الجامعة المستنصرية – كلية الهندسة
قسم الهندسة الكهربائية

Electrical Circuits Lectures

Second Stage

2023- 2024

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Syllabus

1. A Review of A.C. Waveforms and Analysis of A.C. Circuits.
2. Resonance in A.C. Circuits
3. Admittance and Current Locus.
4. The Transient Circuit Analysis.
5. Periodic non-Sinusoidal Signals.
6. Two Port Networks.
7. Coupling Circuits.
8. Polyphase Circuits.
9. Filters.

Intended Learning Outcomes

At the end of this subject (30 Week\ 3 Hours per week) the student will be able to:

1. Develop problem-solving skills.
2. Understands of the analysis of A.C. circuits.
3. Comprehend the concept of transient analysis of electrical circuits.
4. Explain the Admittance and Current Locus.
5. Explain the resonant circuits.
6. Grasp the concept of two port networks.
7. Understand the Periodic non-Sinusoidal Signals.
8. Understand the analysis of Polyphase Circuit.
9. Understand the properties of Coupling Circuits.

References:

1. Charles K. Alexander & Matthew N. O. Sadiku, "Fundamentals of Electric Circuits", McGraw-Hill, New York, Fifth Edition, 2013.
2. Robert L. Boylestad, " Introductory Circuit Analysis", Tenth Edition.

A Review of A.C. Waveforms and Analysis of A.C. Circuits

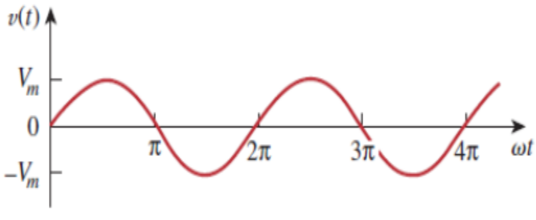
Sinusoids

- A sinusoidal is a signal that has the form of the sine or cosine function.
- A sinusoidal current is usually referred to as *Alternating Current* (*A.C.*).
- For the sinusoidal voltage $v(t) = V_m \sin \omega t$, where:

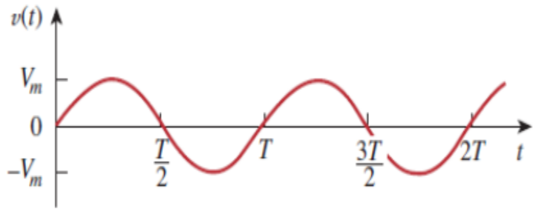
  V_m is the *amplitude* of the sinusoid.

  ω is the *Angular frequency* in radians/s.

  ωt is the *argument* of the sinusoidal wave.



(a)



(b)

From above figure, T is called the ***period*** of the sinusoid.

$$T = \frac{2\pi}{\omega}$$

A periodic function is one that satisfies $f(t) = f(t\ nT)$, for all t and for all integers n.

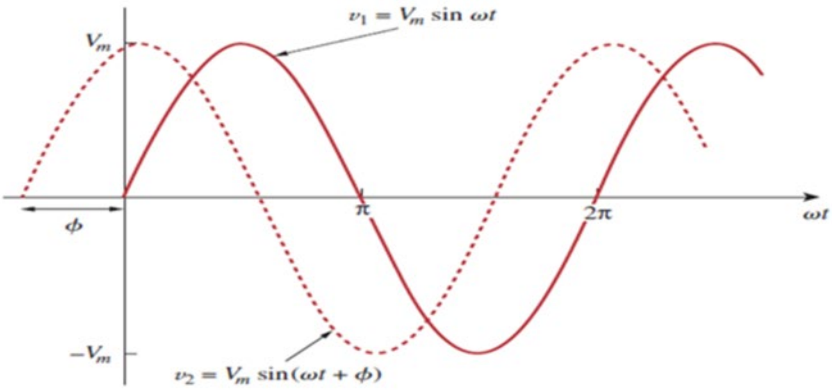
$$f = \frac{1}{T}$$

$$\omega = 2\pi f$$

While ω is in radians per second (rad/s), f is in hertz (Hz).
More general expression for the sinusoid is:

$$v(t) = V_m \sin(\omega t + \phi)$$

Where is the $(\omega t + \phi)$ argument and ϕ is the ***phase***. Both argument and phase can be in radians or degrees.

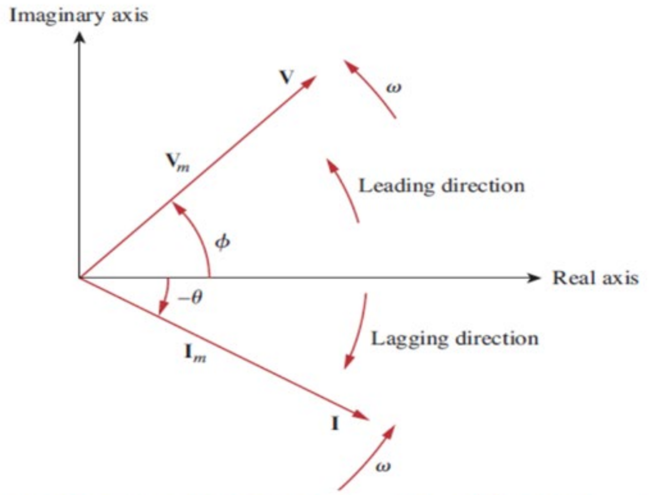


Phasors

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.
- Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)



A phasor diagram showing $\mathbf{V} = V_m \underline{/\phi}$ and $\mathbf{I} = I_m \underline{/-\theta}$.

Complex Numbers

- Complex number z can be written in rectangular form as: $z = x + jy$ where x is the real part of z; y is the imaginary part of z.
- The variables x and y are the real and imaginary parts of z in the **complex plane**.

The complex number z can also be written in polar or exponential form as

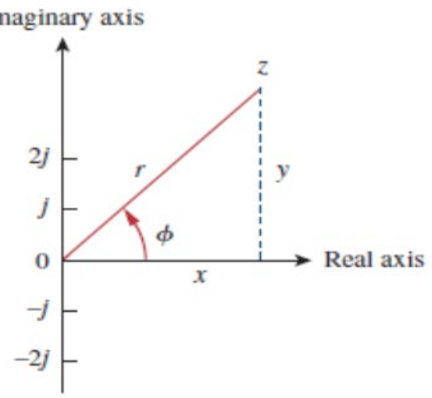
$$z = r\underline{/\phi} = re^{j\phi}$$

where r is the magnitude of z, and is the phase of z. We notice that z can be represented in three ways:

$z = x + jy$	Rectangular form
$z = r\underline{/\phi}$	Polar form
$z = re^{j\phi}$	Exponential form

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi, \qquad y = r\sin\phi$$



Complex Plane

$$z = x + jy = r\underline{/\phi} = r(\cos\phi + j\sin\phi)$$

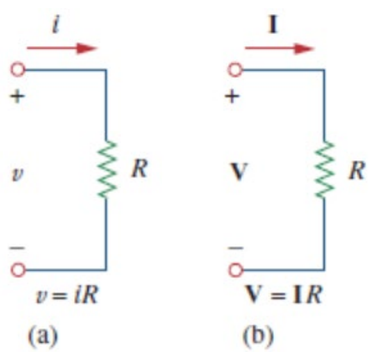
Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \underline{/\phi}$
$V_m \sin(\omega t + \phi)$	$V_m \underline{/\phi - 90°}$
$I_m \cos(\omega t + \theta)$	$I_m \underline{/\theta}$
$I_m \sin(\omega t + \theta)$	$I_m \underline{/\theta - 90°}$

Phasor Relationships for Circuit Elements

Circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the **time domain** to the **frequency domain** for each element.

1. Voltage-Current Relations for a Resistor:



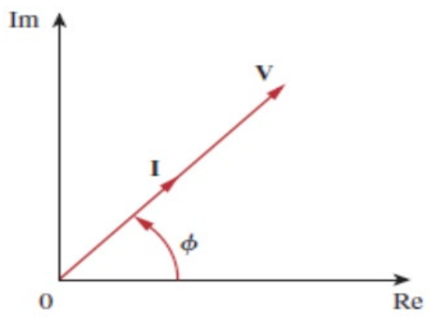
Voltage-current relations for a resistor (a) time domain, (b) frequency domain.

If the current through a resistor R is $i = I_m \cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as:

$$v = iR = RI_m \cos(\omega t + \phi)$$

The phasor form of this voltage is:

$$\mathbf{V} = RI_m \angle \phi$$



Phasor diagram for the resistor.

2. Voltage-Current relations for inductor:

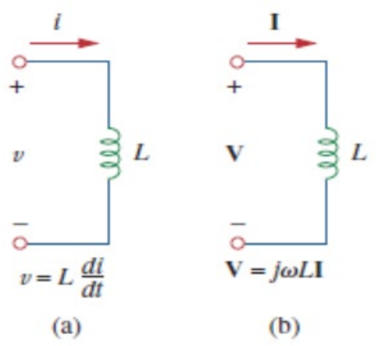
For the inductor L, assume the current through it is $i = I_m \cos(\omega t + \phi)$, the voltage across the inductor is:

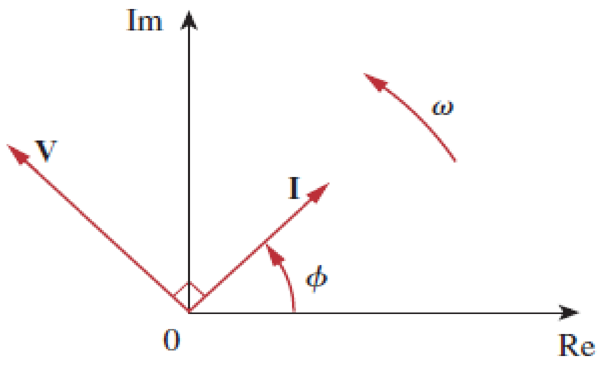
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

OR

$$v = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

$$\mathbf{V} = j\omega L\mathbf{I}$$





Voltage-current relations for inductor

(a) time domain, (b) frequency domain.

