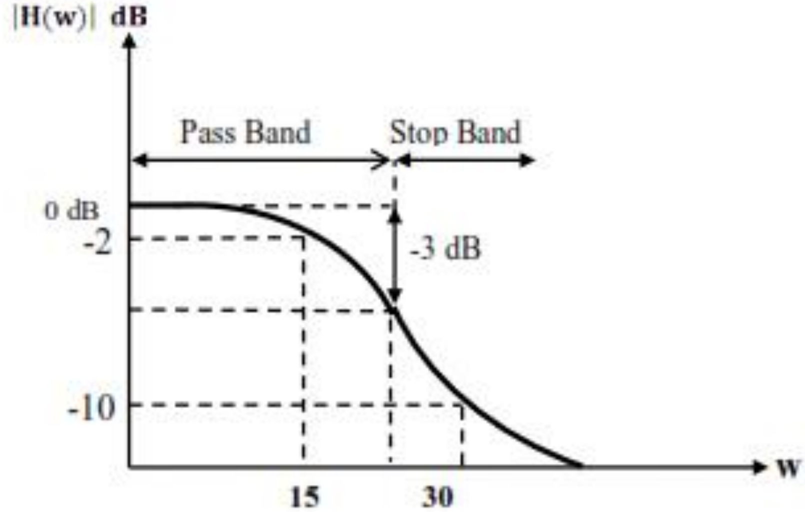
Example Design analog LPF that has pass band attenuation of 2dB at 15 rad/sec and stop band attenuation of 10 dB at 30 rad/sec.

Solution:



$$n = \frac{log_{10}\left[\frac{10^{-k_1/10}-1}{10^{-k_2/10}-1}\right]}{2\,log_{10}\frac{w_1}{w_2}} = 1.97 \cong 2$$

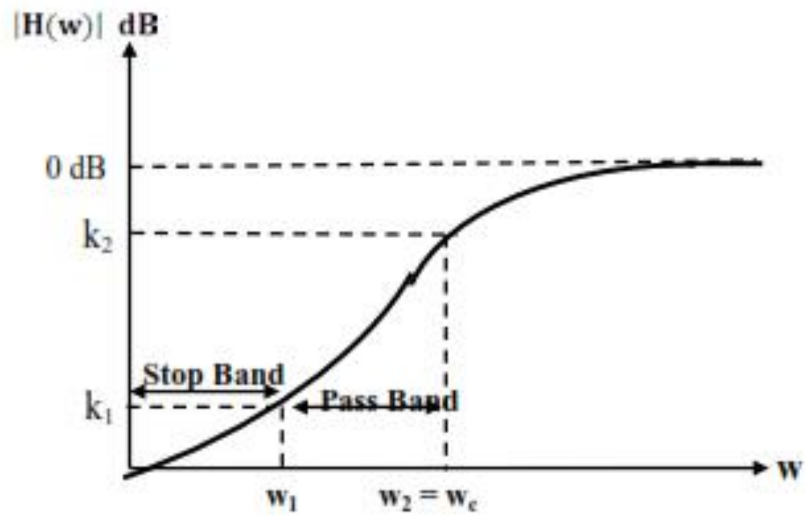
From table, $B(s) = s^2+\sqrt{2}s+1$

$$H(s) = \left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s=\frac{s}{w_c}}$$

$$w_c = \frac{w_1}{(10^{-k_1/10}-1)^{1/2n}} = \frac{15}{(10^{2/10}-1)^{1/4}} = 17.2\,rad/sec$$

$$\therefore H(s) = \frac{1}{\left(\frac{s}{17.2}\right)^2+\sqrt{2}\frac{s}{17.2}+1}$$

b) HPF (high pass filter)



Step1 from the characteristics of the filter in frequency domain, you can find the order of the filter;

$$n = \frac{\log_{10}\left[\frac{10^{-k_1/10}-1}{10^{-k_2/10}-1}\right]}{2\log_{10}\frac{w_2}{w_1}}$$

Step2 from the table, find the function B(s) related to the order n;

n	B(s)
1	s+1
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7653s+1)(s^2+1.8477s+1)$

Step3 find the cutoff frequency;

$w_c = w_2$

Step4 to find the transfer function in S-domain;

$$H(s) = \left.\frac{1}{B(s)}\right|_{s=\frac{w_c}{s}=\frac{w_2}{s}}$$

Example Design Butterworth HP analog filter having 3dB cutoff frequency of 5 rad/sec and an attenuation of 15 dB at frequency of 2 rad/sec.

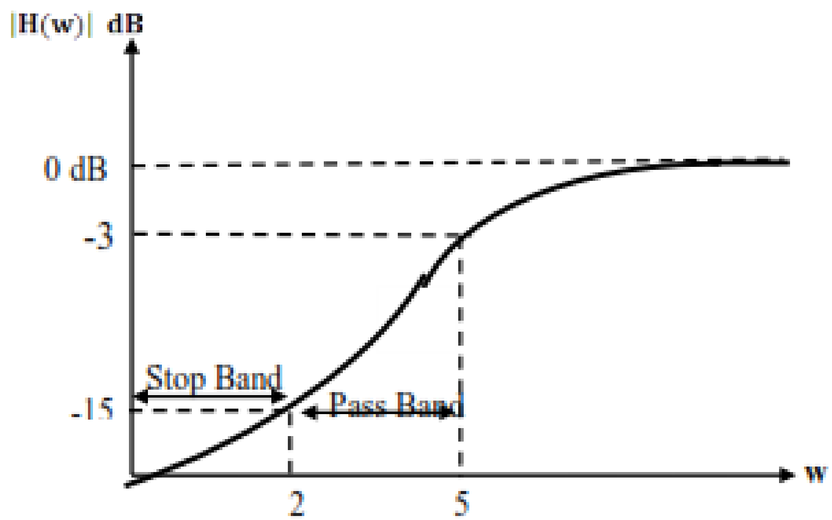
Solution:

$n = 1.87 \cong 2$

From table, $B(s) = s^2+\sqrt{2}s+1$

$$H(s) = \left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s=\frac{w_c}{s}=\frac{w_2}{s}}$$

$$\therefore H(s) = \frac{1}{\left(\frac{5}{s}\right)^2+\sqrt{2}\left(\frac{5}{s}\right)+1} = \frac{s^2}{s^2+5\sqrt{2}s+25}$$



c) BPF (band pass filter)

Step1 from the characteristics of the filter in frequency domain, you can find the order of the filter;

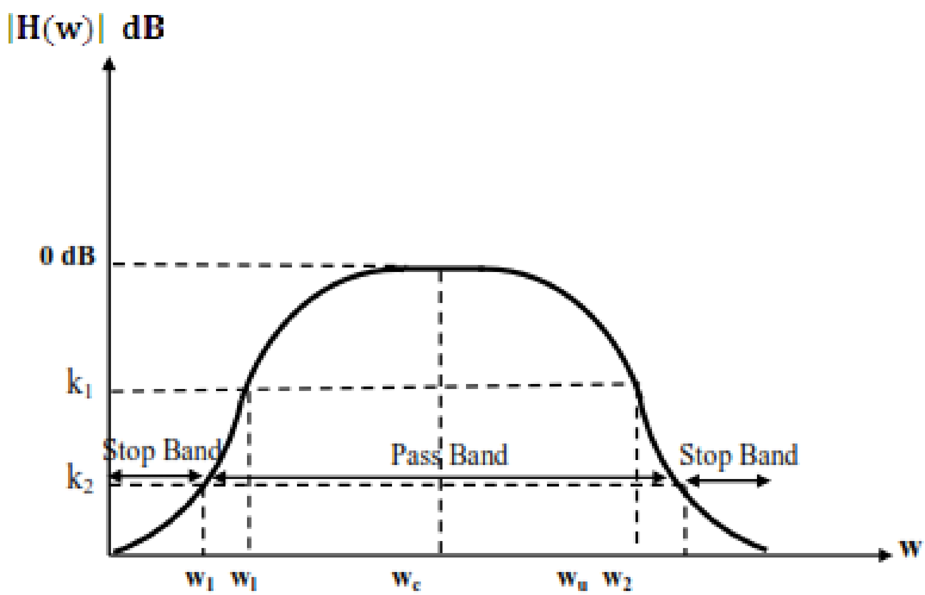
$$n = \frac{log_{10}\left[\frac{10^{-k_1/10}-1}{10^{-k_2/10}-1}\right]}{2log_{10}(P)}$$

$$P = \frac{1}{w_r}$$

$$w_r = min[|A| \; or \; |B|]$$

$$|A| = \frac{-w_1^2 + w_u w_l}{w_1(w_u - w_l)}$$

$$|B| = \frac{w_2^2 - w_u w_l}{w_2(w_u - w_l)}$$



Step2 from the table, find the function B(s) related to the order n;

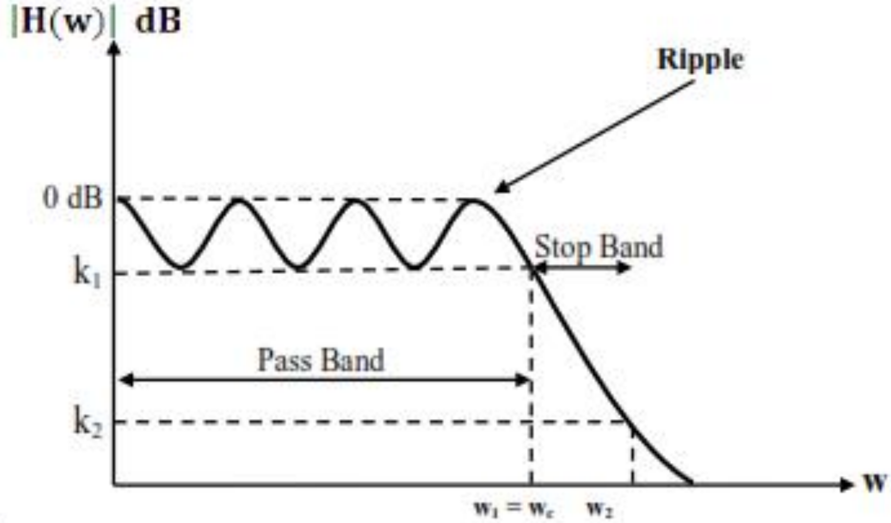
n	B(s)
1	s+1
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7653s+1)(s^2+1.8477s+1)$

Step3 to find the transfer function in S-domain;

$$H(s) = \frac{1}{B(s)}\Bigg|_{s=\frac{s^2+w_l w_u}{s(w_u-w_l)}}$$

2. Chebyshev Filter

a) LPF (Low Pass Filter)



1) $k_1 = -10log_{10}(1+\epsilon^2)$

$k_2 = -20log_{10}A$

$g = \sqrt{\frac{A^2-1}{\epsilon^2}}$

$w_r = \frac{w_2}{w_1}$

2)

$$n = \frac{log_{10}(g+\sqrt{g^2-1})}{log_{10}(w_r+\sqrt{{w_r}^2-1})}$$

3) From table (3.4), we find the coefficients b_0, b_1, b_2….

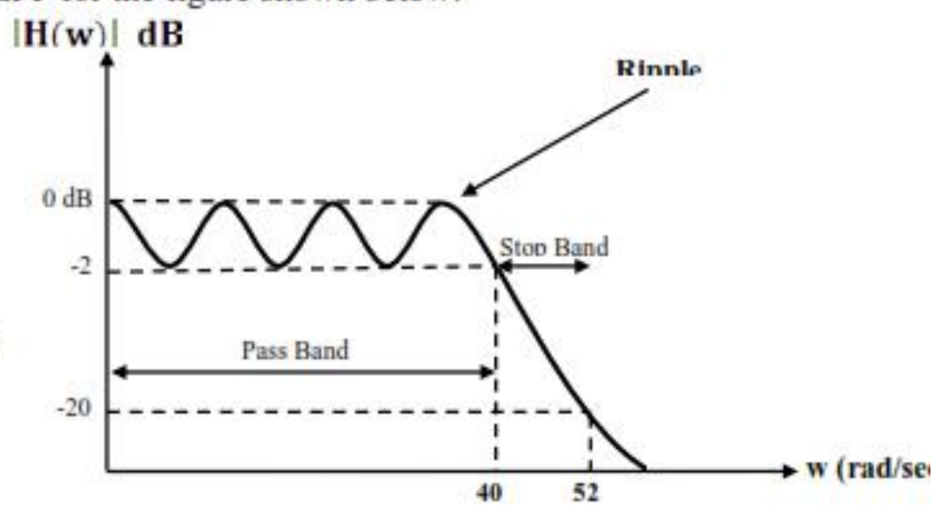
$$H(s) = \frac{k_n}{V_n(s)}$$

$$k_n = \begin{cases} \frac{b_0}{\sqrt{1+\epsilon^2}} & n\ even \\ b_0 & n\ odd \end{cases}$$

$$V_n(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots b_n s^n$$

$$H(s) = \frac{k_n}{V_n(s)}\bigg|_{s=\frac{s}{w_c}}$$

Example: Design Chebyshev LPF for the figure shown below:



Solution:

1. $k_1 = -10log_{10}(1+\epsilon^2)$

$-2 = -10log_{10}(1+\epsilon^2)$

$\epsilon = 0.7647$

$-20 = -20log_{10}A \Rightarrow A = 10$

$g = \sqrt{\frac{A^2-1}{\epsilon^2}} = 13$

$w_r = \frac{w_2}{w_1} = 1.3$

$n = 4.3 \cong 5$

$\because n = 5\ odd \Rightarrow k_n = b_0$

From table (3.4);

