$ α is a constant.
5. $(AB)^T = B^T . A^T$

* A matrix A is said to be <u>invertible</u> if there exist matrix B such that:

$$\boxed{AB = BA = I}$$

matrix B is called the inverse of A denoted by A^{-1}.

$$\boxed{A \cdot A^{-1} = A^{-1} \cdot A = I}$$

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Ex</u>: $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

<u>Notes</u>:

1. $(A^{-1})^{-1} = A$
2. $(A^{T})^{-1} = (A^{-1})^{T}$
3. $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ $\quad \alpha$ is a constant.
4. $(AB)^{-1} = B^{-1} \cdot A^{-1}$
5. if A is invertible (A^{-1} exists), then $AB = 0 \Rightarrow B = 0$.

$AB = 0$.
$A^{-1} \cdot (AB) = A^{-1} 0$
$IB = 0$
$\Rightarrow B = 0$.

Orthogonal Matrix

if $A^T . A = I$ or $A^T = A$, then A is said to be orthogonal matrix.

Ex: $$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T . A = \begin{bmatrix} \cancel{\cos\theta} & \cancel{-\sin\theta} \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 .$$

$\therefore$ A is orthogonal matrix

Determinants

let $A = [a_{ij}]_{N\times N}$, then the determinant of the square matrix A is:

$$|A| = \det(A) = \sum_{K=1}^{N} a_{iK} A_{iK} \quad i=1,2,\dots,N$$

where $A_{ij} = (-1)^{i+j} |M_{ij}|$

Ex: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}_{2\times2} = ad - bc.$

$|a|_{1\times1} = a$ ...

Notes

1. $|A| = |A^T|$.
2. $|AB| = |A||B|$.
3. if $|A| = 0$ then A is said to be singular matrix.
4. if $|A|_{n\times n} \neq 0$ then A is non-singular matrix.

$\Rightarrow A^{-1}$ exist.

$\Rightarrow \text{rank}(A) = n$.

Ex: $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 5 \\ 0 & -1 & -3 \end{bmatrix}_{3\times3}$

Since $|A| = 0 \Rightarrow A$ is Singular.

but $\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3 \neq 0 \Rightarrow \underline{\underline{\text{Rank}(A) = 2}}$

Ex: $B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix}_{3\times3}$

$|B| = 17 \neq 0 \Rightarrow B$ is non-Singular

B^{-1} exist and --

$$B^{-1} = \frac{1}{17}\begin{bmatrix} -4 & 3 & 6 \\ 2 & 7 & -3 \\ -7 & 1 & 2 \end{bmatrix}$$

Since $|B| \neq 0 \Rightarrow \text{rank}(B) = \underline{\underline{3}}$.

5. changing between two rows (or Glumns)
$\Rightarrow$ changes the Sign of determinants

Ex: $\begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & 4 & 5 \end{vmatrix} = 20$ and $\begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 3 & 4 & 5 \end{vmatrix} = -20$

6. $|\alpha A| = \alpha^n |A|$ where $A = [a_{ij}]_{n\times n}$ and α is Const.

Ex: $|A| = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$

then $|2A| = \begin{vmatrix} 2 & 4 \\ -2 & 2 \end{vmatrix} = 4+8 = \underline{\underline{12}}$

$= 2^2 * 3 = 4 * 3 = \underline{\underline{12}}$

7. if two rows (or columns) in a matrix A are equal or are linearly propotional then $|A| = 0$.

Ex: $\begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & 4 & 5 \end{vmatrix} = 20$

and $\begin{array}{c} R_1 \\ R_2 \\ \\ \end{array} \begin{vmatrix} 2 & 1 & -1 \\ 4 & 2 & -2 \\ 3 & 4 & 5 \end{vmatrix} = 0$

because $R_2 = 2R_1$

Applications of Gauss Elimination Method

1. Solving Linear System:

EX: Solve

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= -2 \\ 2x_1 + 3x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 4x_3 &= -1 \end{aligned} \quad — *$$

Sol.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$ Coefficients matrix.

variables array $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$ constant array