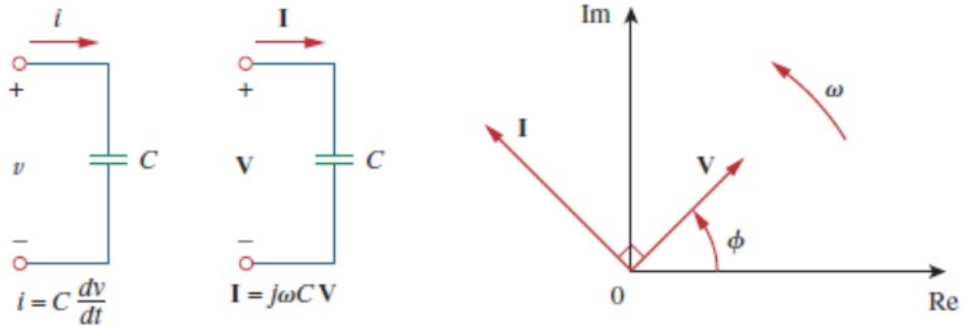
Phasor diagram for the inductor; ***I*** *lag* ***V****.*

3. Voltage-Current relations for Capacitor:

For the capacitor *C*, assume the voltage across it is $v = V_m \cos(\omega t + \phi)$, The current through the capacitor is:

$$i(t) = C\frac{dv}{dt}$$

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$



Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

Impedance and Admittance

- The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).
- The admittance **Y** is the reciprocal of impedance, measured in siemens (S).
- The voltage-current relations for the three passive elements as:

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as:

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{Z}\mathbf{I}$$

where **Z** is a frequency-dependent quantity known as *impedance*, measured in ohms. As a complex quantity, the impedance may be expressed in rectangular form as:

$$\mathbf{Z} = R + jX$$

The impedance may also be expressed in polar form as:

$$\mathbf{Z} = |\mathbf{Z}| \angle \theta$$

$$\boxed{\mathbf{Z} = R + jX = |\mathbf{Z}| \angle \theta}$$

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta$$

- It is sometimes convenient to work with the reciprocal of impedance, known as *admittance*.
- The admittance **Y** is the reciprocal of impedance, measured in siemens (S).

$$\boxed{\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}}$$

As a complex quantity, we write **Y** as

$$\boxed{\mathbf{Y} = G + jB}$$

Where G = Re **Y** is called the ***conductance*** and B = Im **Y** is called the ***susceptance***. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos).

$$G + jB = \frac{1}{R + jX}$$

By rationalization:

$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

Equating the real and imaginary parts gives:

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2}$$

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Resonance in A.C. Circuits

Resonance in A.C. Circuits

The resonant circuit is a combination of *R, L,* and *C* elements having a frequency response characteristic as shown in Fig. (1). The response is a maximum for the frequency f_s, decreasing to the right and left of this frequency.

A network will be in resonance when the voltage and current at input terminals are in phase, so the equivalent impedance and admittance consist of only the real part and the power factor is unity.

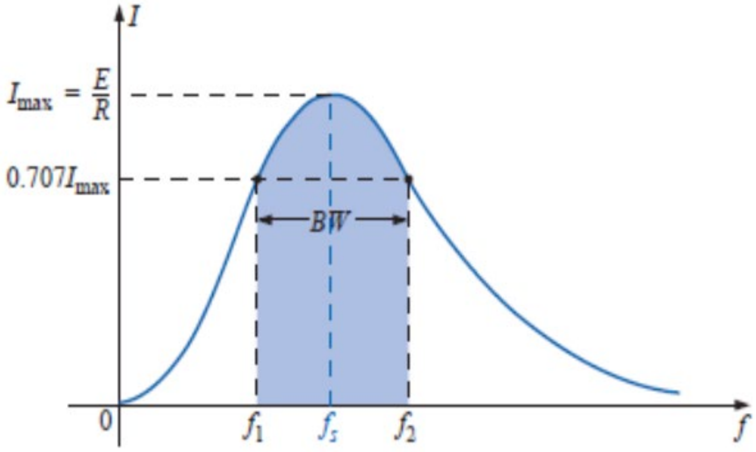


Fig. (1) *current versus frequency for resonant circuit*

- Resonance is a condition in an RLC circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in purely resistive impedance.
- f_s is the resonance or center frequency, f_1& f_2 are the lower and upper frequencies.
- There is a range of frequencies at which the current is near its maximum value and the impedance is at a minimum. Those frequencies corresponding to 0.707 of the maximum current are called the **band frequencies, cutoff frequencies (f_1& f_2)** or **half-power frequencies.**
- The range of frequencies between the f_1 and f_2 is referred to as the **bandwidth** (B.W) of the resonant circuit.
- When resonance occurs due to the application of the proper frequency (f_s), the energy will be oscillating between magnetic field of inductance and electric field of capacitance.
- There are three types of resonant circuits: ***series, parallel and combined.***

1- Series Resonant Circuit

The basic configuration for the series resonant circuit (series or parallel) must have inductive and capacitive elements, as shown in fig. (2-a), and the figure (2-b) is the result of combining the series resistive elements into one total value.

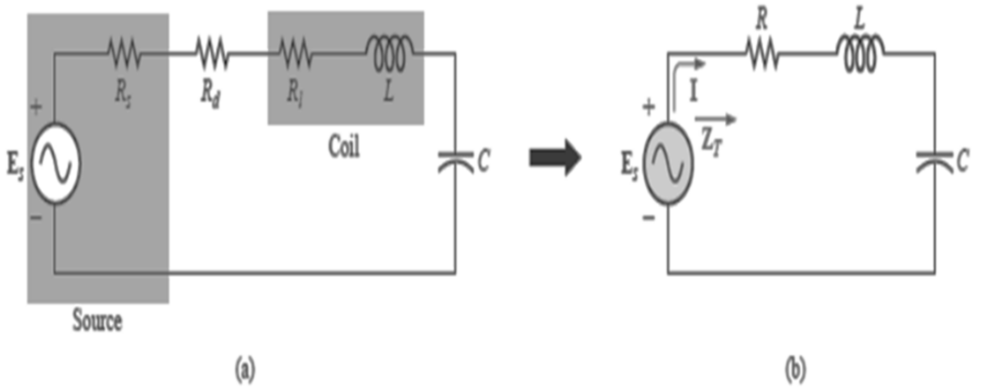


Fig. (2)

$$R = R_s + R_l + R_d$$

The total impedance of this network at any frequency is determined by:

$$\mathbf{Z}_T = R + jX_L - jX_C = R + j(X_L - X_C)$$

The resonant conditions will occur when:

$$X_L = X_C$$

removing the reactive component from the total impedance equation. The total impedance at resonance is then simply:

$$\mathbf{Z}_{T_s} = R$$

- The subscript ***s*** will be employed to indicate series resonant conditions.
- For resonance $X_L = X_C$, Substituting yields:

$$\omega L = \frac{1}{\omega C} \quad \text{and} \quad \omega^2 = \frac{1}{LC}$$

$$\omega_s = \frac{1}{\sqrt{LC}}$$

or

$$f_s = \frac{1}{2\pi\sqrt{LC}}$$

f = hertz (Hz)
L = henries (H)
C = farads (F)

The current through the circuit at resonance is:

$$\mathbf{I} = \frac{E\angle 0^\circ}{R\angle 0^\circ} = \frac{E}{R}\angle 0^\circ$$

Consider also that *the input voltage and current are in phase at resonance.*

- Since the current is the same through the capacitor and inductor, the voltage across each is equal in magnitude but 180° out of phase at resonance:

$$\left.\begin{aligned} \mathbf{V}_L &= (I\angle 0^\circ)(X_L\angle 90^\circ) = IX_L\angle 90^\circ \\ \mathbf{V}_C &= (I\angle 0^\circ)(X_C\angle -90^\circ) = IX_C\angle -90^\circ \end{aligned}\right\} \text{180° out of phase}$$

and, since $X_L = X_C$, the magnitude of V_L equals V_C at resonance; that is,

$$V_{L_s} = V_{C_s}$$

The Quality Factor (*Q*)

The **quality factor** Q of a series resonant circuit is defined as the ratio of the reactive power of the inductor or the capacitor to the average power of the resistor at resonance; that is:

$$Q_s = \frac{\text{reactive power}}{\text{average power}}$$

- The quality factor is also an indication of how much energy is placed in storage (continual transfer from one reactive element to the other) compared to that dissipated.
- For series resonant circuits used in communication systems, Q_s is usually greater than 1.
- The lower the level of dissipation for the same reactive power, the larger the Q_s factor and the more concentrated and intense the region of resonance.

$$Q_s = \frac{I^2 X_L}{I^2 R}$$

and

$$Q_s = \frac{X_L}{R} = \frac{\omega_s L}{R}$$

If the resistance R is just the resistance of the coil (R_l), where:

$$Q_{\text{coil}} = Q_l = \frac{X_L}{R_l} \qquad R = R_l$$

If we substitute:

$$\omega_s = 2\pi f_s$$

and then

$$f_s = \frac{1}{2\pi\sqrt{LC}}$$

We have:

$$Q_s = \frac{\omega_s L}{R} = \frac{2\pi f_s L}{R} = \frac{2\pi}{R}\left(\frac{1}{2\pi\sqrt{LC}}\right)L$$

$$= \frac{L}{R}\left(\frac{1}{\sqrt{LC}}\right) = \left(\frac{\sqrt{L}}{\sqrt{L}}\right)\frac{L}{R\sqrt{LC}}$$

and

$$Q_s = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$V_L = \frac{X_L E}{Z_T} = \frac{X_L E}{R} \quad \text{(at resonance)}$$

and

$$\boxed{V_{L_s} = Q_s E}$$

or

$$V_C = \frac{X_C E}{Z_T} = \frac{X_C E}{R}$$

and

$$\boxed{V_{C_s} = Q_s E}$$

- The total impedance of the series R-L-C circuit is:

$$Z_T = R + jX_L - jX_C \quad \text{or} \quad Z_T = R + j(X_L - X_C)$$

For the capacitor:

$$X_C = \frac{1}{2\pi f C} \quad \text{or} \quad X_C f = \frac{1}{2\pi C}$$

The hyperbolic curve for X_C (f) is plotted in figure (3) below.

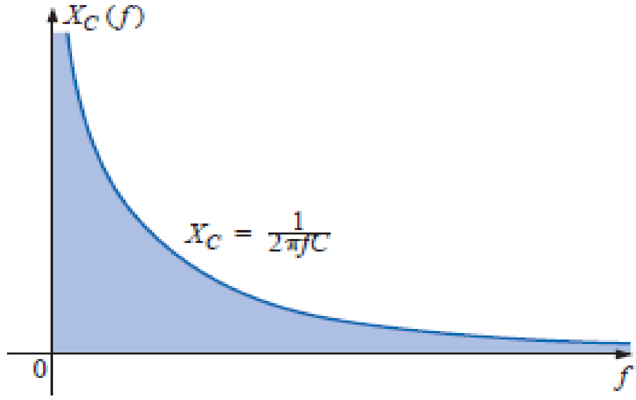


Fig.3 Capacitive reactance versus frequency

If we place the two figures of inductive reactance and capacitive reactance on the same set of axes, we obtain the curves of figure (4).

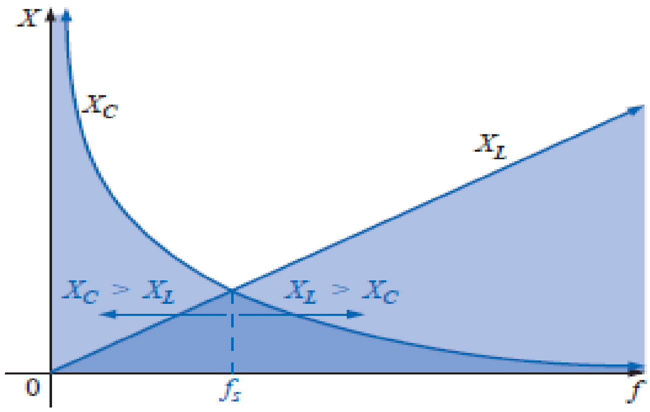


Fig.4 Frequency response of series R-L-C circuit at resonances

The condition of resonance is now clearly defined by the point of intersection in curve in the figure (4) when $X_L = X_C$.

- For frequencies less than f_s, it is clear that the network is primarily capacitive ($X_C > X_L$).
- For frequencies above the resonant condition, $X_L > X_C$, and the network is inductive.
- The minimum impedance occurs at the resonant frequency and is equal to the resistance R.