$V_n(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots b_5 s^5$

$$= 0.1228 + 0.5805s + 0.97439s^2 + 1.6888s^3 + 0.9368s^4 + s^5$$

$$H(s) = \frac{k_n}{V_n(s)}\bigg|_{s=\frac{s}{w_c}}$$

$$H(s) = \frac{0.1228}{0.1228 + 0.5805\left(\frac{s}{40}\right) + 0.97439\left(\frac{s}{40}\right)^2 + 1.6888\left(\frac{s}{40}\right)^3 + 0.9368\left(\frac{s}{40}\right)^4 + \left(\frac{s}{40}\right)^5}$$

b) HPF (High Pass Filter)

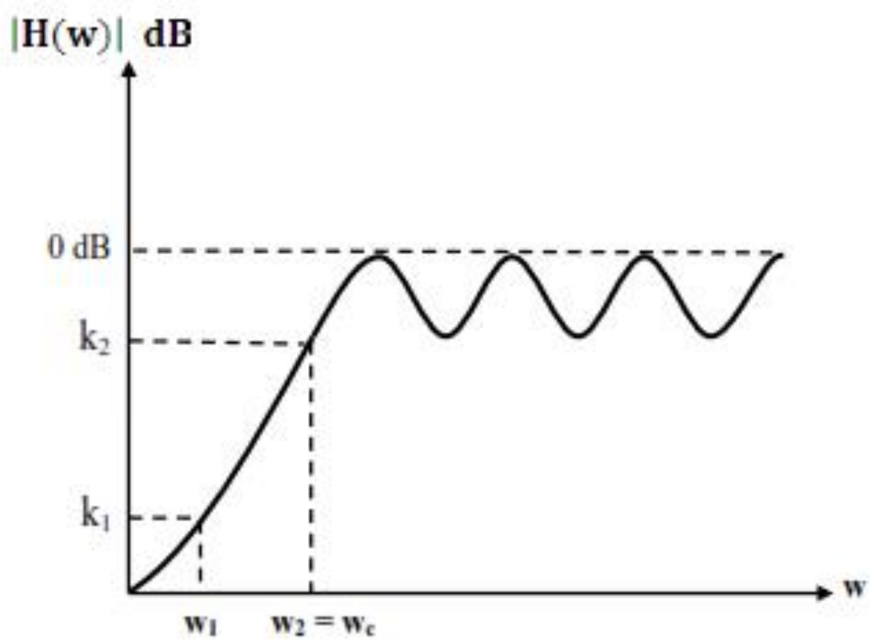
1)

$k_2 = -10log_{10}(1+\epsilon^2)$

$k_1 = -20log_{10}A$

$g = \sqrt{\frac{A^2-1}{\epsilon^2}}$

$w_r = \frac{w_2}{w_1}$



2)

$$n = \frac{log_{10}(g + \sqrt{g^2-1})}{log_{10}(w_r + \sqrt{w_r{}^2 - 1})}$$

3)

$$H(s) = \frac{k_n}{V_n(s)}\bigg|_{s=\frac{w_2}{s}}$$

c) BPF (Band Pass Filter)

1)

$$w_r = min[|A| \ or \ |B|]$$

$$|A| = \frac{-w_1^2 + w_u w_l}{w_1(w_u - w_l)}$$

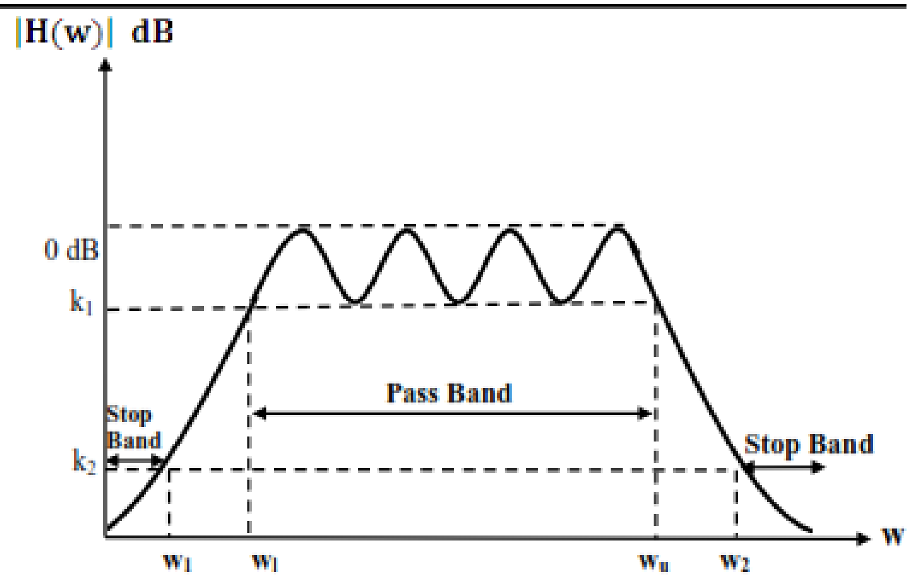
$$|B| = \frac{w_2^2 - w_u w_l}{w_2(w_u - w_l)}$$

2)

$$k_1 = -10log_{10}(1 + \epsilon^2)$$

$$k_2 = -20log_{10}A$$

$$g = \sqrt{\frac{A^2 - 1}{\epsilon^2}}$$



3)

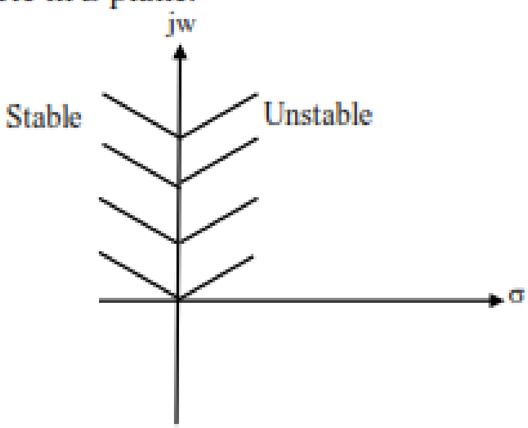
$$n = \frac{log_{10}(g + \sqrt{g^2 - 1})}{log_{10}(w_r + \sqrt{w_r^2 - 1})}$$

4)

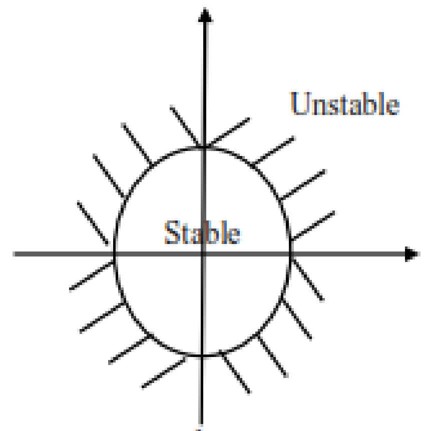
$$H(s) = \frac{k_n}{V_n(s)}\bigg|_{s=\frac{s^2 + w_l w_u}{s(w_u - w_l)}}$$

Bilinear-Transformation (IIR Digital Filter)

To design an IIR digital filter then a suitable transformation called bilinear transformation is used.
This transformation transforms the left hand side of the s-plane into interior inside of the unit circle in z-plane.



s-plane



z-plane

This bilinear transformation transforms the T.F H(s) of the analogue filter

$$s = 2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

f_s : sampling frequency
z^{-1}: delay element by T_s
if $s = jw$, $z^{-1} = e^{-j\lambda}$
where,
λ: Digital frequency (rad)
w: Analogue frequency (rad/sec)
To find a relationship between λ & w

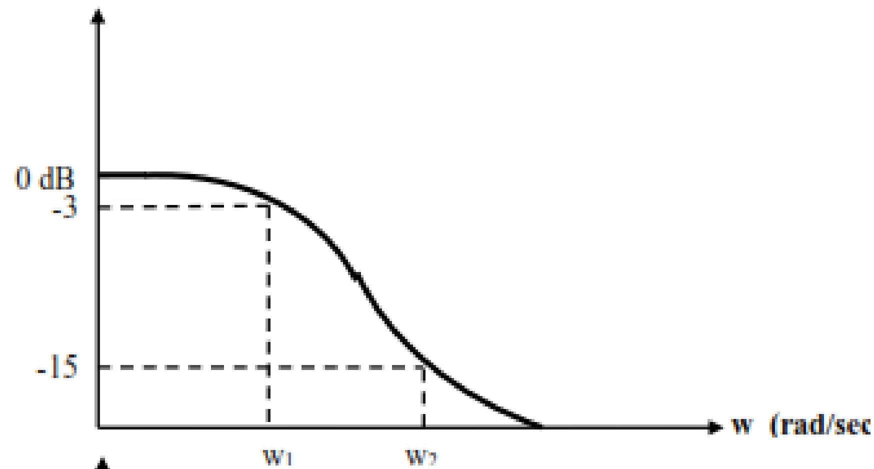
$$s = jw = \frac{2}{T_s}\left(\frac{1-e^{-j\lambda}}{1+e^{-j\lambda}}\right) \cdot \frac{e^{j\lambda/2}}{e^{j\lambda/2}}$$

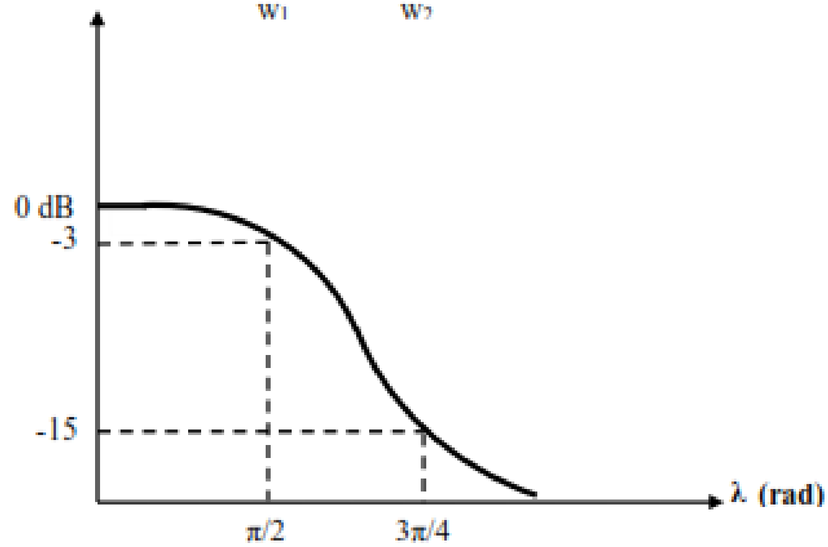
$$jw = \frac{2}{T_s}\left(\frac{e^{j\frac{\lambda}{2}} - e^{-j\frac{\lambda}{2}}}{e^{j\frac{\lambda}{2}} + e^{-j\frac{\lambda}{2}}}\right)$$

$$\therefore w = \frac{2}{T_s} tan\frac{\lambda}{2} \quad \& \quad \lambda = 2tan^{-1}\left(\frac{wT_s}{2}\right)$$

Example: using bilinear transformation design and realize a digital LPF having the following c/cs:
a) Monotone pass band
b) 3 dB cutoff frequency of π/2 rad.
c) Stop band attenuation of 15 dB at 3π/4 rad. (T_s = 1 sec).

Solution:





$$w = \frac{2}{T_s} tan\frac{\lambda}{2}$$

$$w_1 = 2tan\frac{\pi}{4} = 2\ rad/sec$$

$$w_2 = 2tan\frac{3\pi}{8} = 4.8\ rad/sec$$

Monotone →Butterworth

$$n = \frac{log_{10}\left[\frac{10^{3/10}-1}{10^{15/10}-1}\right]}{2log_{10}\frac{2}{4.8}} = 2$$

From table, $B(s) = s^2+\sqrt{2}s+1$

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1}\bigg|_{s=\frac{s}{w_c}}$$

$$w_c = \frac{w_1}{(10^{-k_1/10}-1)^{1/2n}} = 2\ rad/sec$$

$$\therefore H(s) = \frac{1}{\left(\frac{s}{2}\right)^2+\sqrt{2}\frac{s}{2}+1}$$

To design digital filter

$$s = 2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$T_s = 1 \rightarrow f_s = 1$ Hz

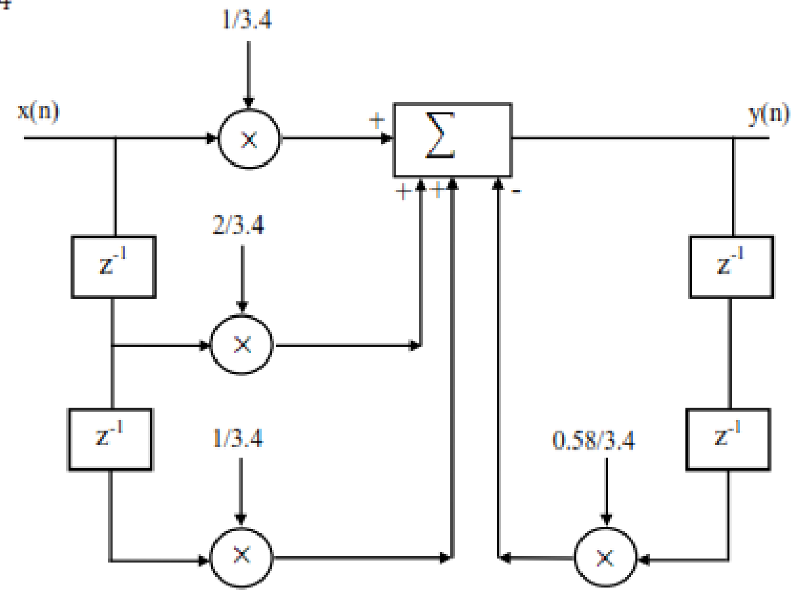
$$s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore H(z) = \frac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+\sqrt{2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1}$$

$$= \frac{1 + 2z^{-1} + z^{-2}}{1 - 2z^{-1} + z^{-2} + \sqrt{2}(1 - z^{-2}) + 1 + 2z^{-1} + z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{3.4 + 0.58z^{-2}}$$

$$\therefore y(n) = \frac{1}{3.4}[x(n) + 2x(n-1) + x(n-2) - 0.58y(n-2)]$$



Example:

(A) Design a digital IIR filter by means of the bilinear transformation from the analog filter with system function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

with a sampling time T =0.5 s.