System (*) can be rewritten as:

$$AX = B$$

and solved using Gauss El. by Expansion Matrix:

$$[A|B] \xrightarrow{\text{convert to}} [I|X]$$

Solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$\begin{array}{c|ccc|c}
 & & A & & B \\
R_1 & 1 & 2 & 3 & -2 \\
R_2 & 2 & 3 & 2 & 0 \\
R_3 & 3 & 3 & 4 & 1
\end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 2 & 3 & 2 & 0 \\ 3 & 3 & 4 & 1 \end{array}\right] \quad \begin{array}{l} \text{reference row} \\ -2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -1 & -4 & 4 \\ 0 & -3 & -5 & 5 \end{array}\right] \xrightarrow{-R_2 \to R_2} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & 1 & 4 & -4 \\ 0 & -3 & -5 & 5 \end{array}\right]$$

$$\begin{array}{l} -2R_2+R_1 \to R_1 \\ 3R_2+R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 6 \\ 0 & 1 & 4 & -4 \\ 0 & 0 & 7 & -7 \end{array}\right] \quad \frac{1}{7}R_3 \to R_3 \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 6 \\ 0 & 1 & 4 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} 5R_3+R_1 \to R_1 \\ -4R_3+R_1 \to R_2 \end{array} \sim \begin{array}{c|c} I_3 & Y \ \text{Solution} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right. & \left.\begin{array}{c} 1 \\ 0 \\ -1 \end{array}\right] \end{array}$$

$$\therefore \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

OR $\left.\begin{array}{l} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{array}\right\}$ is the Solution of the linear system (*)

2. Finding the Inverse of a matrix

Ex: Solve the linear System below using the inverse matrix:

$$x_1 + 2x_2 + 3x_3 = -2$$
$$2x_1 + 3x_2 + 2x_3 = 0$$
$$3x_1 + 3x_2 + 4x_3 = -1$$

Sol. * Convert the system to $\boxed{A\underline{X} = B}$ matrix form

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$

* check $|A| \neq 0 \Rightarrow$ A is non-singular
$\Rightarrow \boxed{A^{-1} \text{ exists.}}$
and $\text{rank}(A) = 3$.

* Use matrix expansion to Find A^{-1}.

$$\underbrace{[A \mid I]}_{\text{identity matrix}} \xrightarrow{\text{Convert to}} [\underbrace{I}_{\text{identity matrix.}} \mid A^{-1}]$$

* the solution is:

$$\boxed{\underline{X} = A^{-1} \cdot B}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -4 & -2 & 1 & 0 \\ 0 & -3 & -5 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & -3 & -5 & -3 & 0 & 1 \end{array}\right] \xrightarrow[3R_2+R_3\to R_3]{-2R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & -3 & 2 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 0 & 7 & 3 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & -3 & 2 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & \frac{-3}{7} & \frac{1}{7} \end{array}\right] \xrightarrow[5R_3+R_1\to R_1]{-4R_3+R_2\to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{5}{7} & \frac{-4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{array}\right] \to A^{-1}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} -6 & -1 & 5 \\ 2 & 5 & -4 \\ 3 & -3 & 1 \end{bmatrix}$$

the solution for the linear system is:

$$\underline{X} = A^{-1} \cdot B$$

or
$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -6 & -1 & 5 \\ 2 & 5 & -4 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore x_1 = 1,\ x_2 = 0,\ x_3 = -1.$

Eigen Values and Eigen Vectors of a Matrix

Let A be an $n \times n$ matrix and $\underline{X}$ be a nonzero vector for which:

$(X \neq 0)$

$$\boxed{\underset{n\times n}{A}\,\underset{n\times 1}{X} = \lambda \underset{n\times 1}{X}} \quad \text{——— (1)}$$

λ is a scalar called an eigenvalue of the matrix A.