In general, therefore, for a series resonant circuit:

$f < f_s$:	network capacitive; **I** leads **E**
$f > f_s$:	network inductive; **E** leads **I**
$f = f_s$:	network resistive; **E** and **I** are in phase

Half-power frequencies are those frequencies at which the power delivered is one-half that delivered at the resonant frequency; that is,

$$P_{max} = I_{max}^2 R$$

and $$P_{HPF} = I^2 R = (0.707 I_{max})^2 R = (0.5)(I_{max}^2 R) = \frac{1}{2} P_{max}$$

$$\boxed{P_{HPF} = \frac{1}{2} P_{max}}$$

In terms of Q_s, if R is larger for the same X_L, then Q_s is less, as determined by the equation $Qs = \omega_s L/R$.

$$Z_T = \sqrt{R^2 + (X_L - X_C)^2}$$

becomes $$\sqrt{2}R = \sqrt{R^2 + (X_L - X_C)^2}$$

or, squaring both sides, that

$$2R^2 = R^2 + (X_L - X_C)^2$$

and $$R^2 = (X_L - X_C)^2$$

Taking the square root of both sides gives:

$$R = X_L - X_C \quad \text{or} \quad R - X_L + X_C = 0$$

Let us first consider the case where $X_L > X_C$, which relates to f_2 or ω_2. Substituting $\omega_2 L$ for X_L and $1/\omega_2 C$ for X_C and bringing both quantities to the left of the equal sign, we have:

$$R - \omega_2 L + \frac{1}{\omega_2 C} = 0 \quad \text{or} \quad R\omega_2 - \omega_2^2 L + \frac{1}{C} = 0$$

which can be written:

$$\omega_2^2 - \frac{R}{L}\omega_2 - \frac{1}{LC} = 0$$

Solving the quadratic, we have:

$$\omega_2 = \frac{-(-R/L) \pm \sqrt{[-(R/L)]^2 - [-(4/LC)]}}{2}$$

and

$$\omega_2 = +\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}$$

with

$$f_2 = \frac{1}{2\pi}\left[\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right] \quad \text{(Hz)}$$

If we repeat the same procedure for $X_C > X_L$, which relates to ω_1 or f_1 such that:

$$Z_T = \sqrt{R^2 + ((X_c - X_L))^2}$$

the solution f_1 becomes:

$$f_1 = \frac{1}{2\pi}\left[-\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right] \quad \text{(Hz)}$$

The bandwidth (BW) is:

$$BW = f_2 - f_1 = \frac{R}{2\pi L}$$

Substituting $R/L = \omega_s/Q_s$ from $Q_s = \omega_s L/R$ and $1/2\pi = f_s/\omega_s$ from $\omega_s = 2\pi f_s$ gives us

$$BW = \frac{R}{2\pi L} = \left(\frac{1}{2\pi}\right)\left(\frac{R}{L}\right) = \left(\frac{f_s}{\omega_s}\right)\left(\frac{\omega_s}{Q_s}\right)$$

or

$$BW = \frac{f_s}{Q_s}$$

$$\frac{f_2 - f_1}{f_s} = \frac{1}{Q_s}$$

- The ratio $(f_2 - f_1)/fs$ is sometimes called the *fractional bandwidth,* providing an indication of the width of the bandwidth compared to the resonant frequency.
- It can also be shown through mathematical manipulations of the pertinent equations that the resonant frequency is related to the geometric mean of the band frequencies; that is:

$$f_s = \sqrt{f_1 f_2}$$

2- Parallel Resonant Circuit

The parallel resonant circuit has the basic configuration of figure (5).

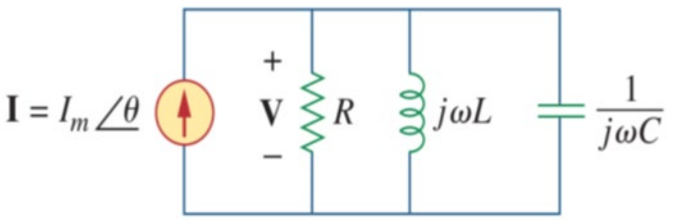


Fig. 5 parallel resonant circuit

- For the parallel resonant circuit, the impedance is relatively high at resonance, producing a significant voltage for $\mathbf{V}_C$ and $\mathbf{V}_L$.
- Resonance will occur when $X_L = X_C$, and the resonant frequency will have the same format obtained for series resonance.

$$\mathbf{Y} = H(\omega) = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

Resonance occurs when the imaginary part of **Y** is zero.

$$\omega C - \frac{1}{\omega L} = 0$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$

In the parallel circuits we are replacing R, L, and C in the expressions for the series circuit with 1/R, C and L respectively, we obtain for the parallel circuit:

$$\boxed{\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}} \qquad \boxed{\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}}$$

$$\boxed{B = \omega_2 - \omega_1 = \frac{1}{RC}} \qquad \boxed{Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}}$$

The half power frequencies in terms of the quality factor are:

$$\boxed{\omega_1 = \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} - \frac{\omega_0}{2Q}, \qquad \omega_2 = \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} + \frac{\omega_0}{2Q}}$$

For high-Q circuits (Q ≥10)

$$\boxed{\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}}$$

3- Combined Resonant Circuits

The combined circuit is mixing of serial and parallel component, for this circuit at resonance , the parameters of quality factor Q , bandwidth (B.W or β), voltages and currents can be calculate by find serial or parallel equivalent of the combined circuit as below:

3-1 Serial to Parallel Conversion

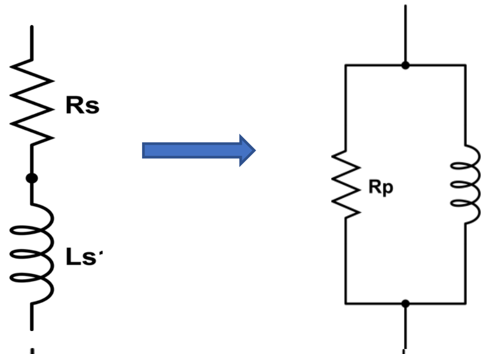
We could convert the serial branch to parallel and determine the Q as shown below.

R-L circuit

$R_P = R_S\,(1 + Q_S^2)$

$L_P = L_S\,((1 + Q_S^2) / Q_S^2)$

Where $Q_S = (\omega_O L_S) / R_S$

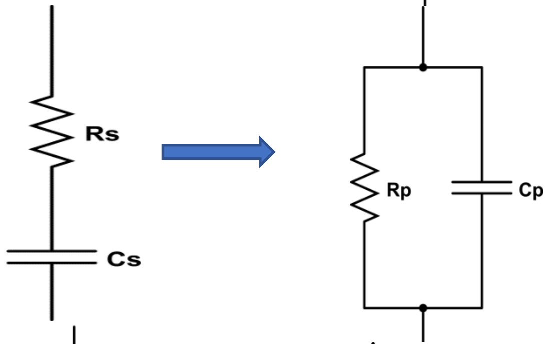


- **R-C circuit**

$R_P = R_S\,(1 + Q_S^2)$

$C_P = C_S\,(Q_S^2 / (1 + Q_S^2))$

Where $Q_S = 1 / \omega_O C_S R_S$



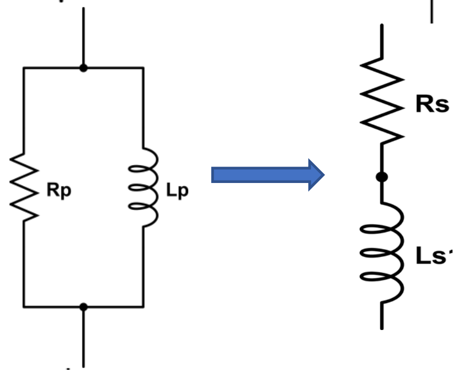
3-2 Parallel to Serial Conversion

- **R-L circuit**

$R_S = R_P\,(1/ (1 + Q_P^2))$

$L_S = L_P\,(Q_P^2 / (1 + Q_P^2))$

Where $Q_P = R_P / (\omega_O L_P)$

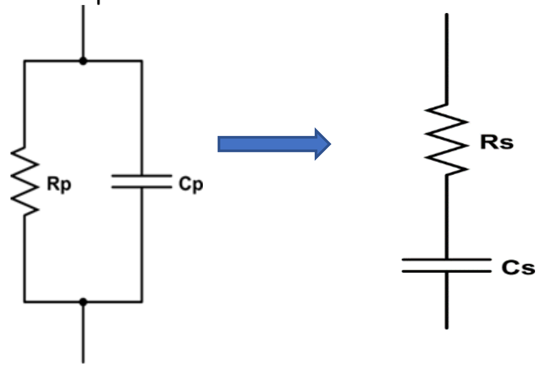


- **R- C circuit**

$R_S = R_P\,(1/(1 + Q_P^2))$

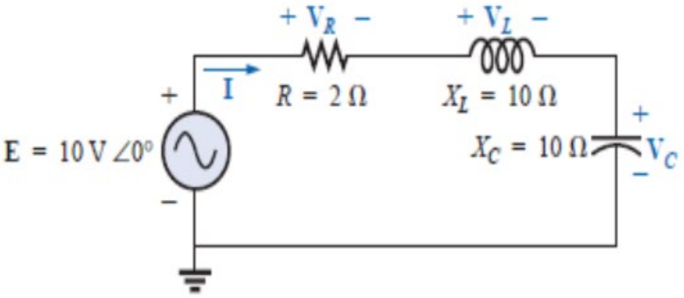
$C_s = C_p\,((1 + Q_P^2) / Q_P^2)$

$\mathbf{Q_p = \omega_O C_P R_P}$



Example.1

a. For the series resonant circuit of below, find **I**, **V**$_R$, **V**$_L$, and **V**$_C$ at resonance.
b. What is the Q_s of the circuit?
c. If the resonant frequency is 5000 Hz, find the bandwidth.
d. What is the power dissipated in the circuit at the half-power frequencies?



Solutions:

a. $\mathbf{Z}_{T_s} = R = 2\,\Omega$

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_{T_s}} = \frac{10\text{ V}\angle 0^\circ}{2\,\Omega\angle 0^\circ} = \mathbf{5\text{ A}\angle 0^\circ}$$

$$\mathbf{V}_R = \mathbf{E} = 10\text{ V}\angle 0^\circ$$

$$\mathbf{V}_L = (I\angle 0^\circ)(X_L\angle 90^\circ) = (5\text{ A}\angle 0^\circ)(10\,\Omega\angle 90^\circ) = \mathbf{50\text{ V}\angle 90^\circ}$$

$$\mathbf{V}_C = (I\angle 0^\circ)(X_C\angle -90^\circ) = (5\text{ A}\angle 0^\circ)(10\,\Omega\angle -90^\circ) = \mathbf{50\text{ V}\angle -90^\circ}$$

b. $$Q_s = \frac{X_L}{R} = \frac{10\,\Omega}{2\,\Omega} = \mathbf{5}$$

c. $$BW = f_2 - f_1 = \frac{f_s}{Q_s} = \frac{5000\text{ Hz}}{5} = \mathbf{1000\text{ Hz}}$$

d. $$P_{HPF} = \frac{1}{2}P_{max} = \frac{1}{2}I_{max}^2 R = \left(\frac{1}{2}\right)(5\text{ A})^2(2\,\Omega) = \mathbf{25\text{ W}}$$

Example.2

The bandwidth of a series resonant circuit is 400 Hz.
a. If the resonant frequency is 4000 Hz, what is the value of Q_s?
b. If R =10Ω, what is the value of X_L at resonance?
c. Find the inductance L and capacitance C of the circuit.

