Infinite Series

The series give us precise ways to express many numbers and functions.

Infinite Sequences

A sequence is a list of numbers:

$$a_1, a_2, a_3, \cdots a_n$$

An infinite sequence of numbers is a function whose domain is the set of positive integers:

$a_n = F(n)$ where:

- a_n: Function of the sequence
- n: Positive integer numbers start from one to infinity ($n = 1$ to ∞) and are called the index of a_n

For example,

1- $a_n = 2n \rightarrow$ the sequence is: $2, 4, 6, 8, \ldots, 2n, \ldots.$
2- $a_n = n^2 \rightarrow$ the sequence is: $1, 4, 9, 16, 25, \ldots., n^2, \ldots.$

Sequences can be described by writing rules that specify their terms, such as:

$a_n = \sqrt{n}$

$b_n = \frac{1}{n}(-1)^{n+1}$

$c_n = \frac{n-1}{n}$

$d_n = \cos(\frac{\pi}{n})$

Or, by listing terms (list of function results)

$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$

$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots., \frac{1}{n}(-1)^{n+1}, \ldots.\right\}$

$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots., \frac{n-1}{n}, \ldots.\right\}$

$\{d_n\} = \left\{\cos(\pi), \cos(\frac{\pi}{2}), \cos(\frac{\pi}{3}), \ldots., \cos(\frac{\pi}{n}), \ldots.\right\}$

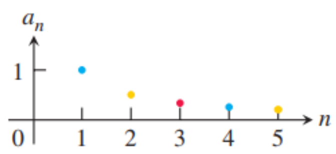
We also sometimes write a sequence as:

$$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$$

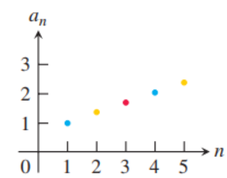
Convergence and divergence of sequences

If a sequence approach to a limit number (L), then the Sequence is convergent, while when it does not approach a limit number or its numbers diverge with an increase in (n), then it is a divergent sequence as shown in the figures:

Converges to 0



diverges to ∞



$a_n = \frac{1}{n}$ (Convergent sequence)

$a_n = \sqrt{n}$ (Divergent sequence)

To find the number (L) Whose the sequence approaches to it, limit law must be used:

$$L = \lim_{n\to\infty} a_n$$

If L is the limit number → the sequence converges

If L is infinity (∞) → the sequence diverges

Example 1: find the first four terms of the following sequences:

1- $a_n = \frac{1-n}{n^2} \rightarrow \left\{0, -\frac{1}{4}, -\frac{2}{9}, -\frac{3}{16}\right\}$

2- $a_n = \frac{1}{n!} \rightarrow \left\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}\right\}$

3- $a_n = 2 + (-1)^n \rightarrow \{1, 3, 1, 3\}$

4- $a_n = \frac{3n}{2^n+1} \rightarrow \left\{1, \frac{6}{5}, 1, \frac{12}{17}\right\}$

Example 2: Find the function formula of the following sequences:

$\{1, -1, 1, -1, \dots\} \rightarrow a_n = (-1)^{n+1}$ an alternating sequence

$\left\{1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \dots\right\} \rightarrow a_n = (-1)^{n+1}\frac{1}{n^2}$ an alternating sequence

$\{1, 5, 9, 13, 17, \dots\} \rightarrow a_n = 4n - 3$

$\{0, 3, 8, 15, 24, \dots\} \rightarrow a_n = n^2 - 1$

Example 3: which of the following sequences converge or diverge, then find the limit of each convergent sequence:

1- $a_n = \frac{-5}{n^2}$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-5}{n^2} = -5 \lim_{n\to\infty} \frac{1}{n^2} = -5\left(\frac{1}{\infty^2}\right) = -5(0) = 0 \ \rightarrow \text{converges from } 0$$

2- $a_n = \frac{n-1}{n}$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1 - \left(\frac{1}{\infty}\right) = 1 \rightarrow \text{converges from } 1$$

