Differential Equation :

Difinition : Any equation involving on derivative or more in the unknown function « the independent verable » is called Differential Equation D.E.

- D.E
  - ordinary D.E
  - partial D.E

Ordinary Differential Equation (O.D.E)

Equation which involves only one independent verable.

$$\frac{dy}{dx} = x + 5 \quad (\text{is an O.D.E})$$

y is dependent verable
x is independent verable

$$y'' + 2(y'')^2 = \cos x \quad (\text{is an O.D.E})$$

y is dependent verable
x is independent verable

Partial Differential Equation (P.D.E)

Equation which involves more than one independent verable.

$$\frac{dz}{dx} + \frac{dz}{dy} = 2z \quad \text{(is a P.D.E)}$$

z is dependent verable
x and y are independent verables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = e^{xy} \quad \text{(is a P.D.E)}$$

u is dependent verable
x and y are independent verables

Difinition: The order of a differential equation is the highest order derevative presents in the D.E.

Difinition: The Degree of a differential equation is the highest power to the highest order derivative in the D.E.

Ex :

1- $\frac{dy}{dx} + 3x = 0$ order 1 Degree 1

2- $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5x = 0$ order 2 Degree 1

3- $\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + xy = 0$ order 2 Degree 3

4- $(y''')^2 + 2(y'')^2 = xy$ order 3 Degree 2

5- $\left(\frac{d^3z}{dy^3}\right)^4 + \left(\frac{dz}{dy}\right)^2 + z = 0$ order 3 Degree 4

Difinition: Any function satisfying the D.E is called solution of D.E.

Ex: show that $y = 3e^{2x} - e^{-2x}$ is a solution for the D.E $y'' - 4y = 0$

$y' = 6e^{2x} + 2e^{-2x}$

$y'' = 12e^{2x} - 4e^{-2x}$

$y'' - 4y = 0 \longrightarrow 12e^{2x} - 4e^{-2x} - 4[3e^{2x} - e^{-2x}] = 0$

$\cancel{12e^{2x}} - \cancel{4e^{-2x}} - \cancel{12e^{2x}} - \cancel{4e^{-2x}} = 0$

Ex: Is this function $y = \sin 2x$ the solution for the D.E $y'' + 4y = 0$?

$y' = 2\cos 2x$

$y'' = -4\sin 2x$

$y'' + 4y = 0 \longrightarrow -4\sin \cancel{2x} + 4\cancel{\sin} 2x = 0$

$\therefore$ the function $y = \sin 2x$ is the solution of the D.E.

Ex: Show that $y = A\cos x + B\sin x$ is a solution of the following D.E
$y'' + y = 0$.

$y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

$y'' + y = 0 \longrightarrow -A\cancel{\cos x} - B\cancel{\sin x} + \cancel{A\cos x} + \cancel{B\sin x} = 0$

Ordinary Differential Equation

- **Non-linear**
  dependent verables and their derivatives are not of degree 1
- **linear**
  Dependent veriables and their derivative are of Degree 1
  - first order
  - second and higher order

1- First order ordinary Differential Equation:

Types of first order O.D.E

- Seprable
- Homogenous
- Exact
- Linear
- Bernoulli

Separable: in this case the D.E can be written in the form integration of both sides gives the general solution.

$$\int f(x)\,dx + \int g(y)\,dy = c$$

Example 1: solve the D.E

$$(x+1)\frac{dy}{dx} = y \qquad \div\ y(x+1)$$

solution:

$$\frac{dy}{y} = \frac{dx}{x+1} \qquad \text{seprable}$$

$$\ln y = \ln(x+1) + c$$

$$y = e^{\ln(x+1) + c} = (x+1) + c$$

Example 2: solve the D.E :

$$x\,(2y-3)\,dx + (x^2+1)\,dy = 0$$

solution

$$\div\ (2y-3)(x^2+1)$$

$$\frac{x}{(x^2+1)}\,dx + \frac{dy}{(2y-3)} = 0 \qquad \text{seprable}$$

$$\frac{1}{2}\int \frac{2x}{(x^2+1)}\,dx + \frac{1}{2}\int \frac{2}{(2y-3)}\,dy = 0$$

$$\frac{1}{2}\ln|x^2+1| + \frac{1}{2}\ln|2y-3| = C$$

Example 3: Solve the D.E

$$x^2 dy = -(x^2-1)y^3 dx$$

solution:

$\div y^3 x^2$

$$\frac{dy}{y^3} = \frac{1-x^2}{x^2} dx \qquad \text{seprable}$$

$$\int y^{-3} dy = \int (x^{-2}-1) dx$$

$$-\frac{1}{2} y^{-2} = -x^{-1} - x + c$$

$$\frac{1}{2y^2} = \frac{1}{x} + x + c$$

$$y = \sqrt{\frac{1}{2(\frac{1}{x}+x)+c}}$$

Example 4: Solve the D.E

$$y' = \frac{\sec^2 y}{1+x^2}$$

solution

$$\frac{dy}{dx} = \frac{1}{\cos^2 y\,(1+x^2)}$$

$$\cos^2 y \, dy = \frac{1}{1+x^2} dx$$ seprable

$$\int \frac{1}{2}(1+\cos 2y) dy = \int \frac{1}{1+x^2} dx$$

$$\frac{1}{2} y + \frac{1}{4} \sin 2y = \tan^{-1} x + c$$

Example 5: Solve the D.E

$$y' = (y^2-1)(1+e^{-x})$$

solution:

$$\frac{dy}{dx} = (y^2-1)(1+e^{-x}) \qquad \times \frac{dx}{y^2-1}$$

$$\int \frac{1}{(y^2-1)} dy = \int (1+e^{-x}) dx$$ seprabel

$$\int \frac{1}{(y^2-1)} dy = \int \left[\frac{A}{(y-1)} + \frac{B}{(y+1)} \right] dy.$$

$$\frac{(y+1)A + (y-1)B}{(y-1)(y+1)} = \frac{1}{(y^2-1)}$$

$$Ay + A + By - B = 1$$
$$(A+B)y + A - B = 1$$

معاملات y

$$A + B = 0 \quad (1)$$
$$A - B = 1 \quad (2)$$

الثوابت

من معادلة (1) و (2)

$$A = \frac{1}{2}$$
$$B = -\frac{1}{2}$$

$$\int \frac{1}{(y^2-1)}dy = \int \left[\frac{\frac{1}{2}}{(y-1)} + \frac{-\frac{1}{2}}{(y+1)} \right] dy$$

$$\int \left[\frac{\frac{1}{2}}{(y-1)} + \frac{-\frac{1}{2}}{(y+1)} \right] dy = \int (1 + e^{-x})\, dx$$

$$\frac{1}{2} \ln|y-1| - \frac{1}{2} \ln|y+1| = x - e^{-x} + C$$

$$\ln\left|\frac{y-1}{y+1}\right| = 2(x - e^{-x} + c)$$

Example 6: solve the D.E

$$x e^{y} dy = -\frac{x^2+1}{y} dx$$

solution: $* \frac{y}{x}$

$$\underbrace{\int y e^{y} dy}_{\text{by } udv} = \int -\frac{x^2+1}{x} dx \qquad \text{seprable}$$

Let $u = y \longrightarrow du = dy$

$dv = e^{y} dy \longrightarrow v = e^{y}$

$$\int u\,dv = vu - \int v\,du$$

$$\int y e^{y} dy = y e^{y} - \int e^{y} dy$$

$$= y e^{y} - e^{y}$$

$$y e^{y} - e^{y} = -\frac{x^2}{2} + \ln x + c$$

Example 7: Solve the D-E

$$\ln x \frac{dx}{dy} = \frac{x}{y} \qquad * \frac{dy}{x}$$

solution:

$$\frac{\ln x}{x} dx = \frac{dy}{y} \qquad \text{seprable}$$

$$\int \frac{1}{x} \ln x \, dx = \int \frac{dy}{y}$$

$$\frac{(\ln x)^2}{2} = \ln y + c$$

Example 8: Solve the D-E

$$x\bar{y} = y + y^2$$

solution:

$$x \frac{dy}{dx} = y(y+1) \qquad * \quad \frac{dx}{xy(1+y)}$$

$$\int \frac{dy}{y(1+y)} = \int \frac{dx}{x} \qquad \text{seprable}$$

$$\int \frac{1}{y(1+y)}\, dy = \int \left[\frac{A}{y} + \frac{B}{(1+y)} \right] dy$$

$$\frac{1}{y(1+y)} = \frac{A(1+y) + By}{y(1+y)}$$

$$\frac{1}{y(1+y)} = \frac{A + Ay + By}{y(1+y)} = \frac{A + (A+B)y}{y(1+y)}$$

$A + B = 0$ ① معامل y

$A = 1$ ② الثوابت

عوض ② في ①

$B = -1$

$$\int \frac{1}{y}\, dy + \int \frac{-1}{1+y}\, dy = \int \frac{dx}{x}$$

$$\ln y - \ln |1+y| = \ln x + C$$

$$\ln \frac{y}{1+y} = \ln x + c$$

Example 9: solve the D.E

$$y' = e^{3x+2y}$$

subjected to the initial condition $y(0)=4$

Solution:

$$\frac{dy}{dx} = e^{3x}\, e^{2y} \qquad * \; \frac{dx}{e^{2y}}$$

$$\frac{1}{e^{2y}}\, dy = e^{3x}\, dx \qquad \text{seprable}$$

$$\int e^{-2y}\, dy = \int e^{3x}\, dx$$

$$-\frac{1}{2} e^{-2y} = \frac{1}{3} e^{3x} + c$$

$$e^{-2y} = -\frac{2}{3} e^{3x} + c$$

ناخذ ln للطرفين

$$-2y = \ln\left(-\frac{2}{3} e^{3x} + c\right)$$

$$y = -\frac{1}{2} \ln\left(-\frac{2}{3} e^{3x} + c\right)$$

$$4 = -\frac{1}{2} \ln\left(-\frac{2}{3} e^{0} + c\right)$$

$$-8 = \ln\left(-\frac{2}{3} + c\right)$$ e بالعرضية

$$e^{-8} = -\frac{2}{3} + c$$

$$c = 0.667$$

$$y = -\frac{1}{2}\ln\left(-\frac{2}{3}e^{3x} + 0.667\right)$$

Example 10 : Solve the D.E

$$\frac{dy}{dx} - x^2y^2 = x^2$$

solution :

$$\frac{dy}{dx} = x^2(1+y^2) \qquad \times \frac{dx}{1+y^2}$$

$$\int \frac{1}{1+y^2}\,dy = \int x^2\,dx$$ seprable

$$\tan^{-1} y = \frac{x^3}{3} + c$$

$$y = \tan\left[\frac{x^3}{3} + c\right]$$

Example 11: solve the D.E

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$$

subjected to the initial condition $y(0)=1$

solution:

$$\int \frac{1+2y^2}{y}\,dy = \int \cos x\,dx \qquad \text{seprable}$$

$$\ln y + y^2 = \sin x + C$$

$y=1$ at $x=0$

$$\ln(1) + (1)^2 = \sin(0) + C$$

$$C = 1$$

$$\ln y + y^2 = \sin x + C$$