Solutions:

a. $$BW = \frac{f_s}{Q_s} \quad \text{or} \quad Q_s = \frac{f_s}{BW} = \frac{4000\text{ Hz}}{400\text{ Hz}} = \mathbf{10}$$

b. $$Q_s = \frac{X_L}{R} \quad \text{or} \quad X_L = Q_s R = (10)(10\,\Omega) = \mathbf{100\,\Omega}$$

c. $$X_L = 2\pi f_s L \quad \text{or} \quad L = \frac{X_L}{2\pi f_s} = \frac{100\,\Omega}{2\pi(4000\text{ Hz})} = \mathbf{3.98\text{ mH}}$$

$$X_C = \frac{1}{2\pi f_s C} \quad \text{or} \quad C = \frac{1}{2\pi f_s X_C} = \frac{1}{2\pi(4000\text{ Hz})(100\,\Omega)}$$

$$= \mathbf{0.398\ \mu F}$$

Example.3

A series *R-L-C* circuit has a series resonant frequency of 12,000 Hz.
a. If R =5 Ω, and if X_L at resonance is 300Ω, find the bandwidth.
b. Find the cutoff frequencies.

Solutions:

a. $Q_s = \frac{X_L}{R} = \frac{300\ \Omega}{5\ \Omega} = 60$

$$BW = \frac{f_s}{Q_s} = \frac{12{,}000\ \text{Hz}}{60} = \mathbf{200\ Hz}$$

b. Since $Q_s \geq 10$, the bandwidth is bisected by f_s. Therefore,

$$f_2 = f_s + \frac{BW}{2} = 12{,}000\ \text{Hz} + 100\ \text{Hz} = \mathbf{12{,}100\ Hz}$$

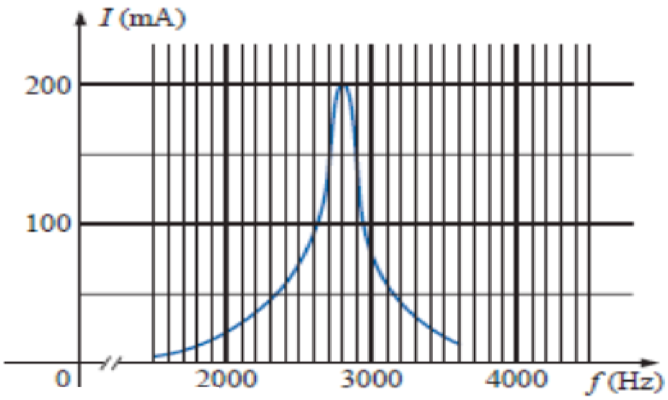
and $f_1 = 12{,}000\ \text{Hz} - 100\ \text{Hz} = \mathbf{11{,}900\ Hz}$

Example.4

a. Determine the Qs and bandwidth for the response curve of figure shown.

b. For C= 101.5 nF, determine L and R for the series resonant circuit.

c. Determine the applied voltage.



Solutions:

a. The resonant frequency is 2800 Hz. At 0.707 times the peak value,

$$BW = \mathbf{200\ Hz}$$

and $$Q_s = \frac{f_s}{BW} = \frac{2800\ \text{Hz}}{200\ \text{Hz}} = \mathbf{14}$$

b. $f_s = \frac{1}{2\pi\sqrt{LC}}$ or $L = \frac{1}{4\pi^2 f_s^2 C}$

$$= \frac{1}{4\pi^2(2.8 \times 10^3\ \text{Hz})^2(101.5 \times 10^{-9}\ \text{F})}$$

$$= \mathbf{31.832\ mH}$$

$Q_s = \frac{X_L}{R}$ or $R = \frac{X_L}{Q_s} = \frac{2\pi(2800\ \text{Hz})(31.832 \times 10^{-3}\ \text{H})}{14}$

$$= \mathbf{40\ \Omega}$$

c. $I_{max} = \frac{E}{R}$ or $E = I_{max}R$

$$= (200\ \text{mA})(40\ \Omega) = \mathbf{8\ V}$$

Example.5

A series *R-L-C* circuit is designed to resonant at ω_s =105 rad/s, have a bandwidth of 0.15ωs, and draw 16 W from a 120 V source at resonance.

a. Determine the value of *R*.
b. Find the bandwidth in hertz.
c. Find the nameplate values of *L* and *C*.
d. Determine the *Qs* of the circuit.
e. Determine the fractional bandwidth.

Solutions:

a. $P = \dfrac{E^2}{R}$ and $R = \dfrac{E^2}{P} = \dfrac{(120\text{ V})^2}{16\text{ W}} = \mathbf{900\ \Omega}$

b. $f_s = \dfrac{\omega_s}{2\pi} = \dfrac{10^5\text{ rad/s}}{2\pi} = 15{,}915.49\text{ Hz}$

$$BW = 0.15 f_s = 0.15(15{,}915.49\text{ Hz}) = \mathbf{2387.32\ Hz}$$

c. Eq. (20.20):

$$BW = \frac{R}{2\pi L} \quad \text{and} \quad L = \frac{R}{2\pi BW} = \frac{900\ \Omega}{2\pi(2387.32\text{ Hz})} = \mathbf{60\ mH}$$

$$f_s = \frac{1}{2\pi\sqrt{LC}} \quad \text{and} \quad C = \frac{1}{4\pi^2 f_s^2 L} = \frac{1}{4\pi^2(15{,}915.49\text{ Hz})^2(60\times10^{-3}\text{ H})} = \mathbf{1.67\ nF}$$

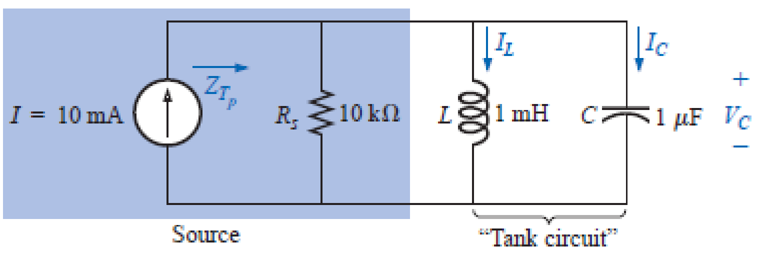
d. $Q_s = \dfrac{X_L}{R} = \dfrac{2\pi f_s L}{R} = \dfrac{2\pi(15{,}915.49\text{ Hz})(60\text{ mH})}{900\ \Omega} = \mathbf{6.67}$

e. $\dfrac{f_2 - f_1}{f_s} = \dfrac{BW}{f_s} = \dfrac{1}{Q_s} = \dfrac{1}{6.67} = \mathbf{0.15}$

Example.6

For the parallel circuit shown below:

a. Determine the resonant frequency f_p.
b. Find the total impedance at resonance.
c. Calculate the quality factor, bandwidth, and cutoff frequencies f_1 and f_2 of the system.
d. Find the voltage V_C at resonance.
e. Determine the currents I_L and I_C at resonance.



Solutions:

a. The fact that R_l is zero ohms results in a very high Q_l (= X_L/R_l), permitting the use of the following equation for f_p:

$$f_p = f_s = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(1\text{ mH})(1\ \mu\text{F})}}$$
$$= \mathbf{5.03\ kHz}$$

b. For the parallel reactive elements:

$$\mathbf{Z}_L \parallel \mathbf{Z}_C = \frac{(X_L \angle 90°)(X_C \angle -90°)}{+j(X_L - X_C)}$$

but $X_L = X_C$ at resonance, resulting in a zero in the denominator of the equation and a very high impedance that can be approximated by an open circuit. Therefore,

$$Z_{T_p} = R_s \parallel \mathbf{Z}_L \parallel \mathbf{Z}_C = R_s = \mathbf{10\ k\Omega}$$

c. $$Q_p = \frac{R_s}{X_{L_p}} = \frac{R_s}{2\pi f_p L} = \frac{10\text{ k}\Omega}{2\pi(5.03\text{ kHz})(1\text{ mH})} = \mathbf{316.41}$$

$$BW = \frac{f_p}{Q_p} = \frac{5.03\text{ kHz}}{316.41} = \mathbf{15.90\ Hz}$$

Eq. (20.39a):

$$f_1 = \frac{1}{4\pi C}\left[\frac{1}{R} - \sqrt{\frac{1}{R^2} + \frac{4C}{L}}\right]$$
$$= \frac{1}{4\pi(1\ \mu\text{F})}\left[\frac{1}{10\text{ k}\Omega} - \sqrt{\frac{1}{(10\text{ k}\Omega)^2} + \frac{4(1\ \mu\text{F})}{1\text{ mH}}}\right]$$
$$= \mathbf{5.025\ kHz}$$

$$f_2 = \frac{1}{4\pi C}\left[\frac{1}{R} + \sqrt{\frac{1}{R^2} + \frac{4C}{L}}\right]$$
$$= \mathbf{5.041\ kHz}$$

d. $$V_C = IZ_{T_p} = (10\text{ mA})(10\text{ k}\Omega) = \mathbf{100\ V}$$

e. $$I_L = \frac{V_L}{X_L} = \frac{V_C}{2\pi f_p L} = \frac{100\text{ V}}{2\pi(5.03\text{ kHz})(1\text{ mH})} = \frac{100\text{ V}}{31.6\ \Omega} = \mathbf{3.16\ A}$$

$$I_C = \frac{V_C}{X_C} = \frac{100\text{ V}}{31.6\ \Omega} = \mathbf{3.16\ A}\ (= Q_p I)$$

Admittance and Current Locus

Admittance and Current Locus

Locus diagrams are the graphical representations of the way in which the response of electrical circuits vary, when one or more parameters are continuously changing.
Electric circuit analysis could be implemented by using engineering locus shapes, where: I=V.Y (ohm's low) and constant voltage "V", so the admittance locus "Y" represents the change in current "I" value with respect to change in element value. So, Locus curves are vector diagrams where only the tip of the vector is dependent on some parameter (ω, R, L & C).
Admittance "Y": It's reciprocal of impedance "Z" or vice versa (Z = 1/Y or Y = 1/Z,), that is a complex number consists of real and imaginary parts, its unit is mho (℧).

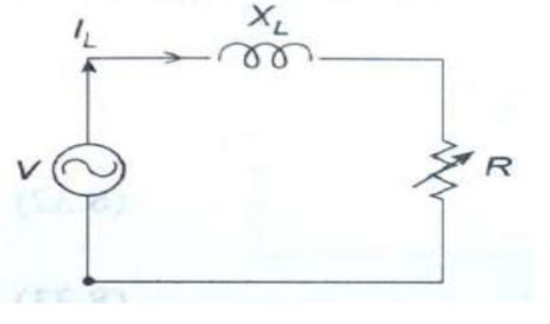
- The **susceptance (B)** is the imaginary part of **admittance (Y)**, where the real part is **conductance (G)**. In SI units, admittance is measured in Siemens.