(B) Realize the designed digital IIR in direct form II architecture

Solution:

Since the sampling time T =0.5 s, thus the desired mapping is

$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

Substitute every s-operator by the above mapping, thus

$$H(z) = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left(4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right)^2+16} = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.0006z^{-1}+0.975z^{-2}}$$

Note that the coefficient of the z^{-1} term in the denominator of $H(z)$ is extremely small and can be approximated by zero. Thus the system function will be

$$H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.975z^{-2}}$$

By long division the IIR filter coefficients can be found:

$H(z) = 0.128 + 0.006z^{-1} - 1.1z^{-2} - z^{-3} + 1.1z^{-4} + z^{-5}$ ...

Therefore, the digital IIR low-pass Butterworth filter has the impulse response $h(n)$ as the coefficients of the system function $H(z)$:

$$h(n) = \{0.128, 0.006, -1.1, -1, +1.1, 1, \ldots\}$$

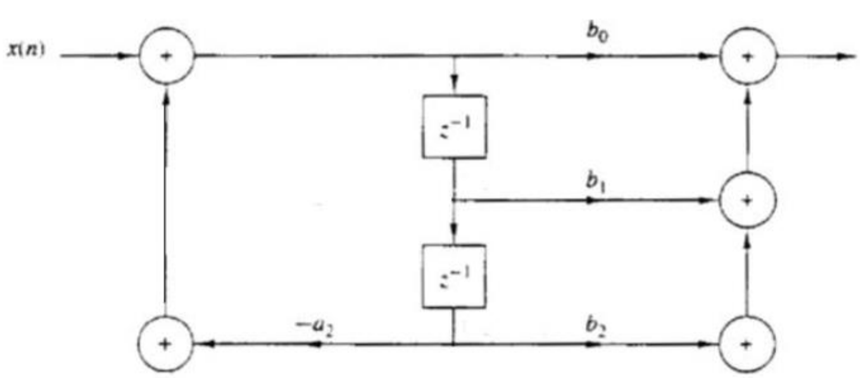
Which can be checked by determining the difference equation of the digital IIR filter

$$H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.975z^{-2}} = \frac{Y(z)}{X(z)}$$

$$(1+0.975z^{-2})Y(z) = (0.128+0.006z^{-1}-0.122z^{-2})X(z)$$

$$y(n) = -0.975y(n-2) + 0.128x(n) + 0.006x(n-1) - 0.122x(n-2)$$

(A) Realize the designed digital IIR in direct form II architecture



Where, b_0=0.128, b_1=0.006, b_2= - 0.122 and $-a_2$= - 0.975

Design by Approximation of Derivatives (backward difference):
Perhaps the simplest method for low-order systems is to use backward-difference approximation to continuous domain derivatives.
The analog filter can be described by the linear constant-coefficient differential equation

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

Where, *x(t)* denotes the input signal and y(t) denotes the output of the filter.
The derivative *dy(t)/dt* at time *t = n T* can be replaced by the difference equation

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT - T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T}$$

The analog differentiator with output *dy(t)/dt* has the system function *H(s) = s,* while the digital system that produces the output [y(n) — y(n — 1)]/T has the system function $H(z) = (1 - z^{-1})/T$. Consequently, the frequency-domain equivalent for the relationship

$$s = \frac{1 - z^{-1}}{T}$$

The second derivative $d^2y(t)/dt^2$ is replaced by the second difference, which is equivalent in the frequency domain to:

$$s^2 = \frac{1 - 2z^{-1} + z^{-2}}{T^2} = \left(\frac{1 - z^{-1}}{T}\right)^2$$

It easily follows from the discussion that the substitution for the k^{th} derivative of *y(t)* results in the equivalent frequency-domain relationship

$$s^k = \left(\frac{1 - z^{-1}}{T}\right)^k$$

Consequently, the system function for the digital IIR filter obtained as a result of the approximation of the derivatives by finite differences is

$$H(z) = H_a(s)|_{s=(1-z^{-1})/T}$$

Where $H_a(s)$ is the system function of the analog filter characterized by the differential Equation

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}$$

Example

Design a digital IIR filter, with a sampling time T =0.1 s, which is equivalent to the analog band-pass filter with system function

$$H_a(s) = \frac{1}{(s+0.1)^2+9}$$

by use of the backward difference for the derivative.

Solution:

Substitution for $s = \frac{1-z^{-1}}{T} = 10 - 10z^{-1}$ into *H(s)* yields

$$H(z) = \frac{1}{(10-10z^{-1}+0.1)^2+9}$$

$$= \frac{1}{(10.01-10z^{-1})^2+9} = \frac{1}{109.2-200.2z^{-1}+100z^{-2}}$$

$$H(z) = \frac{0.009}{1-1.8z^{-1}+0.92z^{-2}} = \frac{Y(z)}{X(z)}$$

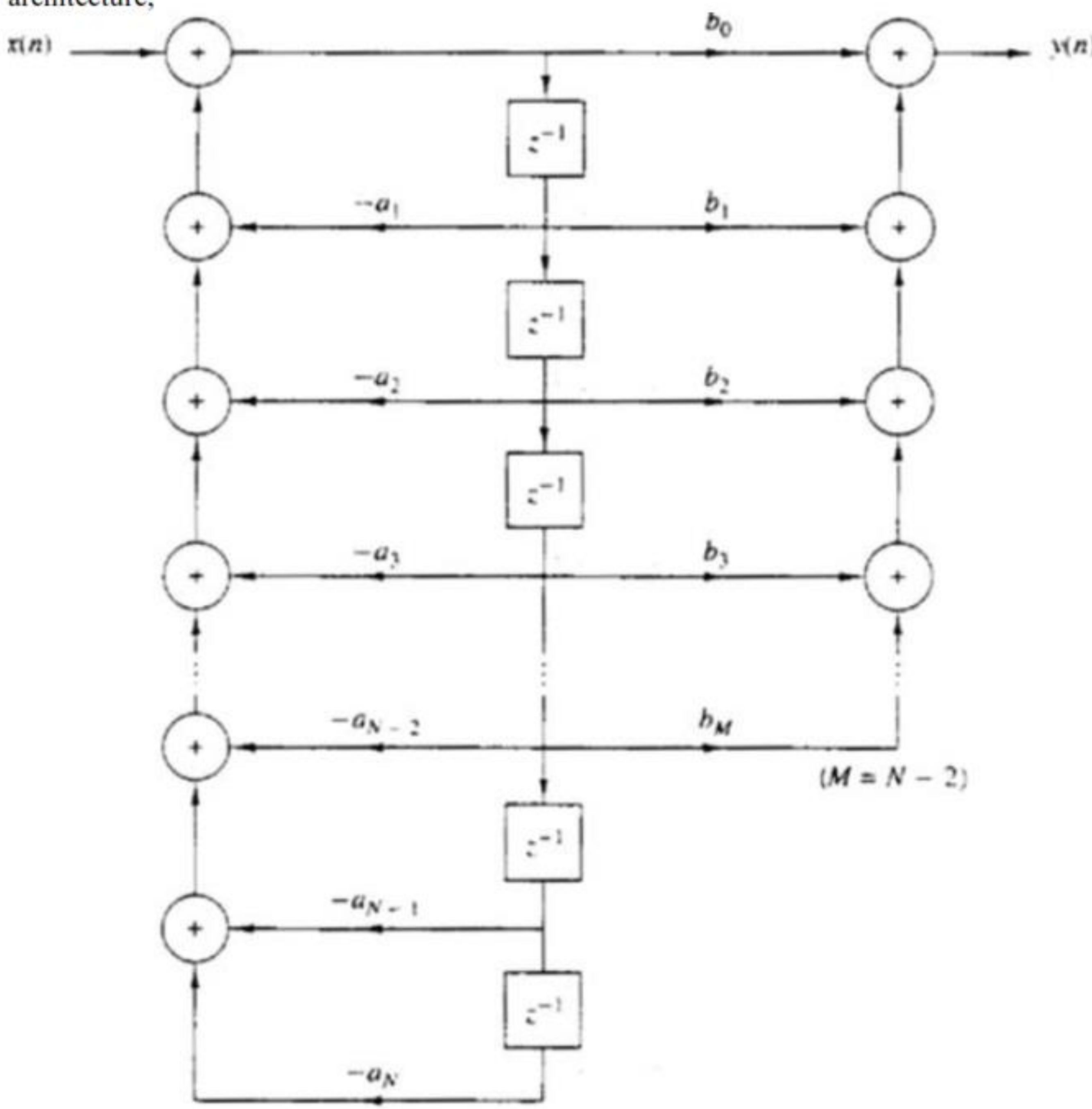
$$(1-1.38z^{-1}+0.92z^{-2})Y(z) = 0.009X(z)$$

$$y(n) = 1.38y(n-1) - 0.92y(n-2) + 0.009x(n)$$

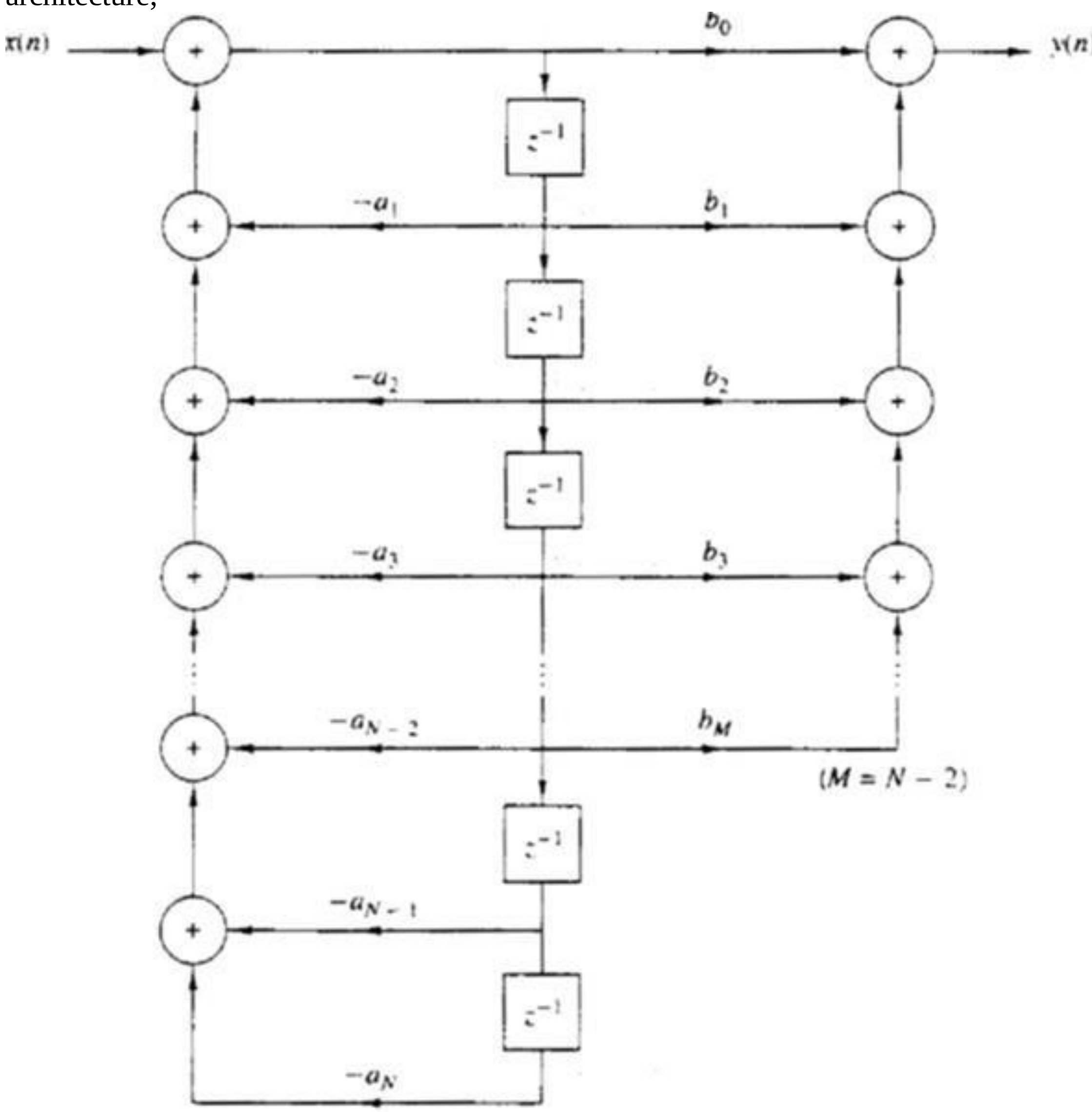
$$H(z) = \frac{1}{(1-(-1-j0.3)z^{-1})(1-(-1+j0.3)z^{-1})}$$

$$H(z) = \frac{1}{(1-(-0.949e^{+j16.5})z^{-1})(1-(-0.949e^{-j16.5})z^{-1})}$$

The designed digital IIR filter can be implemented in hardware as direct form II architecture;



The designed digital IIR filter can be implemented in hardware as direct form II architecture;



Example

Design a digital IIR filter, with a sampling time T =0.5 s, which is equivalent to the analog low-pass Butterworth filter with cut off frequency Ωc = 1 rad/s with system function

$$H_p(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

by use of the backward difference for the derivative.