3- $a_n = \frac{n^2-2n+1}{n-1}$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-2n+1}{n-1}$$

$$= \lim_{n\to\infty} \frac{n^2/_{n^2} - 2n/_{n^2} + 1/_{n^2}}{n/_{n^2} - 1/_{n^2}}$$

$$= \lim_{n\to\infty} \frac{1 - 2/_n + 1/_{n^2}}{1/_n - 1/_{n^2}} = \frac{1 - 2/_\infty + 1/_{\infty^2}}{1/_\infty - 1/_{\infty^2}} = \frac{1-0+0}{0-0} = \frac{1}{0} = \infty$$

$\rightarrow$ diverges (no limit)

Some rules of limit

1- $\lim_{n\to\infty} k = k \ \ where\ k\ is\ constant$

2- $\lim_{n\to\infty} (n)^k = \begin{cases} 1\ if\ k = 0 \\ \infty\ if\ k = positive\ value \\ 0\ if\ k = negative\ value \end{cases}$

3- $\lim_{n\to\infty} (k)^n = \begin{cases} 0 \quad if\ |k| < 1 \\ \infty \quad if\ |k| > 1 \\ 1 \quad if\ k = 1 \end{cases}$

4- $\lim_{n\to\infty} \sqrt[n]{n^k} = 1$ for any value of k

5- $\lim_{n\to\infty} (k)^{\frac{1}{n}} = 1$ for $k > 0$

6- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

7- $\lim_{n\to\infty} (1 + \frac{k}{n})^n = e^k$ for any value of k

8- $\lim_{n\to\infty} \frac{k^n}{n!} = 0$ for any value of k

Infinite series

An infinite series is the sum of an infinite sequence of numbers and is expressed in the following form:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

Where the term (a_n) is the nth term.

Examples:

1- $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$

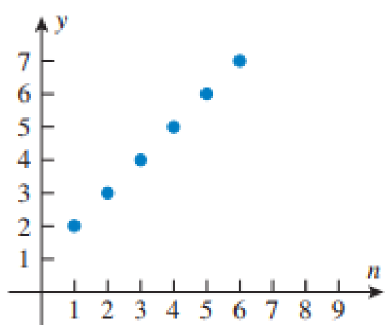
2- $\sum_{n=1}^{\infty} \ln 2n = \ln 2 + \ln 4 + \ln 6 + \cdots + \ln 2n + \cdots$

3- $\sum_{n=1}^{\infty} \frac{n}{1+2^n} = \frac{1}{1+2^1} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \cdots + \frac{n}{1+2^n} + \cdots$

Convergence and Divergence of Series

If the sum of infinite series $(\sum a_n)$ converges to a limit value, we say that the series is convergent, while when this sum diverges and does not converge to a limit value, we say that the series is divergent.

$$a_n = n + 1$$



$\sum a_n$ diverges to ∞

Harmonic series

The harmonic series is a divergent series with decreasing and positive terms. The form of this series is:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} + \cdots$$

The series does not follow from nth term test. The reason it diverges is because there is no upper bound for its partial sum.

P- series

The P- series is the series of decreasing and positive terms. The form of this series is:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \frac{1}{n^p} + \cdots$$

Where: p is a real constant

If $p > 1 \rightarrow$ The p - series is convergent

If $p \leq 1 \rightarrow$ The p - series is divergent

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}$

$= 1 + \frac{1}{4} + \frac{1}{9} + \ldots$

p = 2 > 1 $\therefore$ convergent

Example: $\sum_{n=1}^{\infty} \frac{1}{n^{-2}} = \frac{1}{1^{-2}} + \frac{1}{2^{-2}} + \frac{1}{3^{-2}}$

= 1 + 4 + 9 + ...

p = -2 < 1 $\therefore$ divergent

Alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

n^{th} term test

This type of test is used for detecting the divergent series only.