- The reciprocal of admittance is **impedance (Z)**, where the imaginary part is reactance (X), and the real part is resistance (R).

1. Current and Admittance Locus for R-L Series Circuit

Case 1: when R is varied, and X_L is constant.

Refer to the series R-L circuit shown in the figure (a) below with constant X_L and varying R.



- The current I_L lags the applied voltage V by a phase angle:
  $\Phi = \tan^{-1}(X_L/R)$ for a given value of R.
- When R=0, the current is maximum equal to V/X_L and lies along the I axis with phase angle equal to 90°.
- When R is increased from zero to infinity the current gradually reduces from V/X_L to zero, and phase angle also reduces from 90° to 0°.

$$Z = R + jX_L = \frac{1}{G+jB} = \frac{1}{G+jB} * \frac{G-jB}{G-jB} = \frac{G-jB}{G^2+B^2} = \frac{G}{G^2+B^2} - j\frac{B}{G^2+B^2}$$

So:

$$X_L = \frac{-B}{G^2+B^2} \quad and \quad R = \frac{G}{G^2+B^2}$$

From X_L relationship (Constant Element): $G^2 + \frac{B}{X_L} + B^2 = 0$

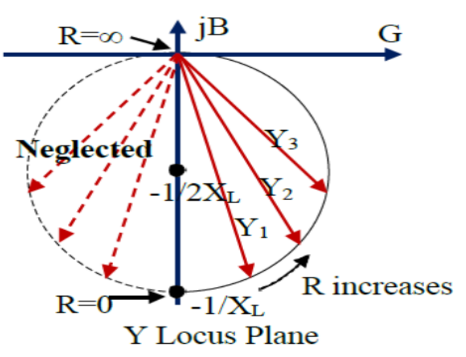
Or

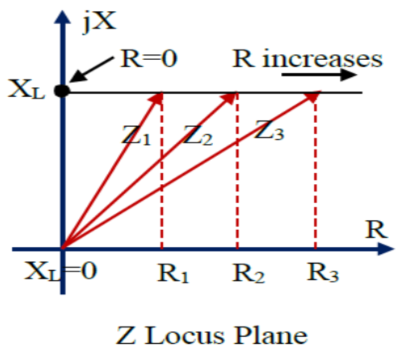
$$G^2 + \frac{B}{X_L} + B^2 + \left(\frac{1}{2X_L}\right)^2 = \left(\frac{1}{2X_L}\right)^2$$

$$G^2 + \left(B + \frac{1}{2X_L}\right)^2 = \left(\frac{1}{2X_L}\right)^2$$

The above equation is a circle (as compared with the general circle equation $(x - h)^2 + (y - k)^2 = r^2$) with center $(0, -\frac{1}{2X_L})$ and radius $\frac{1}{2X_L}$.

From real part of the equation (variable element), G must be always positive to give (because) R always positive, so the admittance locus is half right circle.





Note: Current locus is the same as admittance locus multiplied by the applied voltage "V" (I=V*Y), so there are three cases:

(V=1) current & admittance locus are the same,

(V>1) current > admittance locus

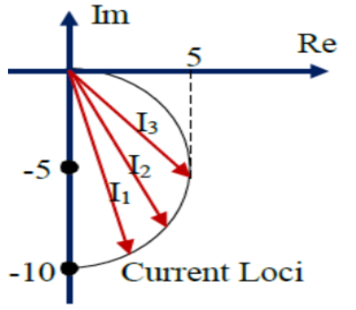
(V<1) current < admittance locus.

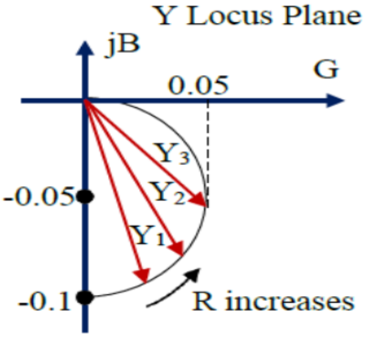
Example:

100V applied to series R&L circuit with variable "R" and X_L=10Ω. Draw the admittance and current locus.

Solution:

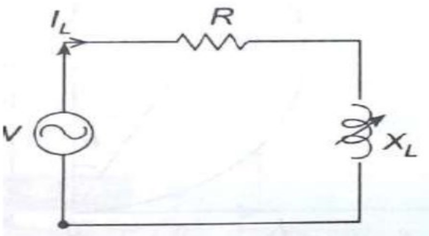
R is variable and X_L=10Ω, so the admittance locus its half right circle with:
Center = (0, -1/2X_L) = (0, -1/20 ℧)
And radius = (-1/20 ℧)
And the current locus (I=V.Y) is half right circle too with:
Center = (0, -100/20) = (0, -5 A) and radius = -5A
Note Y_1, Y_2, Y_3,..., etc. represents complex value of total admittance for the circuit. While I_1, I_2, I_3,, etc. represents complex value of total current in the circuit.





Case 2: When X_L is varied, and R is constant.

The current I_L lags behind the applied voltage V by a phase angle $\Phi = \tan^{-1}(X_L/R)$. When $X_L = 0$, the current is maximum equal to V/R and lies along the +ve V axis with phase angle equal to 0°. When X_L is increased from zero to infinity the current gradually reduces from V/R to 0 and phase angle increases from 0^0 to 90^0.



A few other points will confirm the semicircular locus, with the center at 1/2R and radius 1/2R.

Or in other way:

$$Z = R + jX_L = \frac{1}{G + jB} = \frac{1}{G + jB} * \frac{G - jB}{G - jB} = \frac{G - jB}{G^2 + B^2} = \frac{G}{G^2 + B^2} - j\frac{B}{G^2 + B^2}$$

$$X_L = \frac{-B}{G^2 + B^2} \quad and \quad R = \frac{G}{G^2 + B^2}$$

From R relationship (Constant Element):

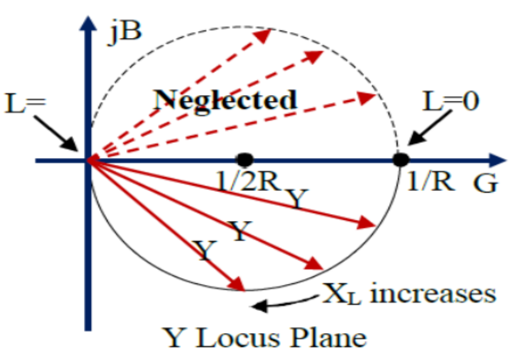
$$G^2 + B^2 = \frac{G}{R}$$

$G^2 - \frac{G}{R} + B^2 = 0$ or $G^2 - \frac{G}{R} + B^2 + \left(\frac{1}{2R}\right)^2 = \left(\frac{1}{2R}\right)^2$

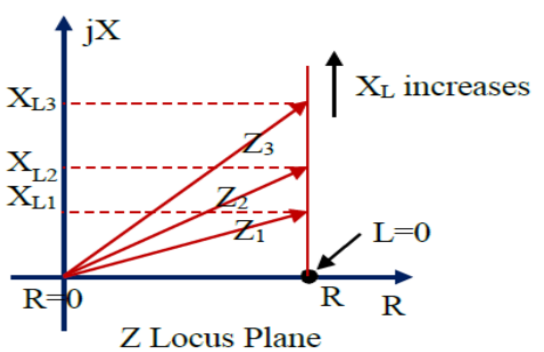
$$B^2 + \left(G - \frac{1}{2R}\right)^2 = \left(\frac{1}{2R}\right)^2$$

The above equation is a circle with center $(\frac{1}{2R}, 0)$ and radius $\frac{1}{2R}$

From imaginary part of the equation (variable element X_L), B must be always negative to make (because) X_L always positive, so the admittance loci are half lower circle.



Y Locus Plane



Z Locus Plane

Example:

20V applied to series R-L circuit with variable "L" and R=5Ω. Draw the admittance and current locus.

Solution:

X_L is variable and R =10Ω the admittance locus, its half lower circle with:

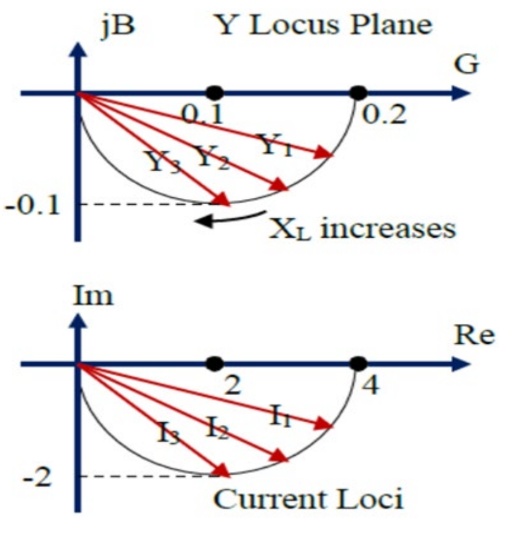
$$\mathbf{Y} = \frac{1}{R + j\omega L} = \frac{1}{\sqrt{R^2 + (\omega L)^2}} \angle \tan^{-1}(-\omega L/R)$$

Note that for $\omega L = 0$, $\mathbf{Y} = (1/R)\angle 0^\circ$; and for $\omega L \to \infty$, $\mathbf{Y} \to 0\angle -90^\circ$. When $\omega L = R$,

$$\mathbf{Y} = \frac{1}{R\sqrt{2}} \angle -45^\circ$$

Centered at $(0, \frac{1}{2R})$ = (0, 0.1 ℧) and radius $\frac{1}{2R} = 0.1$℧

The current locus (I=V.Y Ohm's law) is half lower circle too with center = (0, V/2R) = (0, 20/10)= (0, 2 A). And radius = V/2R=2A

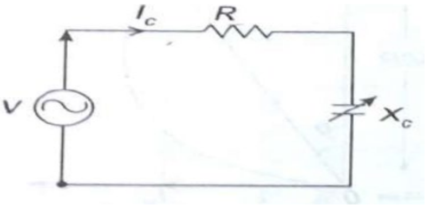


Note Y_1, Y_2, Y_3…etc. represents complex value of total admittance for the circuit. While I_1, I_2, I_3,...,etc. represents complex value of total current in the circuit.

2. Current and Admittance Locus for R-C Series Circuit

Case 1: when X_C is varied and R constant.