$\underline{X}$ is called an eigenvector of A associated with λ.

we can rewrite (1) as:

$$A\underline{X} - \lambda \underline{X} = 0$$

$$\text{or} \quad [A - \lambda I]\underline{X} = 0 \quad \text{——— (2)}$$

we could write (2) as:

$$\boxed{[\lambda I - A]\,\underline{X} = 0} \quad \text{——— (3)}$$

which is more commonly used.

if the matrix $[\lambda I - A]$ is invertible (inverse exists) then $\underline{X} = 0$. Howevere, we required that $\underline{X} \neq 0$. therefor, $[\lambda I - A]$ cannot have an inverse, i.e, it's singular matrix, then

is called characteristic equation ← $\boxed{|\lambda I - A| = 0}$ or $|A - \lambda I| = 0$

is called characteristic polynomial ← $\boxed{C(\lambda) = |\lambda I - A| = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_i)^{m_i}}$

$m_1 + m_2 + \cdots + m_i = n$

m_i is called the algebric multiplicity of λ_i

Ex: Let $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$, Find its eigenvalues and eigenvectors associated.

Sol.

First step: Finding λ: —

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} \lambda+5 & -2 \\ 7 & \lambda-4 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+5 & -2 \\ 7 & \lambda-4 \end{vmatrix} = (\lambda+5)(\lambda-4) + 14.$$

$$= \lambda^2 - 4\lambda + 5\lambda - 20 + 14 = 0$$

$$= \lambda^2 + \lambda - 6 = 0.$$

$$\therefore |\lambda I - A| = (\lambda - 2)(\lambda + 3) = 0.$$

Second step: Finding Vectors

$$\therefore \lambda_1 = 2 \text{ and } \lambda_2 = -3$$

– Using equation (1): $AX = \lambda X$ or $[\lambda I - A]X = 0$

for $\lambda_1 = 2 \Rightarrow \begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 7 & -2 & 0 \\ 7 & -2 & 0 \end{array}\right] \overset{\frac{1}{7}R_1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{2}{7} & 0 \\ 7 & -2 & 0 \end{array}\right] \overset{-7R_1+R_2}{\sim}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{2}{7} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 - \frac{2}{7} x_2 = 0$$

let $x_2 = s$.

and s is any constant

$\therefore \quad x_1 = \frac{2}{7} x_2 = \frac{2}{7} s.$

the eigenvector associated with $\lambda_1 = 2$ is:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7}s \\ s \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix} s.$$

let $s = 7 \Rightarrow X_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

For $\lambda_2 = -3$,

$$[\lambda_2 I - A] \cdot X = 0$$

$$\left[-3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ Solu us: GE

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 7 & -7 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 7 & -7 & 0 \end{array}\right] \xrightarrow{-7R_1 + R_2}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 - x_2 = 0$$