1. Determine a_n
2. Take $\lim_{n\to\infty} a_n$
3. Check with the theorem

- If $\lim_{n\to\infty} a_n \neq 0$ or it fails to exist, then:

$\sum_{n=1}^{\infty} a_n$ is divergent series

- If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive

Example: use n^{th} -term test to show that the following series is divergent or that the test is inconclusive.

1- $\sum_{n=1}^{\infty} n^2$: $\lim_{n\to\infty} n^2 = \infty^2 = \infty \neq 0$ $\therefore$ divergent

2 − $\sum_{n=1}^{\infty} \frac{n+1}{n}$: $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \left(1 + \frac{1}{\infty}\right) = 1 \neq 0$ $\therefore$ divergent

3-$\sum_{n=1}^{\infty}(-1)^{n+1}$: $\lim_{n\to\infty}(-1)^{n+1} = (-1)^{\infty+1}$

does not exist $\therefore$ divergent

4- $\sum_{n=1}^{\infty}\frac{n^2}{n^2+3}$: $\lim_{n\to\infty}\left(\frac{n^2}{n^2+3}\right) = \lim_{n\to\infty}\left(\frac{n^2/n^2}{n^2/n^2+3/n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{3}{n^2}}\right)$

$$= \frac{1}{1+\frac{3}{\infty^2}} = \frac{1}{1+0} = 1 \neq 0 \therefore \text{divergent}$$

5- $\sum_{n=1}^{\infty}\frac{1}{2^n}$: $\lim_{n\to\infty}\frac{1}{2^n} = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$ $\therefore$ inconclusive

6- $\sum_{n=1}^{\infty}\frac{n}{\ln n^n+n}$: $\lim_{n\to\infty}\frac{n}{\ln n^n+n} = \lim_{n\to\infty}\frac{n}{n\ln n+n} = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}(\ln n+1)}$

$$= \lim_{n\to\infty}\frac{1}{\ln n+1} = \frac{1}{\ln\infty+1} = \frac{1}{\infty+1} = 0 \therefore \text{inconclusive}$$

Homework #1

Exercise 1: Find the first four terms of the following sequences:

1- $a_n = \frac{(-1)^{n+1}}{2n-1}$ 2- $a_n = \frac{n+3}{n^2+5n+6}$

3- $a_n = \frac{2^n}{2^{n+1}}$ 4- $a_n = n! - n^2$

Exercise 2: Find the function formula (sequence's formula) for the nth term of the following sequences:

1- $\{2, 6, 10, 14, 18, \ldots\}$ hint: every other even positive integer
2- $\{2, 5, 8, 11, 14, \ldots\}$
3- $\{2, 8, 18, 32, 50, \ldots\}$
4- $\left\{\frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \ldots\right\}$

Exercise 3: Find the limit number of the following sequences and which of them is convergent or divergent:

1- $a_n = 2 + (0.1)^n$ 2- $a_n = \frac{1-5n^4}{n^4+8n^3}$ 3- $a_n = (1+\frac{7}{n})^n$

4- $a_n = \sqrt[n]{10n}$ 5- $a_n = \frac{1+n^3}{70+4n^2}$ 6- $a_n = \left(\frac{3n+1}{3n-1}\right)^n$

Exercise 4: Find the nature of the following series using n^{th} - term test.

1- $\sum_{n=1}^{\infty}\cos\frac{1}{n}$ 2 – $\sum_{n=1}^{\infty}\ln\frac{1}{n}$ 3 – $\sum_{n=1}^{\infty}\frac{n(n+1)}{(n+2)(n+3)}$

4 – $\sum_{n=1}^{\infty}\frac{e^n}{e^n+n}$ 5 – $\sum_{n=1}^{\infty}\frac{n}{n^2+3}$ 6 – $\left(\frac{1}{2}+\frac{3}{3}+\frac{5}{4}+\frac{7}{5}\right)$