Solution:

Substitution for $s = \frac{1-z^{-1}}{T} = 2 - 2z^{-1}$ into *H(s)* yields

$$H(z) = \frac{1}{(2-2z^{-1})^2 + \sqrt{2}(2-2z^{-1}) + 1}$$
$$= \frac{0.25}{1.957 - 2.707z^{-1} + z^{-2}}$$

By long division the IIR filter coefficients can be found:

$$H(z) = 0.128 + 0.177z^{-1} + 0.18z^{-2} + 0.157z^{-3} - 0.793z^{-4} \ldots$$

Therefore, the digital IIR low-pass Butterworth filter has the impulse response *h(n)* as the coefficients of the system function *H(z)*:

$$h(n) = \{0.128, 0.177, 0.18, 0.157, -0.793, \ldots\}$$

Which can be checked by determining the difference equation of the digital IIR filter

$$H(z) = \frac{0.128}{1 - 1.38z^{-1} + 0.51z^{-2}} = \frac{Y(z)}{X(z)}$$
$$(1 - 1.38z^{-1} + 0.51z^{-2})Y(z) = 0.128X(z)$$
$$y(n) = 1.38y(n-1) - 0.51y(n-2) + 0.128x(n)$$

To study the stability of the digital filter, the System function can be factorized into

$$H(z) = \frac{1}{(1-(-1-j0.3)z^{-1})(1-(-1+j0.3)z^{-1})}$$
$$H(z) = \frac{1}{(1-(-e^{+j16.5})z^{-1})(1-(-e^{-j16.5})z^{-1})}$$

Then the system is at the verge of stability!

The IIR filter design technique (Approximation of Derivatives) have a severe limitation of being appropriate only for low pass filters and a limited class of band-pass filters.

6.5.1 Design by the Bilinear Transform:

The **bilinear transformation** is a conformal mapping from the s-plane to the z-plane. That transforms the jΩ-axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Furthermore, all points in the LHP of *s* are mapped inside the unit circle in the z-plane and all points in the RHP of *s* are mapped in to corresponding points outside the unit circle in the z-plane.

The ideal mapping of a prototype analog filter to the z-plane is

$$H_p(s) \longrightarrow H(z)|_{z=e^{sT}} \Rightarrow s \longrightarrow \frac{1}{T}\ln(z) \Rightarrow H(z) = H_p(s)|_{s=\frac{1}{T}\ln(z)}$$

The Laurent series expansion for *ln(z)* is

$$\ln(z) = 2\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \cdots\right] \qquad \text{for } \Re\{z\} \geq 0, z \neq 0$$

The bilinear transform method uses the truncated series approximation

$$s \longrightarrow \frac{1}{T}\ln(z) \approx \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

In a more general sense, any transformation of the form

$$s = A\left(\frac{z-1}{z+1}\right) \quad \text{which implies} \quad z = -\left(\frac{s+A}{s-A}\right)$$

is a bilinear transform. In particular, when $A = 2/T$ the method is known as Tustin's method.

With this transformation the digital filter is designed from the prototype using

$$H(z) = H_p(s)\big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

Example:

(A) Design a digital IIR filter by means of the bilinear transformation from the analog filter with system function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

with a sampling time T =0.5 s.

(B) Realize the designed digital IIR in direct form II architecture

Solution:

Since the sampling time T =0.5 s, thus the desired mapping is

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Substitute every s-operator by the above mapping, thus

$$H(z) = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left(4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right)^2+16} = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.0006z^{-1}+0.975z^{-2}}$$

Note that the coefficient of the z^{-1} term in the denominator of *H(z)* is extremely small and can be approximated by zero. Thus the system function will be

$$H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.975z^{-2}}$$

By long division the IIR filter coefficients can be found:

$H(z) = 0.128 + 0.006z^{-1} - 1.1z^{-2} - z^{-3} + 1.1z^{-4} + z^{-5}$ ...

Therefore, the digital IIR low-pass Butterworth filter has the impulse response *h(n)* as the coefficients of the system function *H(z)*:

$$h(n) = \{0.128, 0.006, -1.1, -1, +1.1, 1, ...\}$$

Which can be checked by determining the difference equation of the digital IIR filter

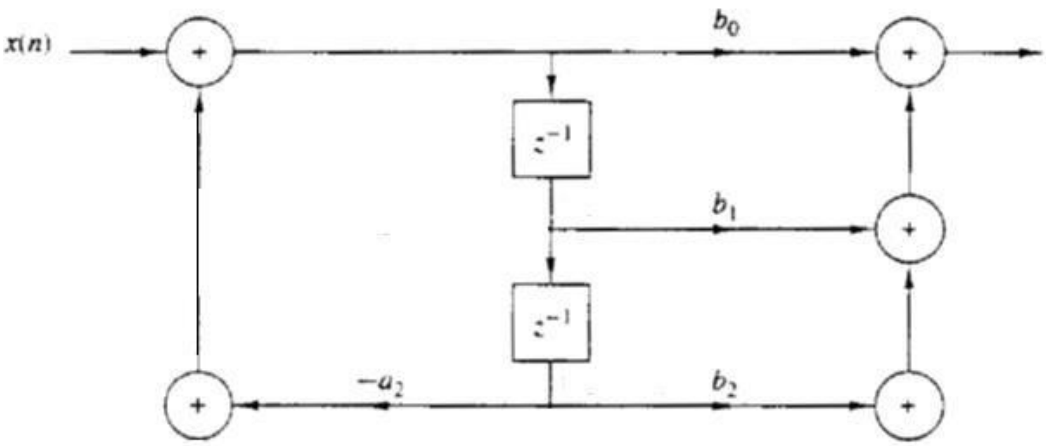
$$H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.975z^{-2}} = \frac{Y(z)}{X(z)}$$

$$(1+0.975z^{-2})Y(z) = (0.128+0.006z^{-1}-0.122z^{-2})X(z)$$

$$y(n) = -0.975y(n-2) + 0.128x(n) + 0.006x(n-1) - 0.122x(n-2)$$

(A) Realize the designed digital IIR in direct form II architecture

Where



Where, b_0=0.128, b_1=0.006, b_2= - 0.122 and $-a_2$= - 0.975

HW:

(A) Design a digital IIR low pass filter by means of the bilinear transformation from the analog Butterworth filter with system function

$$H(s) = \frac{0.1716}{s^2 + 0.5858s + 1.7577}$$

with a sampling time T =2 s.

(B) Realize the designed digital IIR in direct form II architecture

Digital FIR filter Design

The frequency response of an ideal low-pass filter with linear phase and a cutoff frequency w_c is given by;

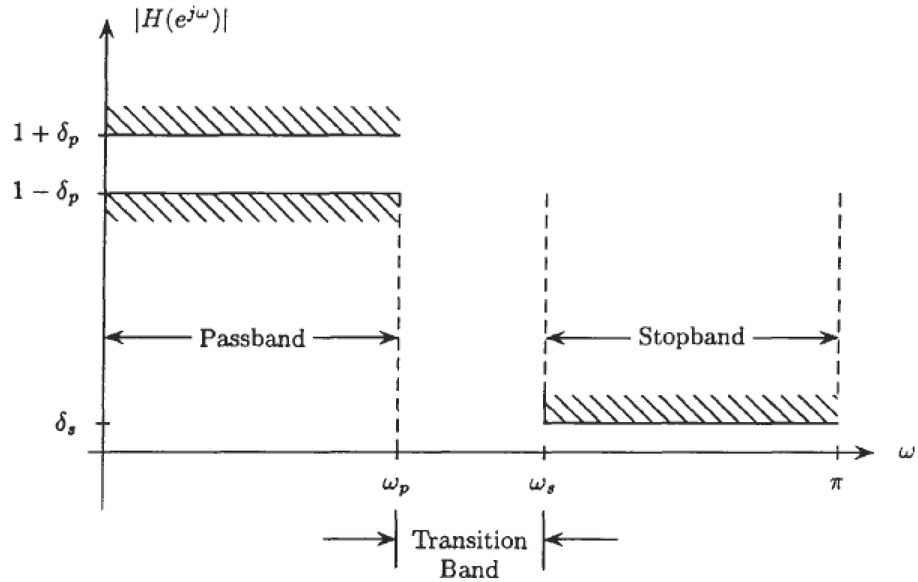
$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

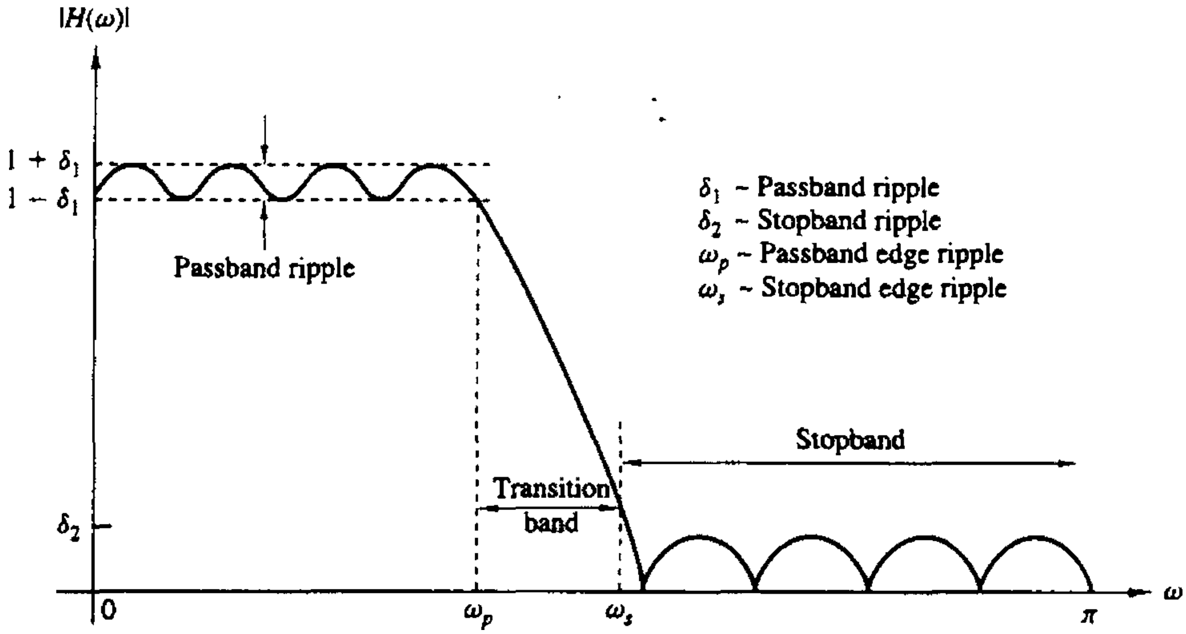
$$h_d(n) = \frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)}$$

Where ***d*** stands for the desired response and α is constant representing the center point of the response in time.
Because this filter is unrealizable (non-causal and unstable), it is necessary to relax the ideal constraints on the frequency response and allow some deviation from the ideal response. The specifications for a low-pass filter will typically have the form;

$$1-\delta_p < |H(e^{j\omega})| \le 1+\delta_p \qquad 0 \le |\omega| < \omega_p$$

$$|H(e^{j\omega})| \le \delta_s \qquad \omega_s \le |\omega| < \pi$$





Once the filter specifications have been defined, the next step is to design a filter that meets these specifications. The frequency response of an N[th] order causal FIR filter is;

$$H(e^{j\omega}) = \sum_{n=0}^{N} h(n)e^{-j\omega n}$$

The design of a FIR filter involves finding the coefficients *h(n)* that result in a frequency response that satisfies a given set of filter specifications. FIR filters have two important advantages over IIR filters. **First,** FIR filters are guaranteed to be **stable**, even after the filter coefficients have been quantized. **Second,** FIR filters may be easily constrained to have (generalized) linear phase. Because FIR filters are generally designed to have **linear phase**, in the following we consider the design of linear phase FIR filters.