Fixing the resistance and change the capacitance will give straight line parallel to the imaginary axis in the Z-plane (Impedance Locus) while it will be half upper circle in the Y-plane (Admittance Locus). Also, it will be half upper circle in the current Locus.



$$Z = R - jX_C = \frac{1}{G - jB} = \frac{1}{G + jB} * \frac{G - jB}{G - jB} = \frac{G - jB}{G^2 + B^2} = \frac{G}{G^2 + B^2} - j\frac{B}{G^2 + B^2}$$

$$X_C = \frac{B}{G^2 + B^2} \quad and \quad R = \frac{G}{G^2 + B^2}$$

From R relationship (Constant Element):

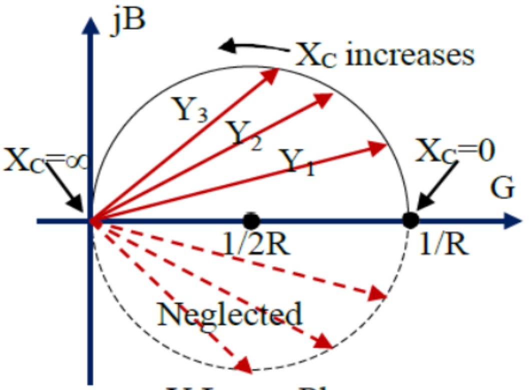
$$G^2 + B^2 = \frac{G}{R}$$

$G^2 - \frac{G}{R} + B^2 = 0$ or $G^2 - \frac{G}{R} + B^2 + \left(\frac{1}{2R}\right)^2 = \left(\frac{1}{2R}\right)^2$

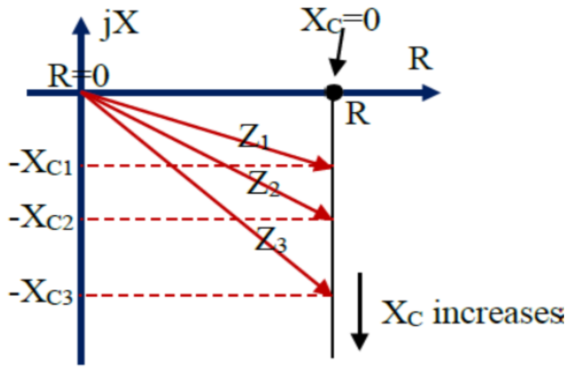
$$\left(G - \frac{1}{2R}\right)^2 + B^2 = \left(\frac{1}{2R}\right)^2$$

The above equation is a circle with center ($\frac{1}{2R}$, 0) and radius $\frac{1}{2R}$

From imaginary part of the equation (variable element), B must be always positive to make (because) X_C always positive, so the admittance locus is half upper. circle.



Y Locus Plane



Z Locus Plane

Example:

50V applied to series R-C circuit with variable "C" and R=20Ω. Draw the admittance and current locus.

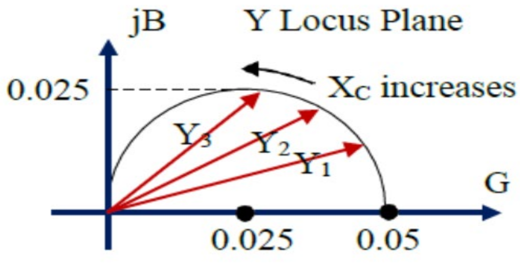
Solution:

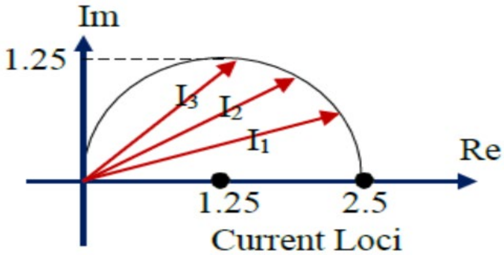
Variable "C" and R=20Ω the admittance locus is half upper circle with:

Center = (1/2R, 0) = (0.025 ℧, 0), and radius = 0.025 ℧

While the current locus is half upper circle with:

Center = (V/2R, 0) = (1.25 A, 0), and radius = 1.25 A

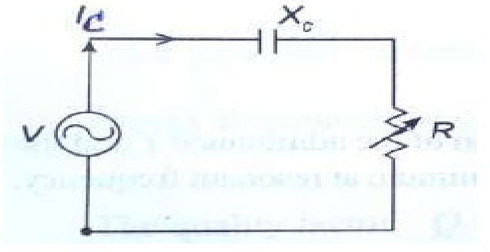




Current Loci

Case 2: when R is varied and X_C constant.

Fixing the capacitance and change the resistance will give straight line parallel to the real axis in the Z-plane (Impedance Locus) while it will be half right circle in the Y-plane (Admittance Locus). Also, it will be half right circle in the current Locus.



$$Z = R - jX_C = \frac{1}{G+jB} = \frac{1}{G+jB} * \frac{G-jB}{G-jB} = \frac{G-jB}{G^2+B^2} = \frac{G}{G^2+B^2} - j\frac{B}{G^2+B^2}$$

So: $X_C = \frac{B}{G^2+B^2}$ and $R = \frac{G}{G^2+B^2}$

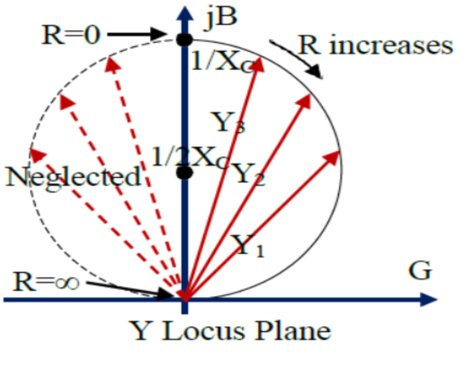
From X_C relationship (Constant Element): $G^2 - \frac{B}{X_C} + B^2 = 0$

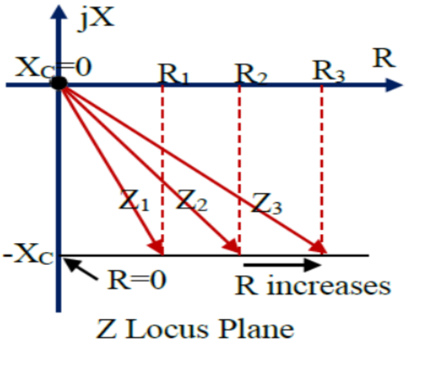
$$\text{Or: } G^2 - \frac{B}{X_C} + B^2 + \left(\frac{1}{2X_C}\right)^2 = \left(\frac{1}{2X_C}\right)^2$$

$$G^2 + \left(B - \frac{1}{2X_C}\right)^2 = \left(\frac{1}{2X_C}\right)^2$$

The above equation is a circle with center $(0, \frac{1}{2X_C})$ and radius $\frac{1}{2X_C}$.

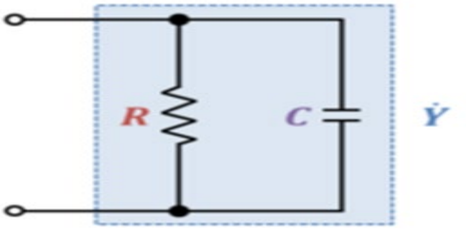
From real part of the equation (variable), G must be always positive to give (because) R always positive, so the admittance loci are half right circle.



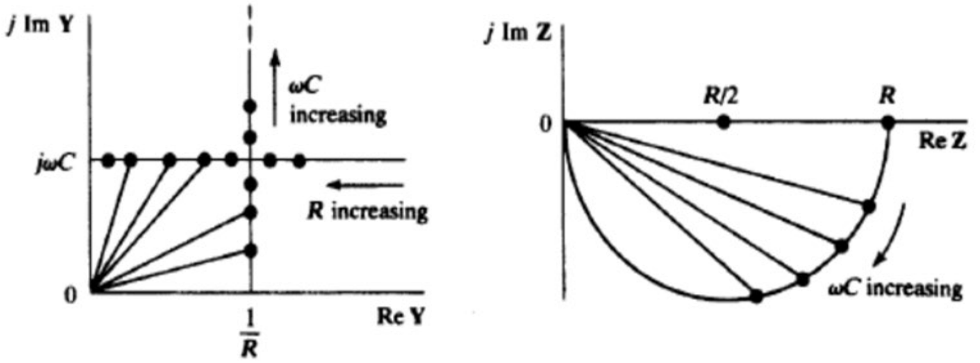


3. A parallel RC circuit with Variable X_C

Fixing the resistance and change the capacitance will give straight line parallel to the imaginary axis in the Y-plane (Impedance Locus) while it will be half lower circle in the Z-plane (Admittance Locus). Also, it will be half upper circle in the current Locus.



$$\mathbf{Y} = \frac{1}{R} + j\omega C \quad \text{and} \quad \mathbf{Z} = \frac{R}{\sqrt{1+(\omega CR)^2}} \underline{/\tan^{-1}(-\omega CR)}$$