or $x_1 = x_2$

let $x_2 = t \Rightarrow x_1 = t$. t is a scalar

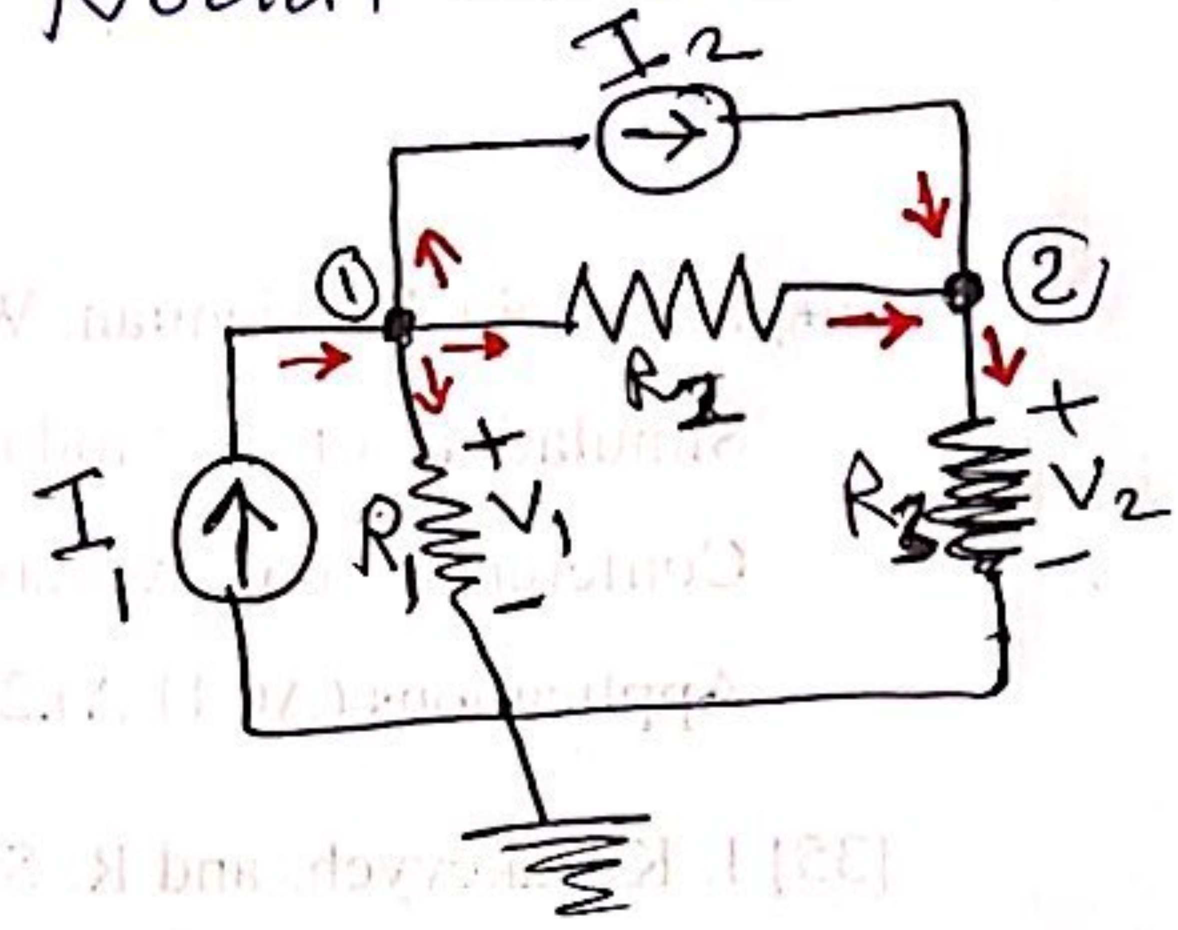
$\therefore$ the eigen vector associated with $\lambda_2 = -3$ is

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t.$$

if $t = 1 \Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

DC circuit analysis

Nodal and Mesh



EX: Use Nodal analysis to solve for V_1 and V_2 using ① GE, ② Inverse,

Sol. at node ①: $I_1 = I_2 + \frac{V_1 - V_2}{R_2} + \frac{V_1 - 0}{R_1}$ — ①

or $I_1 - I_2 = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_1 - \frac{1}{R_2}V_2$ — ②

or can use conductance ($G = 1/R$):

$$I_1 - I_2 = (G_1 + G_2)V_1 - G_2 V_2 \quad \text{— ③}$$

at node ②: $I_2 + \frac{V_1 - V_2}{R_2} = \frac{V_2}{R_3}$ — ④

$$I_2 = -\frac{1}{R_2}V_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)V_2 \quad \text{— ⑤}$$

OR $I_2 = -G_2 V_1 + (G_2 + G_3)V_2$ — ⑥

from ② and ⑤

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix} \quad \text{— ⑦}$$

OR

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix} \quad \text{— ⑧}$$

Assumme $G_1 = 2$, $G_2 = 2$, $G_3 = 4$

$I_1 = 7$, $I_2 = 5$.

rewrite (8):

$A X = B$

$$\underbrace{\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{B}$$

(1) using GE

$$\left[\begin{array}{cc|c} 4 & -2 & 2 \\ -2 & 6 & 5 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ -2 & 6 & 5 \end{array}\right] \xrightarrow{2R_1+R_2}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 5 & 6 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{6}{5} \end{array}\right] \xrightarrow{\frac{1}{2}R_2+R_1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{11}{10} \\ 0 & 1 & \frac{6}{5} \end{array}\right] \Rightarrow \begin{array}{l} V_1 = \frac{11}{10} \\ V_2 = \frac{6}{5}. \end{array}$$

(2) using GE and A^{-1}

$$\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \times 6 - 4} \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

$\therefore$ Solution is $(X = A^{-1}B)$:

$$\frac{1}{20} \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 22 \\ 24 \end{bmatrix} = \begin{bmatrix} \frac{11}{10} \\ \frac{6}{5} \end{bmatrix}.$$

$\therefore V_1 = \frac{11}{10}$ and $V_2 = \frac{6}{5}$.