FIR Filter Design (Linear Phase Condition)

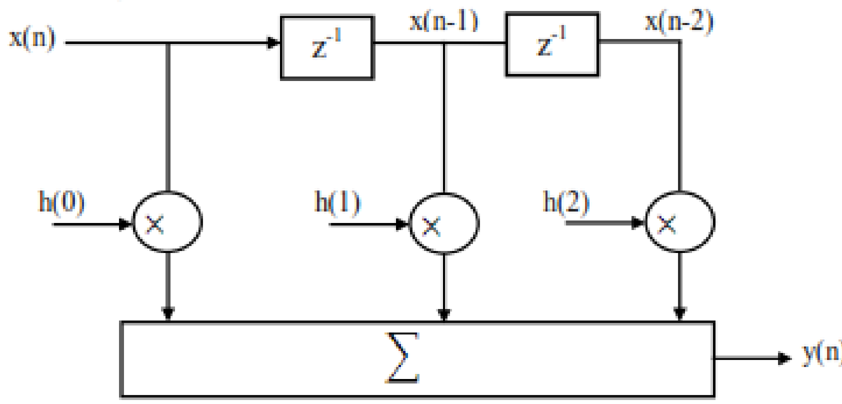
For a certain FIR filter to be linear phase, the following equations must be satisfied:

$h(n) = h(N-1-n) \qquad n = 0,1,2,\dots N-1$

And N is odd (N: number of coefficient)

- For N = 3

h(n) = h(2-n)
h(0) = h(2)
h(1) = h(1)
h(2) = h(0)



Procedures:

1) Convert the units of w from rad/sec to rad by:

$$w_{1new} = w_1 \cdot T_s = w_1 \cdot \frac{1}{f_s}$$

$$w_{2new} = w_2 \cdot T_s = w_2 \cdot \frac{1}{f_s}$$

$$w_c = w_{1new}$$

2)

$$N = \frac{2\pi k}{w_{2new} - w_{1new}}$$

N: number of samples must be odd number

3)

$$\alpha = \frac{N-1}{2}$$

4)

$$h(n) = \frac{\sin[w_c(n-\alpha)]}{\pi(n-\alpha)} \cdot w_n$$

5) We can find w_n from table (1)

Note:

- If N =2 become N = 3 (odd)
- If N = 53.2 become N = 55 (odd)

TABLE 8.1 WINDOW FUNCTIONS FOR FIR FILTER DESIGN

Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \dfrac{M-1}{2}\right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{M-1}{2}\right)^2 - \left(n - \dfrac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\dfrac{M-1}{2}\right)\right]}$
Lanczos	$\left\{\dfrac{\sin\left[2\pi\left(n - \dfrac{M-1}{2}\right)\Big/(M-1)\right]}{2\pi\left(n - \dfrac{M-1}{2}\right)\Big/\left(\dfrac{M-1}{2}\right)}\right\}^L \quad L > 0$
Tukey	$1, \quad \left\lvert n - \dfrac{M-1}{2}\right\rvert \le \alpha\dfrac{M-1}{2} \qquad 0 < \alpha < 1$ $\dfrac{1}{2}\left[1 + \cos\left(\dfrac{n - (1+\alpha)(M-1)/2}{(1-\alpha)(M-1)/2}\pi\right)\right]$ $\alpha(M-1)/2 \le \left\lvert n - \dfrac{M-1}{2}\right\rvert \le \dfrac{M-1}{2}$

Window	Factor (k)	w_n	
Rectangular	2	1	$0 \le n \le N1$
		0	elsewhere
Barlet	4	$\frac{n}{\alpha}$	$0 \le n \le \alpha$
		$2-\frac{n}{\alpha}$	$\alpha < n \le N-1$
		0	elsewhere
Hanning	4	$0.5-0.5\cos\left(\frac{2\pi n}{N-1}\right)$	$0 \le n \le N-1$
		0	elsewhere
Hamming	4	$0.54-0.46\cos\left(\frac{2\pi n}{N-1}\right)$	$0 \le n \le N-1$
		0	elsewhere
Blackman	6	$0.42-0.5\cos\left(\frac{2\pi n}{N-1}\right)+0.08\cos\left(\frac{4\pi n}{N-1}\right)$	$0 \le n \le N-1$
		0	elsewhere

Example: design LP digital filter having linear phase (FIR) with 3 dB cutoff frequency of 1 KHz and stop band attenuation of 28 dB at 2 KHz. Use f_s = 8 KHz and Hanning window.

Solution:

$$w_{1new} = 2\pi \times 10^3 \times \frac{1}{8\times 10^3} = 0.7854 = 0.25\pi \quad rad$$

$$w_{2new} = 4\pi \times 10^3 \times \frac{1}{8\times 10^3} = 1.57 = 0.5\pi \quad rad$$

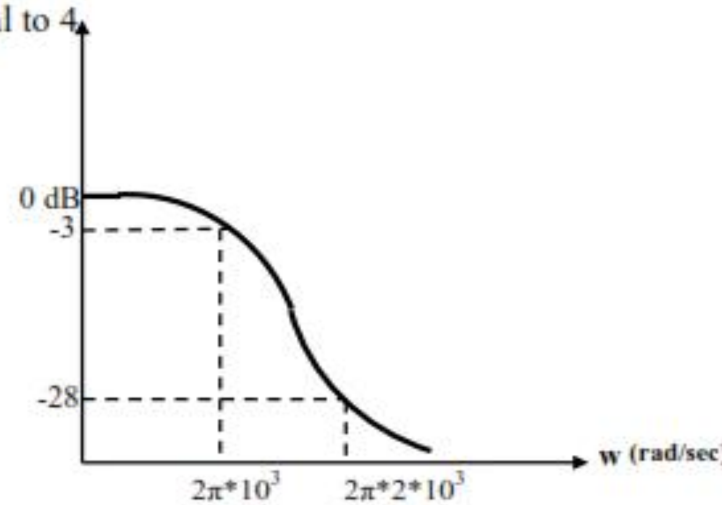
$$w_c = w_{1new} = 0.25\pi \quad rad$$

From table (1) the k for Hanning window equal to 4

$$N = \frac{2\pi \times 4}{0.5\pi - 0.25\pi} = 32 \cong 33\ (odd)$$

$$\alpha = \frac{N-1}{2} = \frac{33-1}{2} = 16$$

$$w(n) = \begin{cases} 0.5-0.5\cos\left(\frac{\pi n}{16}\right) & 0 \le n \le 32 \\ 0 & \text{elsewhere} \end{cases}$$



$$h(n) = \frac{sin[w_c(n-\alpha)]}{\pi(n-\alpha)} \cdot w_n$$

$$h(n) = \frac{sin[0.25\pi(n-16)]}{\pi(n-16)} \cdot (0.5 - 0.5cos\left(\frac{\pi n}{16}\right))$$

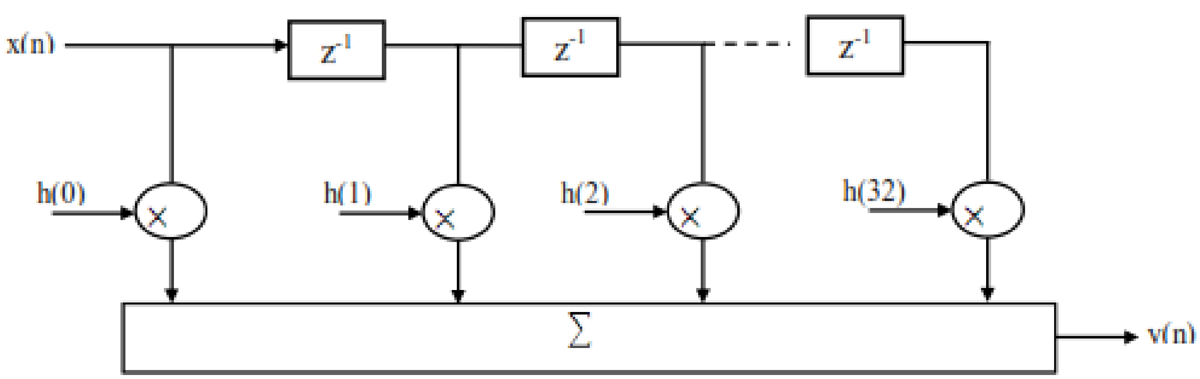
h(n) = h(N-1-n) = h(32-n)

h(0) = h(32), h(1) = h(31), h(2) = h(30), h(3) = h(29), h(4) = h(28), h(5) = h(27), h(6) = h(26), h(7) = h(25), h(8) = h(24), h(9) = h(23), h(10) = h(22), h(11) = h(21), h(12) = h(20), h(13) = h(19), h(14) = h(18), h(15) = h(17), h(16) = h(16), h(17) = h(15), h(18) = h(14), h(19) = h(13), h(20) = h(12), h(21) = h(11), h(22) = h(10), h(23) = h(9), h(24) = h(8), h(25) = h(7), h(26) = h(6), h(27) = h(5), h(28) = h(4), h(29) = h(3), h(30) = h(2), h(31) = h(1), h(32) = h(0).

n	0	1	2	3	4	5	6
h(n)	0	$-1.44*10^{-4}$	$-8.65*10^{-4}$	$-1.459*10^{-3}$	0	$4.547*10^{-3}$	$9.825*10^{-3}$

n	7	8	9	10	11	12	13
h(n)	$10*10^{-3}$	0	$-19*10^{-3}$	$-30*10^{-3}$	$-35*10^{-3}$	0	$63*10^{-3}$

n	14	15	16
h(n)	$153.1*10^{-3}$	$223*10^{-3}$	0.25



Digital FIR Filter Design Procedure Using a Fixed Window:

The only design parameters available when using a fixed window are

(1) The low-pass **cut-off frequency** ω_c,

(2) The choice of **window type**, and

(3) The **filter length** *M*.

Once these choices are made, the **Design Procedure** is as follows

(a) Form the samples of the ideal low-pass filter of length *M*.

$$h_d(n) = \frac{\sin(w_c n)}{\pi n} \qquad for \quad -\frac{M-1}{2} \le n \le \frac{M-1}{2}$$

(b) Form the length *M* window *w(n)* of the chosen type.
(c) Form the impulse response *h(n)* where $h(n)=h_d(n)w(n)$

(d) Shift all samples to the right by $\frac{M-1}{2}$ samples.

Example: Design a low-pass FIR filter of (41) length with cut-off frequency $\omega_c = 0.4\pi$ using a Hamming window, $(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1}; 0 \le n \le M-1$.

Solution:

The available design parameters; (1) the low-pass cut-off frequency $\omega_c = 0.4\pi$, (2) Hamming window, and (3) the filter length M =41. Once these choices are made, the procedure is as follows

(a) Form the samples of the ideal low-pass filter of length M = 41.

$$h_d(n) = \frac{\sin(w_c n)}{\pi n} \quad for \quad -\frac{M-1}{2} \le n \le \frac{M-1}{2}$$

$$h_d(n) = \frac{\sin(0.4\pi n)}{\pi n} \quad for \quad -20 \le n \le 20$$

(b) Form the length M window *w(n)* of the chosen type.

$$w(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1} = 0.54 - 0.46\cos\frac{2\pi n}{40}$$

(c) Form the impulse response *h(n)* where $h(n)=h_d(n)w(n)$

$$h(n) = \frac{\sin(0.4\pi n)}{\pi n}\left(0.54 - 0.46\cos\frac{2\pi n}{40}\right)$$

(d) Shift all samples to the right by $\frac{M-1}{2} = 20$samples.

$$h(n) = \frac{\sin 0.4\pi(n-20)}{\pi(n-20)}\left(0.54 - 0.46\cos\frac{2\pi}{40}(n-20)\right) \quad for\ 0 \le n \le 40$$

Evaluate the impulse response of the digital band-stop FIR filter with $h(20) = \frac{w_c}{\pi}$

$$h(n) = \{0, -0.0158, -0.0102, 0.0105, 0.0173, 0, -0.0175, -0.0108, 0.0106, 0.0168, 0, -0.0157, 0.0093, 0.0089, 0.0136, 0, -0.0127, -0.0081, 0.0096, 0.0259, \mathbf{0.4}, 0.0259, 0.0096, -0.0081, -0.0127, 0, 0.0136, 0.0089, -0.0093, -0.0157, 0, 0.0168, 0.0106, -0.0108, -0.0175, 0, 0.0173, 0.0105, -0.0102, -0.0158, 0\}$$

Example: Design a linear phase low-pass FIR filter of a length 31 with the specifications:

$$0.99 \le |H(e^{j\omega})| \le 1.01 \qquad 0 \le |\omega| \le 0.19\pi$$
$$|H(e^{j\omega})| \le 0.01 \qquad 0.21\pi \le |\omega| \le \pi$$

$$1-\delta_p < |H(e^{j\omega})| \le 1+\delta_p \qquad 0 \le |\omega| < \omega_p$$
$$|H(e^{j\omega})| \le \delta_s \qquad \omega_s \le |\omega| < \pi$$

Use a Bartlett window, $w(n) = 1 - \frac{2\left|n - \frac{M-1}{2}\right|}{M-1}; 0 \le n \le M-1.$

Solution:

$w_c = (w_s + w_p)/2 = 0.2\pi$.

(a) Form the samples of the ideal low-pass filter of length *M* = 31.

$$h_d(n) = \frac{\sin(w_c n)}{\pi n} \quad for \quad -\frac{M-1}{2} \le n \le \frac{M-1}{2}$$

$$h_d(n) = \frac{\sin(0.2\,\pi\, n)}{\pi n} \quad for \quad -15 \le n \le 15$$

(b) Form the length *M* window *w(n)* of the chosen type.

$$w(n) = 1 - \frac{2\left|n - \frac{M-1}{2}\right|}{M-1} = 1 - \frac{2}{30}|n-15|\text{-}$$

(c) Form the impulse response *h(n)* where $h(n)=h_d(n)w(n)$

$$h(n) = \frac{\sin(0.2\,\pi\, n)}{\pi n}\left(1 - \frac{2}{30}|n-15|\right)$$

(d) Shift all samples to the right by $\frac{M-1}{2} = 15$ samples.