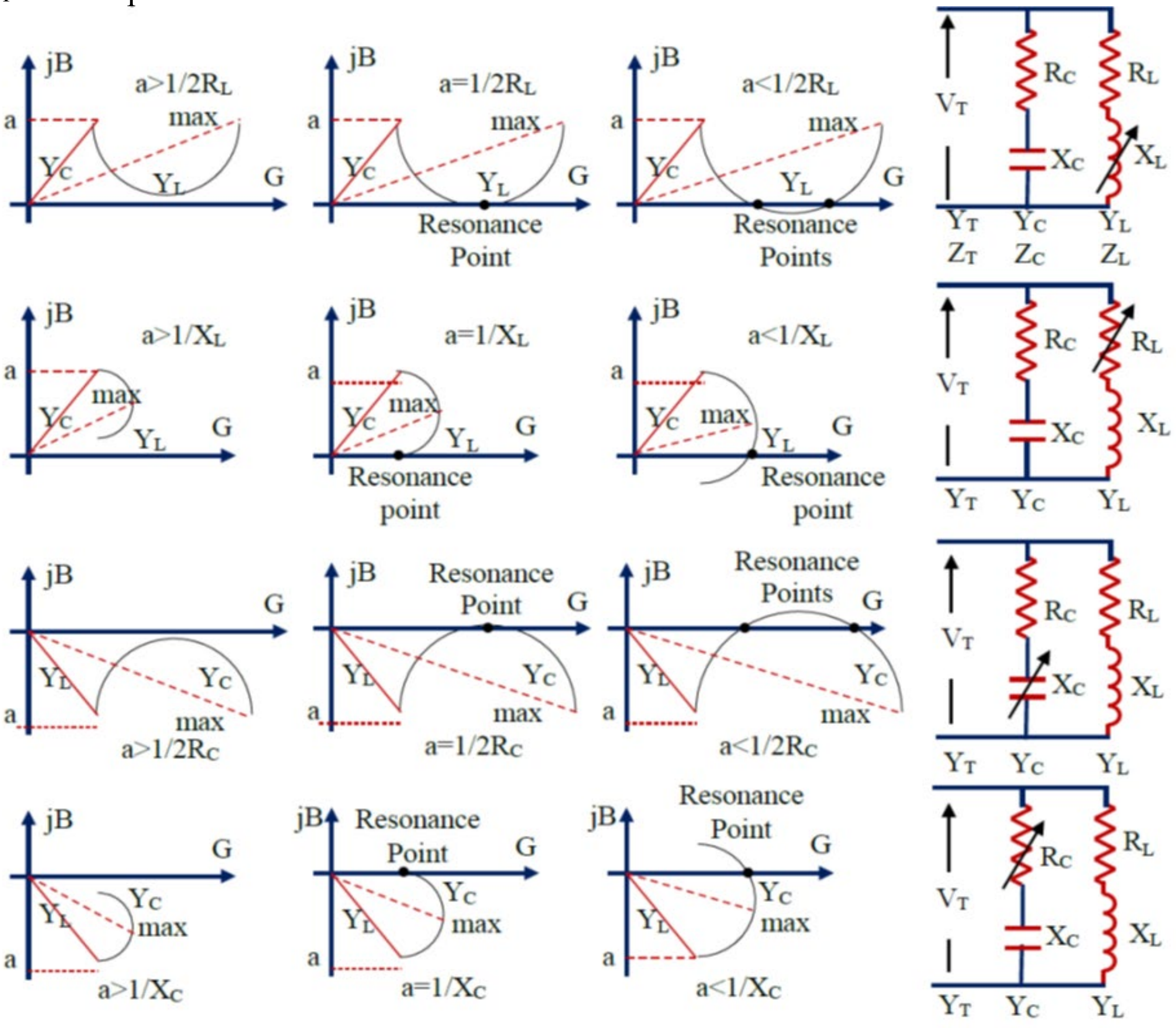


4. Admittance Locus for two Parallel Branch Electric Circuits with One Variable Element:

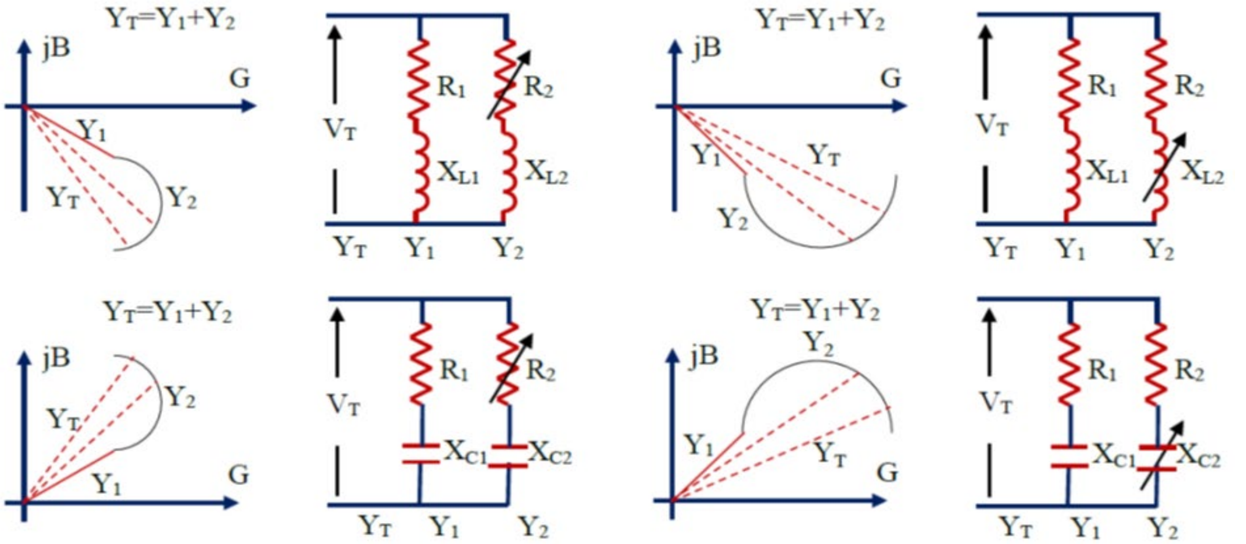
The total admittance for the two parallel branch electric circuits equal to the sum of the individual admittances for each branch as follows:

$$Y_T = Y_1 + Y_2 \text{ or } \quad \frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

So, the total admittance is any line from the origin to that point on the half circle. The resonance points for the locus could be found by intersecting the locus with the real axis (G) (i.e., real part of Y_T), while the maximum and minimum points for the locus could be found by deriving the equation of Y_T with respect to the variable element in the circuit.

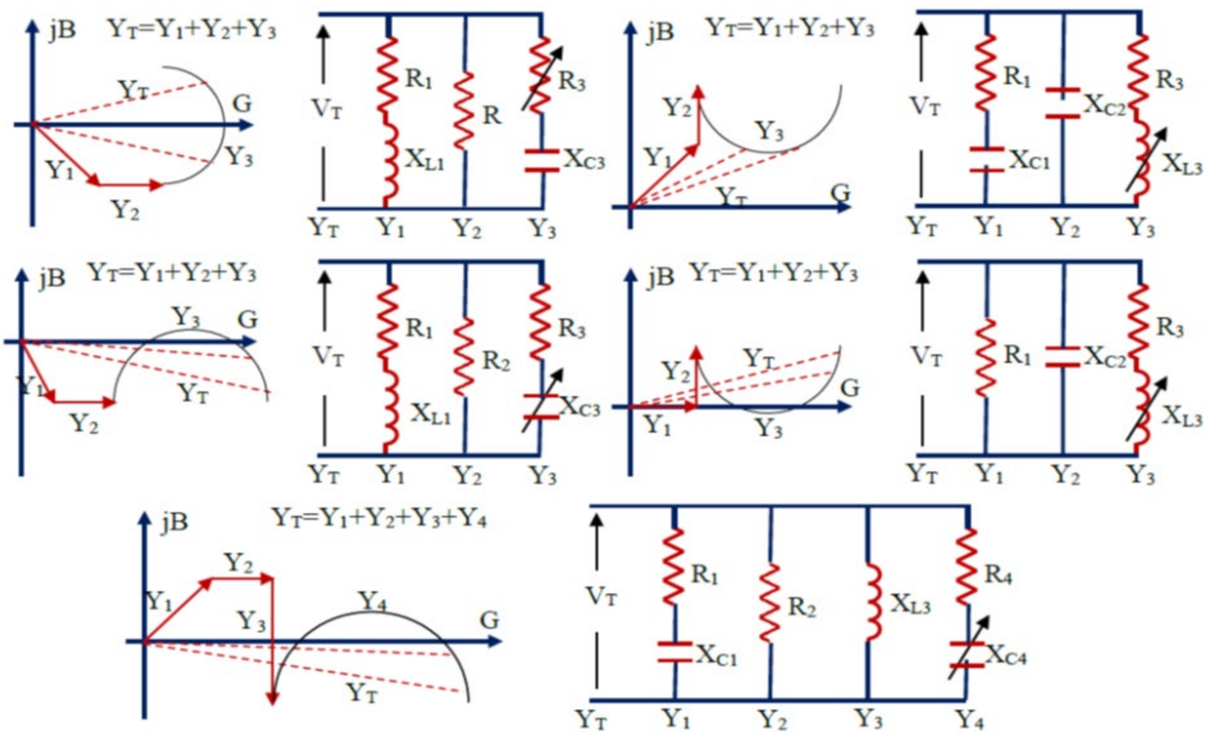


Note: In the previous circuits the resonance could be happened when the admittance locus intersects the real axis (G), then the total current will be in- phase with the total applied voltage. In the next circuits there are no resonance case (the admittance locus does not intersect the real axis).



5. Admittance Locus for Circuits with More Than Two Parallel Branches

The total admittance for these parallel electric circuits is equal to the sum of the admittances of each branch. Also, the resonance may be happened or not depend on the intersection of the locus with the real axis.



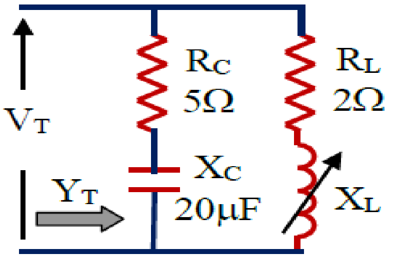
Note: The total impedance is the reciprocal of the total admittance for the circuit, and the total current, power & power factor for the circuit could be implemented.

Example:

For the circuit shown:

1. Derive an equation for suitable value of RL that give resonance at two points.
2. Find the value of "L" that makes the circuit in resonance?
3. Find the value of total impedance at resonance for the values found in point "2"?

Note: Explain the resonance using admittance locus.

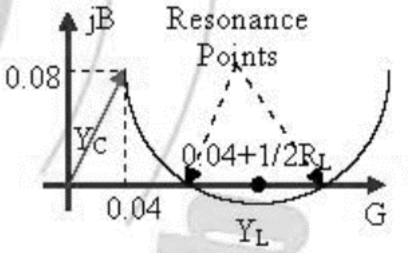


ω_O=5000 rad/sec

Solution:

1. For two points at resonance the radius of the circle must be greater than the imaginary part of admittance locus (Y_C).

$$\therefore \frac{1}{2R_L} > \frac{X_C}{R_C^2+X_C^2} \Rightarrow R_L < \frac{1}{2}\left(\frac{R_C^2}{X_C}+X_C\right)$$



2. Draw the admittance locus with the following points:

Y_C is a straight line with 0.04 real part and 0.08 imaginary part.

Y_L is half down circle with 0.25 radius and (0.25,0) center. The Y_L locus is drawn after the Y_C locus end point as seen in the locus shown.

$$X_C = \frac{1}{\omega C} = \frac{1}{500 \times 20 \times 10^{-6}} = 10\Omega$$

$$Y_T = Y_C + Y_L = \left(\frac{5}{125}+\frac{2}{4+X_L^2}\right) + j\left(\frac{10}{125}-\frac{X_L}{4+X_L^2}\right)$$

At resonance the imaginary part equal to zero.

$$\therefore \frac{10}{125} = \frac{X_L}{4+X_L^2} \Rightarrow X_L^2 - 12.5X_L + 4 = 0$$

Or $(X_L-0.33)(X_L-12.12)=0$

At $X_L=0.33\Omega \Rightarrow L_{O1}=0.33/5000=0.066mH$

At $X_L=12.12\Omega \Rightarrow L_{O2}=12.12/5000=2.43mH$

3. At $X_L=0.33\Omega$ $\quad Y_{O1} = \frac{5}{125} + \frac{2}{4+(0.33)^2} = 0.52\mho \Rightarrow Z_{O1} = \frac{1}{Y_{O1}} = 1.92\Omega$

At $X_L=12.12\Omega$ $\quad Y_{O2} = \frac{5}{125} + \frac{2}{4+(12.12)^2} = 0.053\mho \Rightarrow Z_{O2} = \frac{1}{Y_{O2}} = 18.8\Omega$

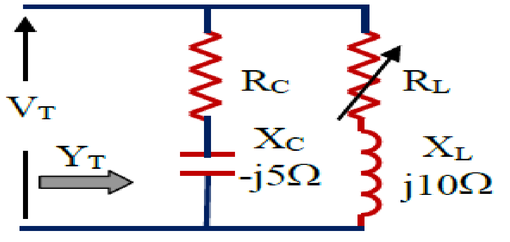
Note: As shown there are two resonance points from the locus and by using the equation of total admittance for the circuit seen by the source.

Example:

For the circuit shown:

1. Find the value of "RL" that makes the circuit in resonance at $R_C=10\Omega$.
2. Find the value of "RL" that makes the circuit in resonance at $R_C=4\Omega$. If not, what changes should be made in the circuit elements value?

Note: Prove the resonance using admittance locus.



Solution:

1. $Y_T = Y_C + Y_L = \left(\frac{10}{125} + \frac{R_L}{R_L^2+100}\right) + j\left(\frac{5}{125} - \frac{10}{R_L^2+100}\right)$

At resonance the imaginary part equal to zero.