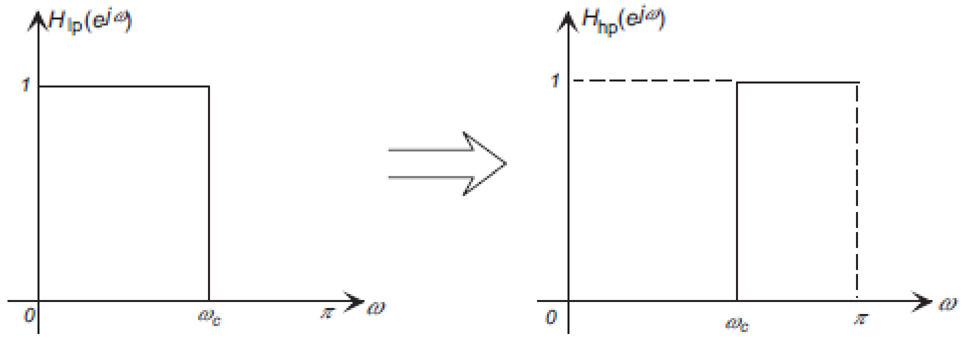
$$h(n) = \frac{\sin 0.2\,\pi(\,n-15)}{\pi(\,n-15)}\left(1 - \frac{2}{15}|n-30|\right)$$

Evaluate the impulse response of the digital band-stop FIR filter, but $h(15) = \frac{w_c}{\pi}$

High-Pass Filter:

Given an ideal low-pass filter $H_{lp}(e^{j\omega})$, a high-pass filter $H_{hp}(e^{j\omega})$ may be created:

$H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$



Then the impulse response of the filter is

$$\{h_{hp}(n)\} = \text{IDFT}\{1\} - \text{IDFT}\{H_{lp}(e^{j\omega})\}$$
$$= \delta(n) - \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n}$$

After windowing to a length *M*

$$h(n) = w(n)\left(\delta(n) - \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n}\right), \qquad |n| \le M\text{-}1/2.$$

The impulse response is then shifted to the right by $\frac{M-1}{2}$ samples to make it causal as before.

Example:

(A) Design a high-pass FIR filter with total number of filter coefficients = 11and cut-off low-pass frequency ωc =0.25π using a Rectangular window, $w(n) = 1; 0 \le n \le M-1$.

(B) Implement the designed digital FIR filter in hardware.

Solution:

(A) The available design parameters; (1) the low-pass cut-off frequency $\omega_c = 0.25\pi$, (2) Rectangular window, $w(n) = 1; 0 \leq n \leq 10$, and (3) the filter length M =11. Then, the Design procedure is as follows

(a) Form the samples of the ideal high-pass filter of length M = 11.

$$h_d(n) = \delta(n) - \frac{\sin(w_c n)}{\pi n} \qquad for \quad -\frac{M-1}{2} \leq n \leq \frac{M-1}{2}$$

$$h_d(n) = \delta(n) - \frac{\sin(0.25\pi n)}{\pi n} \qquad for \quad -5 \leq n \leq 5$$

(b) Form the length M window *w(n)* of the chosen type.

$$w(n) = 1; 0 \leq n \leq 10$$

(c) Form the impulse response *h(n)* where $h(n)=h_d(n)w(n)$

$$h(n) = \left(\delta(n) - \frac{\sin(0.25\pi n)}{\pi n}\right) w(n)$$

(d) Shift all samples to the right by $\frac{M-1}{2} = 5$ samples.

$$h(n) = \delta(n-5) - \frac{\sin 0.25\pi(n-5)}{\pi(n-5)} \quad for\ 0 \leq n \leq 10$$

Evaluate the impulse response of the digital band-stop FIR filter, but $h(5) = 1 - \frac{w_c}{\pi}$

$$h(n) = 1 - \{-0.045, 0, 0.075, 0.159, 0.225, \mathbf{0.25}, 0.225, 0.159, 0.075, 0, -0.045\}$$
$$h(n) = \{1.045, 1, 0.925, 0.841, 0.775, \mathbf{0.75}, 0.775, 0.841, 0.925, 1, 1.045\}$$

(B) The Figure A below illustrates the direct realization of designed FIR filter, whereas Figure B illustrates the optimized realization of designed FIR filter, which is based on the fact that all FIR filter coefficients are, for the sake of linear phase characteristic, symmetric about their middle element.

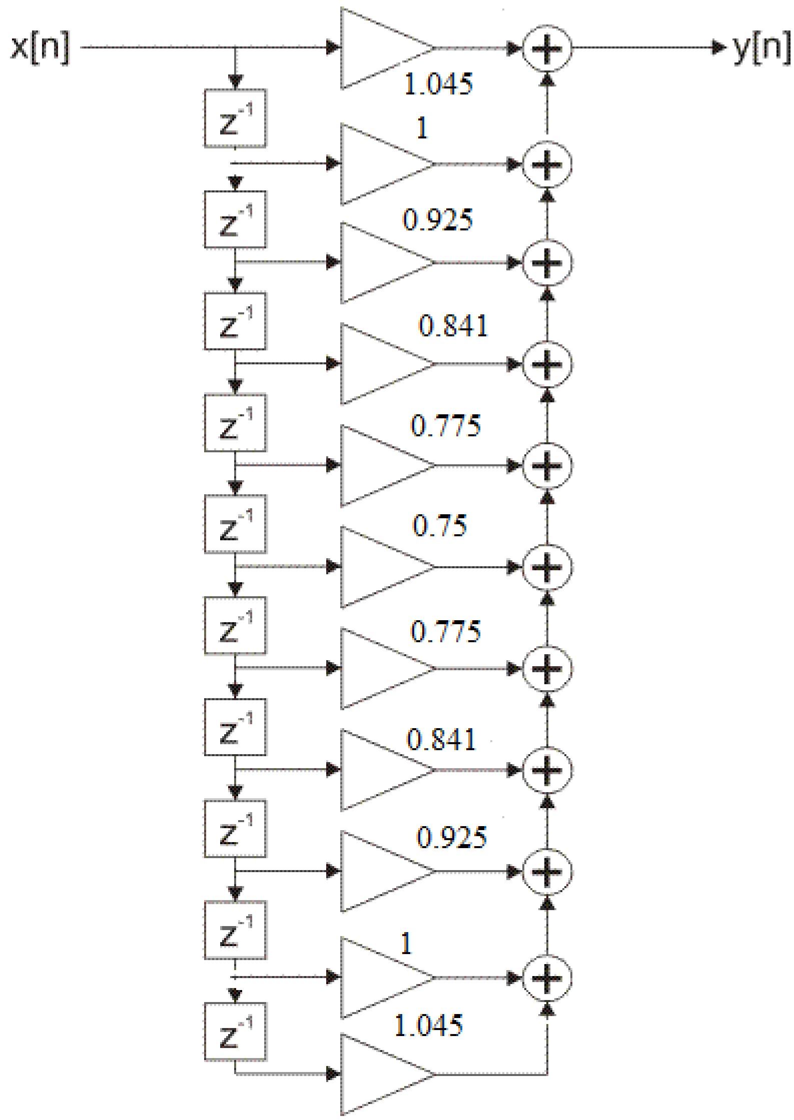


Figure A. FIR filter direct realization

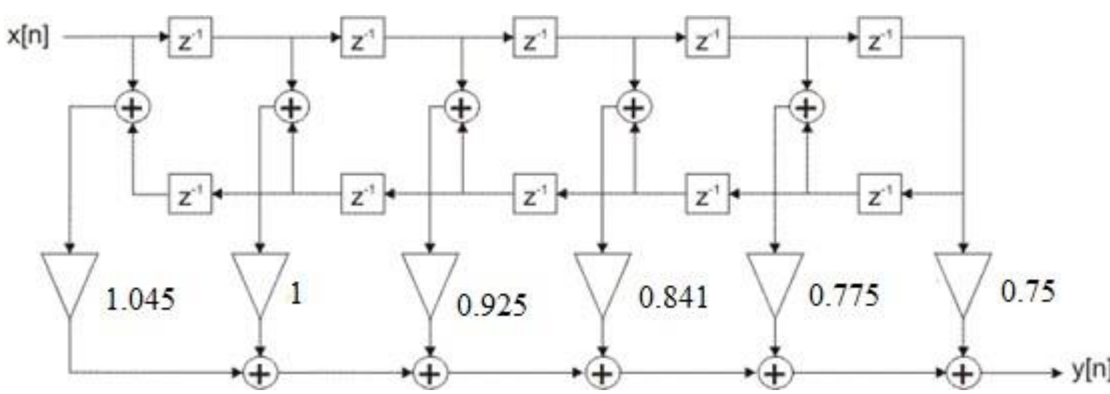
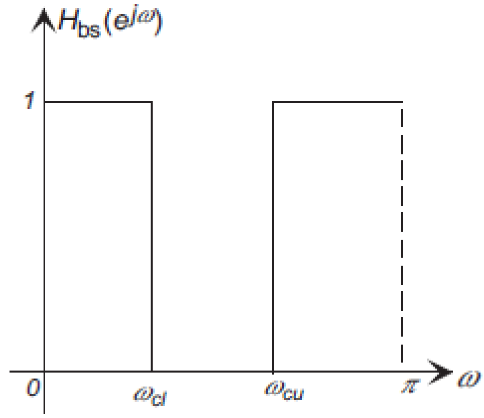


Figure B. Optimized realization structure of FIR filter

Band-Stop Filter: A band-stop or rejection filter $H_{bs}(e^{j\omega})$ may be designed from a pair of low-pass filters $H_{lpu}(e^{j\omega})$ and $H_{lpl}(e^{j\omega})$ with cut-off frequencies ω_{cu} and ω_{cl} respectively,

$$H_{bs}(e^{j\omega}) = H_{lpu}(e^{j\omega}) + (1 - H_{lpl}(e^{j\omega})).$$



Then

$$h_{bs}(n) = w(n)\left(\frac{\omega_{cu}}{\pi}\frac{\sin(\omega_{cu}n)}{\omega_{cu}n} + \delta(n) - \frac{\omega_{cl}}{\pi}\frac{\sin(\omega_{cl}n)}{\omega_{cl}n}\right), \qquad |n| \le M\text{-}1/2.$$

Example:

(A) Design a band-stop FIR filter with total number of filter coefficients = 11and cut-off low-pass frequencies ω_{cl} =0.1π and ω_{cu} =0.25π using a Rectangular window, $w(n) = 1; 0 \le n \le M-1$.

(B) Implement the designed digital FIR filter in hardware.

Solution:

(A) The available design parameters; (1) the low-pass cut-off frequencies ω_{cl} =0.1π and ω_{cu} =0.25π, (2) Rectangular window, $w(n) = 1; 0 \le n \le 10$, and (3) the filter length M =11. Then, the Design procedure is as follows

(a) Form the samples of the ideal high-pass filter of length M = 11.

$$h_d(n) = \frac{\sin(w_{cu}n)}{\pi n} + \delta(n) - \frac{\sin(w_{cl}n)}{\pi n} \qquad for \quad -\frac{M-1}{2} \le n \le \frac{M-1}{2}$$

$$h_d(n) = \frac{\sin(0.25\pi n)}{\pi n} + \delta(n) - \frac{\sin(0.1\pi n)}{\pi n} \qquad for \quad -5 \le n \le 5$$

(b) Form the length M window *w(n)* of the chosen type.

$$w(n) = 1; 0 \le n \le 10$$

(c) Form the impulse response *h(n)* where $h(n) = h_d(n)w(n)$

$$h(n) = w(n)\left(\frac{\sin(0.25\pi n)}{\pi n} + \delta(n) - \frac{\sin(0.1\pi n)}{\pi n}\right)$$

(d) Shift all samples to the right by $\frac{M-1}{2} = 5$ samples.

$$h(n) = \frac{\sin\big(0.25\pi(n-5)\big)}{\pi(n-5)} + \delta(n-5) - \frac{\sin 0.1\pi(n-5)}{\pi(n-5)} \quad for\ 0 \le n \le 10$$

Evaluate the impulse response of the digital band-stop FIR filter, but $h(5) = 1 - \frac{w_{cu} - w_{cl}}{\pi}$

$$h(n) = \{0.891, 0.924, 0.989, 1.066, 1.127, 0.85, 1.127, 1.066, 0.989, 0.924, 0.891\}$$

(B) The Figure A below illustrates the direct realization of designed FIR filter, whereas Figure B illustrates the optimized realization of designed FIR filter, which is based on the fact that all FIR filter coefficients are, for the sake of linear phase characteristic, symmetric about their middle element.

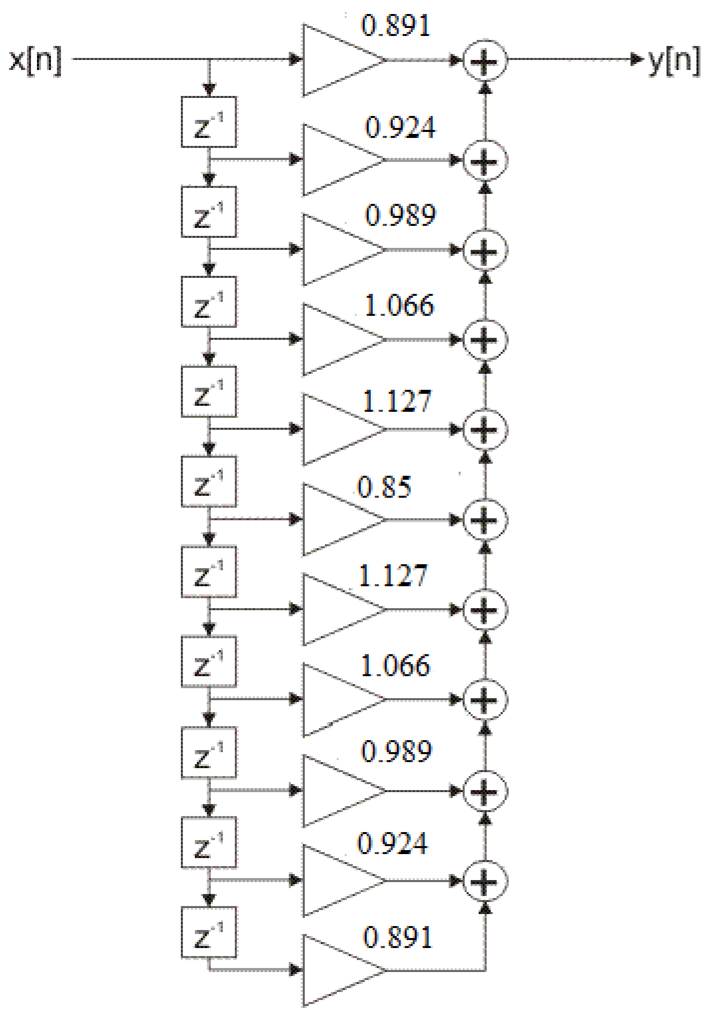


Figure A. FIR filter direct realization

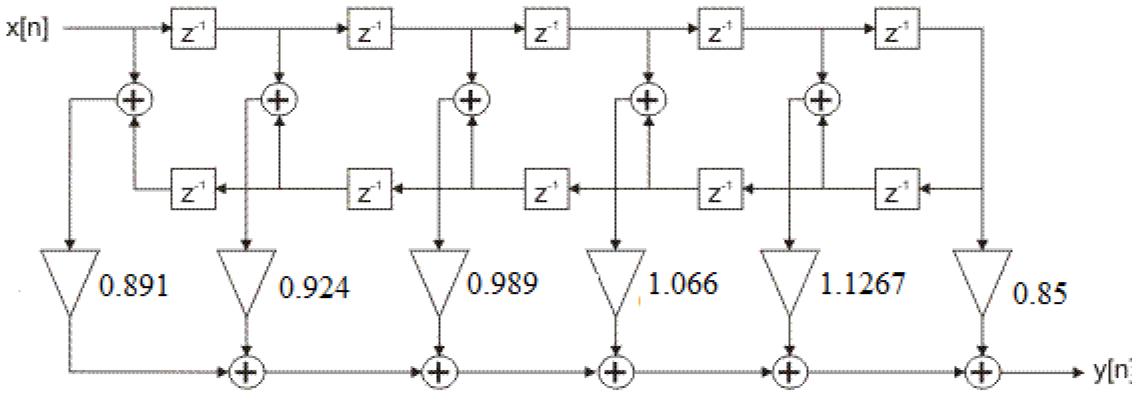
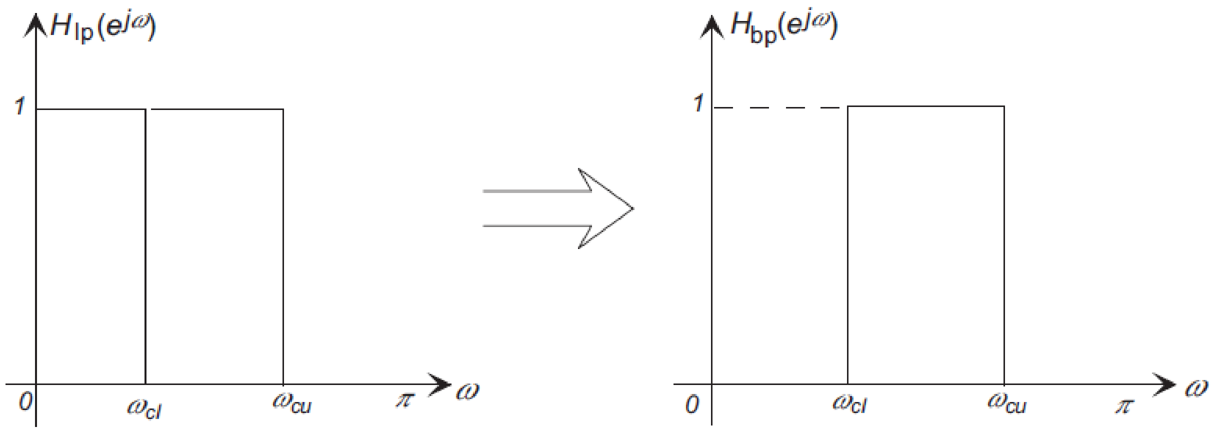


Figure B. Optimized realization structure of FIR filter

Band-Pass Filter: A band-pass filter $H_{bp}(e^{j\omega})$ may be designed from a pair of low-pass filters $H_{lpu}(e^{j\omega})$ and $H_{lpl}(e^{j\omega})$ with cut-off frequencies ω_{cu} and ω_{cl} respectively,

$H_{bp}(e^{j\omega}) = H_{lpu}(e^{j\omega}) - H_{lpl}(e^{j\omega})$.



Then,

$$h_{bp}(n) = w(n)\left(\frac{\omega_{cu}}{\pi}\frac{\sin(\omega_{cu}n)}{\omega_{cu}n} - \frac{\omega_{cl}}{\pi}\frac{\sin(\omega_{cl}n)}{\omega_{cl}n}\right), \qquad |n| \leq M/2.$$

Example:

(A) Design a band-pass FIR filter with total number of filter coefficients = 11and cut-off low-pass frequencies $\omega_{cu} = 0.25\pi$ and $\omega_{cl} = 0.1\pi$ using a Rectangular window, $w(n) = 1;\ 0 \leq n \leq M-1$.

(B) Implement the designed digital FIR filter in hardware.

Solution:

(A) The available design parameters; (1) the low-pass cut-off frequencies ω_{cu} =0.1π and ω_{cl} =0.25π, (2) Rectangular window, $w(n) = 1; 0 \leq n \leq 10$, and (3) the filter length M =11. Then, the Design procedure is as follows

(a) Form the samples of the ideal band-pass filter of length M = 11.

$$h_d(n) = \frac{\sin(w_{cu}n)}{\pi n} - \frac{\sin(w_{cl}n)}{\pi n} \quad for \quad -\frac{M-1}{2} \leq n \leq \frac{M-1}{2}$$

$$h_d(n) = \frac{\sin(0.25\pi n)}{\pi n} - \frac{\sin(0.1\pi n)}{\pi n} \quad for \quad -5 \leq n \leq 5$$

(b) Form the length M window *w(n)* of the chosen type.

$$w(n) = 1; 0 \leq n \leq 10$$

(c) Form the impulse response *h(n)* where $h(n)=h_d(n)w(n)$

$$h(n) = w(n)\left(\frac{\sin(0.25\pi n)}{\pi n} - \frac{\sin(0.1\pi n)}{\pi n}\right)$$

(d) Shift all samples to the right by $\frac{M-1}{2} = 5$ samples.

$$h(n) = \frac{\sin(0.25\pi(n-5))}{\pi(n-5)} - \frac{\sin 0.1\pi(n-5)}{\pi(n-5)} \quad for\ 0 \leq n \leq 10$$

Evaluate the impulse response of the digital band-stop FIR filter, but $h(5) = \frac{w_{cu}-w_{cl}}{\pi}$

$$h(n) = \{-0.1087, -0.0757, -0.0108, 0.0656, 0.1267, 0.15, 0.1267, 0.0656, -0.0108, -0.0757, -0.1087\}$$

(B) The Figure A below illustrates the direct realization of designed FIR filter, whereas Figure B illustrates the optimized realization of designed FIR filter, which is based on the fact that all FIR filter coefficients are, for the sake of linear phase characteristic, symmetric about their middle element.

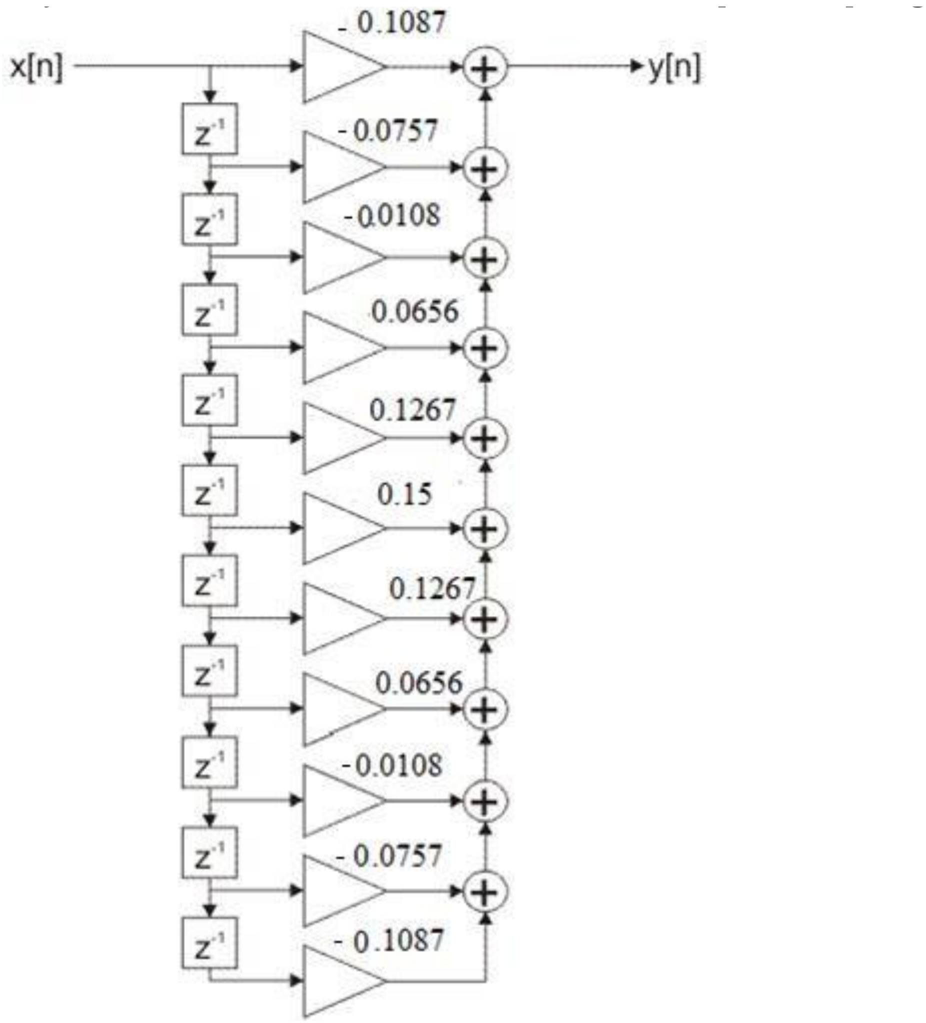


Figure A. FIR filter direct realization

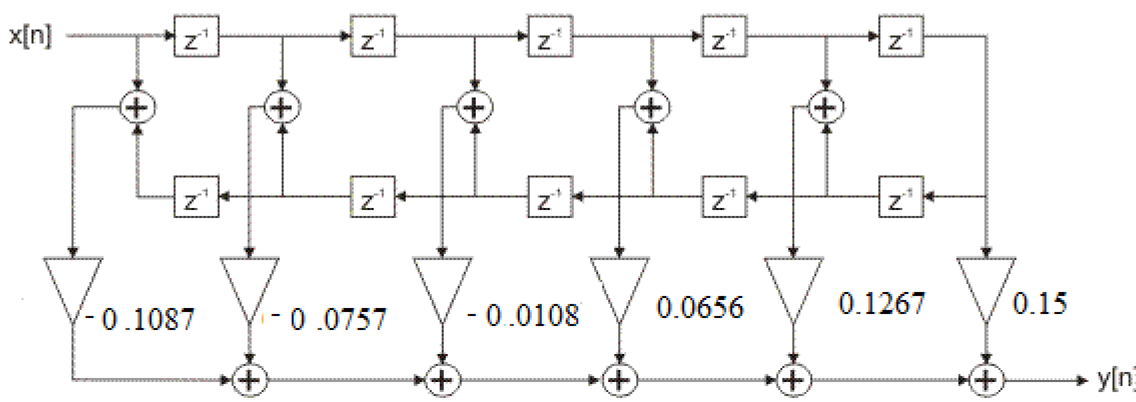


Figure B. Optimized realization structure of FIR filter

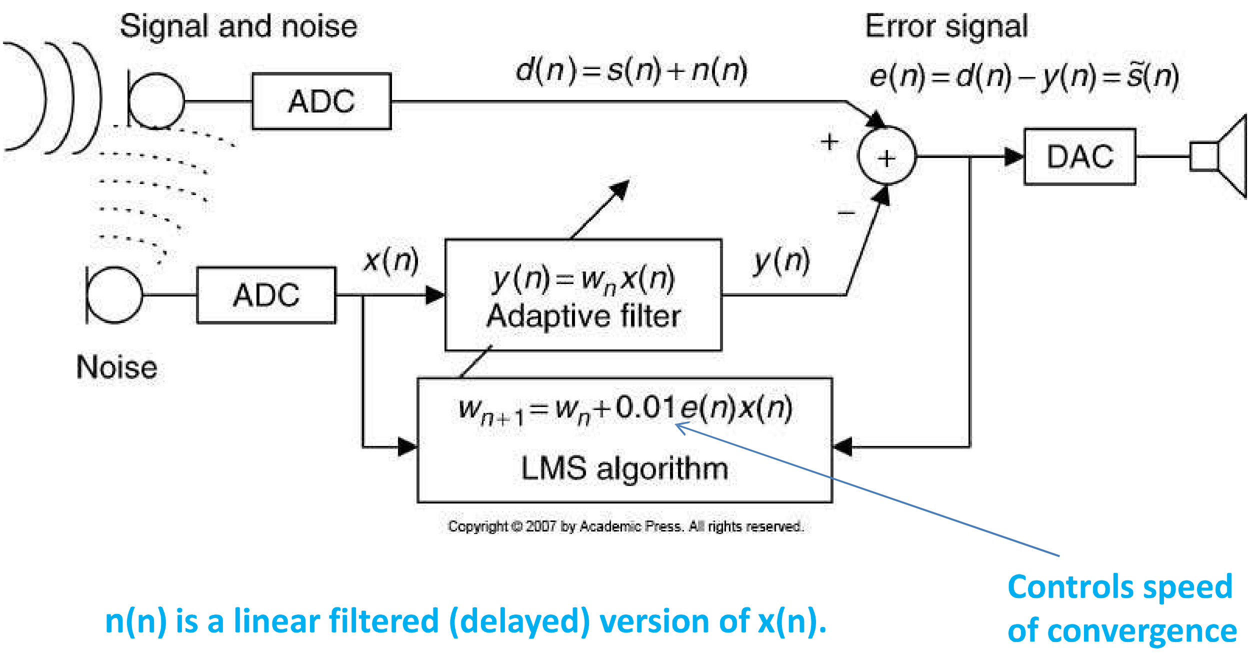
Adaptive Filter

A digital filter that automatically adjusts its coefficients to adapt input signal via an adaptive algorithm.

Applications:

- **Signal enhancement**
- **Active noise control**
- **Noise cancellation**
- **Telephone echo cancellation**

Simplest Noise Canceller



n(n) is a linear filtered (delayed) version of x(n).

Simplest Noise Canceller – contd.

$$y(n) = w_n x(n)$$
$$e(n) = d(n) - y(n)$$
$$w_{n+1} = w_n + 0.01 e(n) x(n).$$

Initial coefficient $w_0 = 0.3$

n	$d(n)$	$x(n)$
0	−0.2947	−0.5893
1	1.0017	0.5893

Measured

$n = 0$, $y(0) = w_0 x(0) = 0.3 \times (-0.5893) = -0.1768$

$$e(0) = d(0) - y(0) = -0.2947 - (-0.1768) = -0.1179 = \tilde{s}(0)$$

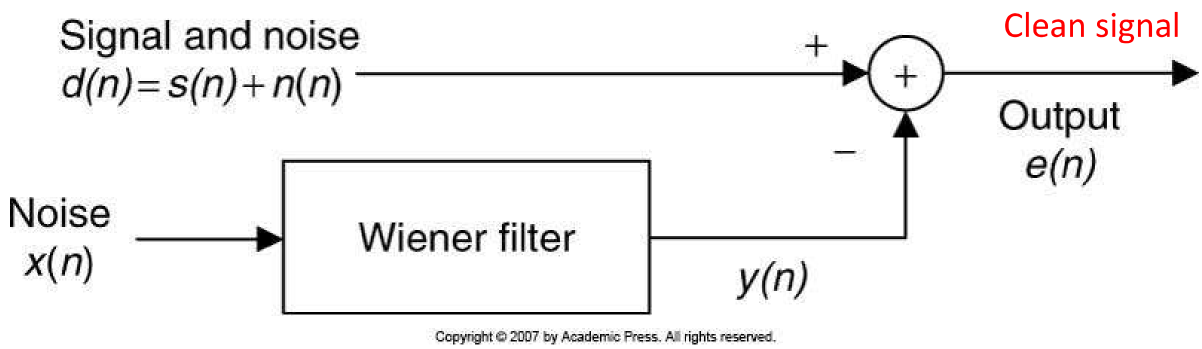
$$w_1 = w_0 + 0.01 e(0) x(0) = 0.3 + 0.01 \times (-0.1179) \times (-0.5893) = 0.3007$$

$n = 1$, $y(1) = w_1 x(1) = 0.3007 \times 0.5893 = 0.1772$

$$e(1) = d(1) - y(1) = 1.0017 - 0.1772 = 0.8245 = \tilde{s}(1)$$

$$w_2 = w_1 + 0.01 e(1) x(1) = 0.3007 + 0.01 \times 0.8245 \times 0.5893 = 0.3056$$

Wiener Filter & LMS Algorithm



Consider, single weight case, $y(n) = wx(n)$

Error signal, $e(n) = d(n) - wx(n)$

Now we have to solve for the best weight w*

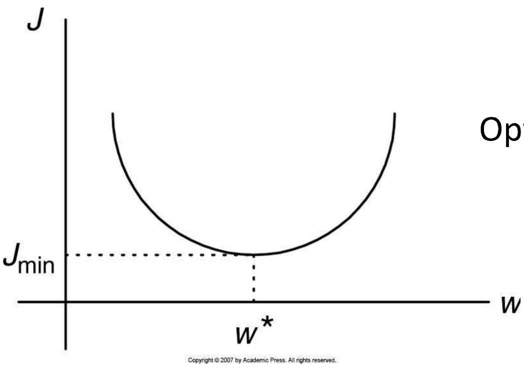
$$e^2(n) = (d(n) - wx(n))^2 = d^2(n) - 2d(n)wx(n) + w^2x^2(n)$$

LMS Algorithm

Taking Expectation of squared error signal

$$E(e^2(n)) = E(d^2(n)) - 2wE(d(n)x(n)) + w^2E(x^2(n))$$

$$J = E(e^2(n)) = \text{MSE (mean squared error)}$$
$$\sigma^2 = E(d^2(n)) = \text{power of corrupted signal}$$
$$P = E(d(n)x(n)) = \text{cross-correlation between } d(n) \text{ and } x(n)$$
$$R = E(x^2(n)) = \text{autocorrelation}$$



For large N, $J = \sigma^2 - 2wP + w^2R$

Optimal w* is found when minimum J is achieved

$$\frac{dJ}{dw} = -2P + 2wR = 0.$$

$$w^* = R^{-1}P$$

LMS Algorithm - Example

Given MSE function for the Wiener filter:

$$\frac{dJ}{dw} = -20 + 10 \times 2w = 0$$

Solving for optimal, we get $w^* = 1$

Finally we get

$$J_{\min} = J|_{w=w^*} = 40 - 20w + 10w^2|_{w=1} = 40 - 20 \times 1 + 10 \times 1^2 = 30$$

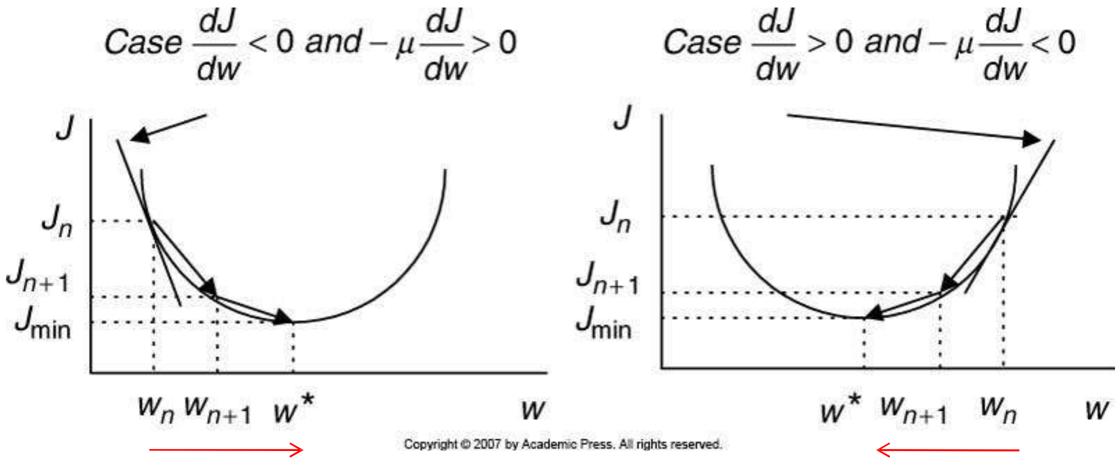
large N $w^* = R^{-1}P$ R^{-1}: Matrix inversion

Makes real-time implementation difficult

Steepest Decent Algorithm

$$w_{n+1} = w_n - \mu \frac{dJ}{dw}$$

μ = constant controlling the speed of convergence.



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Steepest Decent Algorithm: Example

Given: $J = 40 - 20w + 10w^2$

$\mu = 0.04 \qquad w_0 = 0$

Iteration three times

Find optimal solution for w*

Solution:

$$\frac{dJ}{dw} = -20 + 10 \times 2w_n$$

For n = 0,

$$\mu\frac{dJ}{dw} = 0.04 \times (-20 + 10 \times 2w_0)|_{w_0=0} = -0.8$$

$$w_1 = w_0 - \mu\frac{dJ}{dw} = 0 - (-0.8) = 0.8$$

For n = 1,

$$\mu\frac{dJ}{dw} = 0.04 \times (-20 + 10 \times 2w_1)|_{w_1=0.8} = -0.16$$

$$w_2 = w_1 - \mu\frac{dJ}{dw} = 0.8 - (-0.16) = 0.96$$

For n = 2,

$$\mu\frac{dJ}{dw} = 0.04 \times (-20 + 10 \times 2w_2)|_{w_2=0.96} = -0.032$$

$$w_3 = w_2 - \mu\frac{dJ}{dw} = 0.96 - (-0.032) = 0.992.$$

$$w^* \approx w_3 = 0.992$$

$$J_{min} \approx 40 - 20w + 10w^2|_{w=0.992} = 30.0006$$

Steepest Decent Algorithm – contd1.

To make it sample-based processing, we need to take out estimation.

$$J = e^2(n) = (d(n) - wx(n))^2$$
$$\frac{dJ}{dw} = 2(d(n) - wx(n))\frac{d(d(n) - wx(n))}{dw} = -2e(n)x(n)$$

Updating weight $w_{n+1} = w_n + 2\mu e(n)x(n)$

For multiple tap FIR filter:

$$y(n) = w_n(0)x(n) + w_n(1)x(n-1) + \cdots + w_n(N-1)x(n-N+1)$$
for $i = 0, \ldots, N-1$
$$w_{n+1}(i) = w_n(i) + 2\mu e(n)x(n-i).$$

Choose convergence constant as $0 < \mu < \frac{1}{NP_x}$ P_x : maximum input power

Steepest Decent Algorithm – contd2.

Steps:

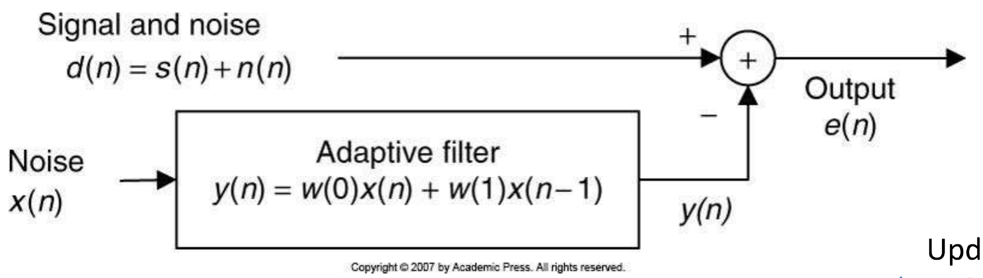
1. Initialize $w(0), w(1), \ldots, w(N-1)$ to arbitrary values.
2. Read $d(n)$, $x(n)$, and perform digital filtering:
$$y(n) = w(0)x(n) + w(1)x(n-1) + \cdots + w(N-1)x(n-N+1).$$
3. Compute the output error:
$$e(n) = d(n) - y(n).$$
4. Update each filter coefficient using the LMS algorithm:
$$\text{for } i = 0, \ldots, N-1$$
$$w(i) = w(i) + 2\mu e(n)x(n-i).$$

Noise Canceller Using Adaptive Filter-1

Perform adaptive filtering to obtain outputs $e(n) = n = 0,\ 1,\ 2$

Given: $x(0) = 1,\ x(1) = 1x(2) = -1,$ $d(0) = 2,\ d(1) = 1,\ d(2) = -2$

Initial weights: $w(0) = w(1) = 0,$ $\mu = 0.1$



Updating weights:

Solution:

Filtering: $y(n) = w(0)x(n) + w(1)x(n-1)$

Output: $e(n) = d(n) - y(n)$

$$w(0) = w(0) + 2\mu e(n)x(n)$$
$$w(1) = w(1) + 2\mu e(n)x(n-1)$$

Noise Canceller Using Adaptive Filter

For $n = 0$

Digital filtering:

$$y(0) = w(0)x(0) + w(1)x(-1) = 0 \times 1 + 0 \times 0 = 0$$

Computing the output:

$$e(0) = d(0) - y(0) = 2 - 0 = 2$$

Updating coefficients:

$$w(0) = w(0) + 2 \times 0.1 \times e(0)x(0) = 0 + 2 \times 0.1 \times 2 \times 1 = 0.4$$
$$w(1) = w(1) + 2 \times 0.1 \times e(0)x(-1) = 0 + 2 \times 0.1 \times 2 \times 0 = 0.0$$

For $n = 1$

Digital filtering:

$$y(1) = w(0)x(1) + w(1)x(0) = 0.4 \times 1 + 0 \times 1 = 0.4$$

Computing the output:

$$e(1) = d(1) - y(1) = 1 - 0.4 = 0.6$$

Updating coefficients:

$$w(0) = w(0) + 2 \times 0.1 \times e(1)x(1) = 0.4 + 2 \times 0.1 \times 0.6 \times 1 = 0.52$$
$$w(1) = w(1) + 2 \times 0.1 \times e(1)x(0) = 0 + 2 \times 0.1 \times 0.6 \times 1 = 0.12$$

Noise Canceller Using Adaptive Filter

For $n = 2$

Digital filtering:

$$y(2) = w(0)x(2) + w(1)x(1) = 0.52 \times (-1) + 0.12 \times 1 = -0.4$$

Computing the output:

$$e(2) = d(2) - y(2) = -2 - (-0.4) = -1.6$$

Updating coefficients:

$$w(0) = w(0) + 2 \times 0.1 \times e(2)x(2) = 0.52 + 2 \times 0.1 \times (-1.6) \times (-1) = 0.84$$
$$w(1) = w(1) + 2 \times 0.1 \times e(2)x(1) = 0.12 + 2 \times 0.1 \times (-1.6) \times 1 = -0.2.$$

Output (noise-cleaned signal):

$$e(0) = 2,\ e(1) = 0.6,\ e(2) = -1.6$$