$\frac{5}{125} = \frac{10}{R_L^2+100} \Rightarrow R_L^2 + 100 = 250$

$\therefore R_L = 12.25\Omega \Rightarrow Y_{T_O} = 0.129\mho$ and $Z_{T_O} = 7.75\Omega$

2. $Y_T = Y_C + Y_L = \left(\frac{4}{41} + \frac{R_L}{R_L^2+100}\right) + j\left(\frac{5}{41} - \frac{10}{R_L^2+100}\right)$

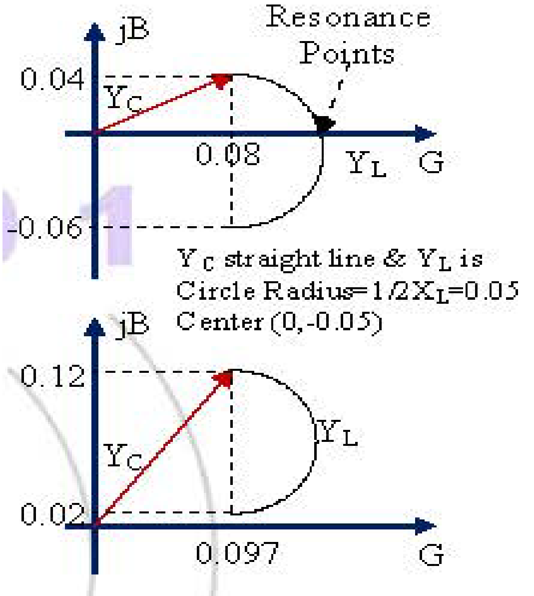
At resonance the imaginary part equal to zero.

$\frac{5}{41} = \frac{10}{R_L^2+100} \Rightarrow R_L^2 + 100 = 82$ or $R_L^2 = -18$

$\therefore$ The resonance cannot be happened, to solve this problem the value of X_L must be changed to make the circuit work at resonance as follows:

$1/X_L=0.12 \Rightarrow X_L=8.3\Omega$

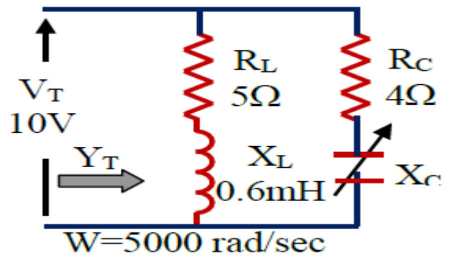
or $X_L \le 8.3\Omega$



Example:

The circuit shown work at resonance:

1. Find the values of "c".
2. Draw the admittance locus.
3. Find the total current.



Solution:

1. $Y_T = Y_C + Y_L = \left(\frac{4}{16+X_C^2} + \frac{5}{34}\right) + j\left(\frac{X_C}{16+X_C^2} - \frac{3}{34}\right)$

At resonance the imaginary part equal to zero.

$\frac{X_C}{16+X_C^2} = \frac{3}{34} \quad \Rightarrow \quad X_C^2 - 11.3X_C + 16 = 0$

$\therefore X_C = 9.68\Omega$

and $c = 1/500\times 9.68 = 20.6\mu F$

$\therefore X_C = 1.65\Omega$

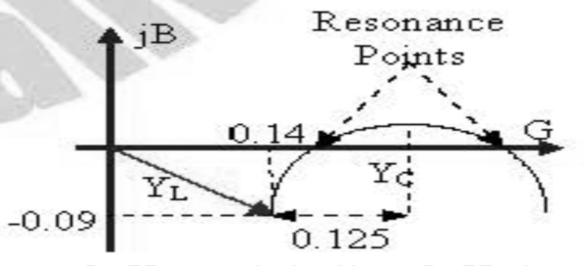
and $c = 1/500\times 1.65 = 121\mu F$

3. at $X_C = 9.68\Omega$

$Y_{O1} = \frac{4}{16+(9.68)^2} + \frac{5}{34} = 0.18\mho \quad \Rightarrow \quad I_{O1} = VY_{O1} = 1.8A$

at $X_C = 1.65\Omega$

$Y_{O2} = \frac{4}{16+(1.65)^2} + \frac{5}{34} = 0.36\mho \quad \Rightarrow \quad I_{O2} = VY_{O2} = 3.6A$



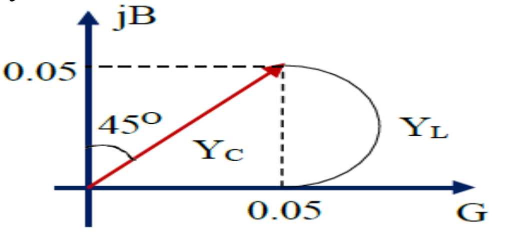
2. Y_L straight line & Y_C is Circle Radius $= 1/2R_C = 0.125$

Center (0.125,0)

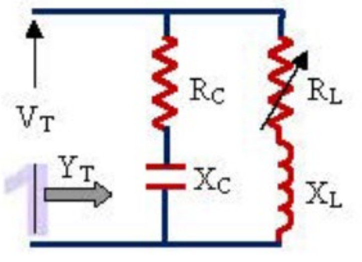
Example:

For the total admittance locus shown that represent an electrical circuit with two parallel branches, find the value of each element in the circuit.

Note: Use the angular frequency ω=500 rad/sec.



Solution:



From the locus we find that:

$$Y_C = 0.05 + j0.05 = 0.0707\angle 45^{o}\ \mho$$

$$Z_C = \frac{1}{Y_C} = 14.14\angle -45^{o}\,\Omega = (10 - j10)\Omega$$

∴ From real and imaginary parts we found that:

$R_C = 10\Omega$ and $X_C = 10\Omega$

$$c = \frac{1}{\omega X_C} = \frac{1}{500 \times 10} = 200\mu F$$

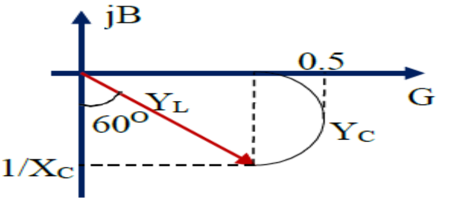
Again from the locus we find that:

Diameter=$1/X_L = 0.05\mho$ ⇒ $X_L = 20\Omega$

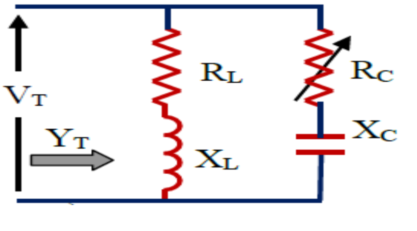
$$\therefore\ L = \frac{X_L}{\omega} = \frac{20}{500} = 40mH$$

Example:

For the total admittance locus shown that represent an electrical circuit with two parallel branches. The maximum value of real part of total admittance is equal to (0.5), what is the value and type of each element in the circuit.



Solution:



From the locus we find that:

$$\frac{1}{X_C} \times \frac{1}{\tan 30^o} + \frac{1}{2X_C} = 0.5$$

Solving this equation for X_C we get: X_C=4.464Ω

Again from the locus we find that:

$$Y_L = \frac{1}{X_C} - j\left(\frac{1}{X_C} \times \frac{1}{\tan 30^o}\right)$$

Were $\frac{1}{X_C}$ representing the real part of Y_L and $\left(\frac{1}{X_C} \times \frac{1}{\tan 30^o}\right)$ representing the imaginary part of Y_L.

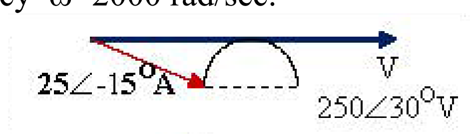
Substituting the value of X_C to get: Y_L=(0.224-j0.388)℧

$$\therefore Z_L = \frac{1}{Y_L} = (1.938 + j1.119)\Omega = R_L + jX_L$$

Example:

For the total current locus shown that represent an electrical circuit with two parallel branches, find the value of each element in the circuit.

Note: Use the angular frequency ω=2000 rad/sec.



Solution:

$$Z_L = \frac{V}{I_L} = \frac{250\angle 30^o}{25\angle -15^o} = 10\angle 45^o\Omega = (7.07 + j7.07)\Omega$$

∴ From real and imaginary parts we found that:

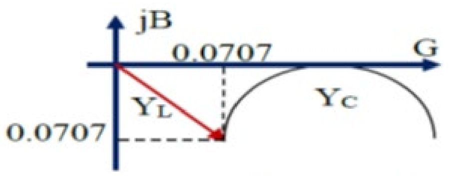
R_L=7.07Ω and X_L=7.07Ω

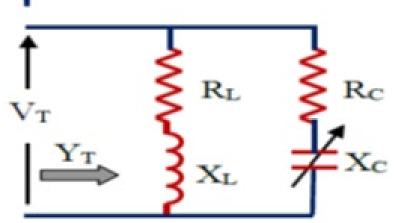
$$\therefore L = \frac{X_L}{\omega} = \frac{7.07}{2000} = 3.53\text{mH}$$

From the admittance locus we find that:

$$Y_L = \frac{1}{Z_L} = \frac{1}{10\angle 45^o} = 0.1\angle -45^o ℧ = (0.0707 + j0.0707)℧$$

Radius=1/2R_C=0.0707℧ ⇒ R_C=7.07Ω

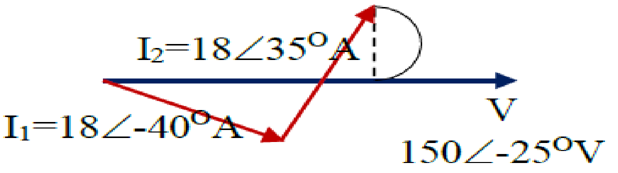




Example:

For the total current locus shown that represent an electrical circuit with three parallel branches, find the value of each element in the circuit.

Note: Use the angular frequency ω=5000 rad/sec.



Solution

$$Z_1 = \frac{V}{I_1} = \frac{150\angle -25^o}{18\angle -40^o} = 8.33\angle 15^o\ \Omega = (8.05 + j2.16)\Omega$$

∴ From real and imaginary parts we found that:

$R_{L1}=8.05\Omega$ and $X_{L1}=2.16\Omega$

$$\therefore L_1 = \frac{X_{L1}}{\omega} = \frac{2.16}{5000} = 0.433\text{mH}$$

$$Z_2 = \frac{V}{I_2} = \frac{150\angle -25^o}{18\angle 35^o} = 8.3\angle -60^o\ \Omega = (4.16 - j7.22)\Omega$$

∴ From real and imaginary parts we found that:

$R_C=4.16\Omega$ and $X_C=7.22\Omega$

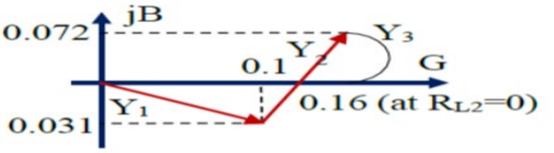
$$\therefore c = \frac{1}{\omega X_C} = \frac{1}{5000 \times 7.22} = 27.7\mu\text{F}$$

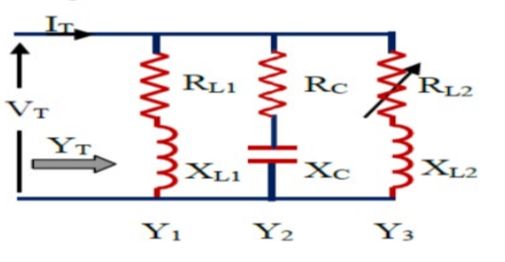
From the admittance locus we find that:

Diameter=$1/X_{L2}$=Im(Y_2)-Im(Y_1)=0.072℧ ⇒ $X_{L2}=13.7\Omega$

Using: $Y_1=1/Z_1$ and $Y_2=1/Z_2$

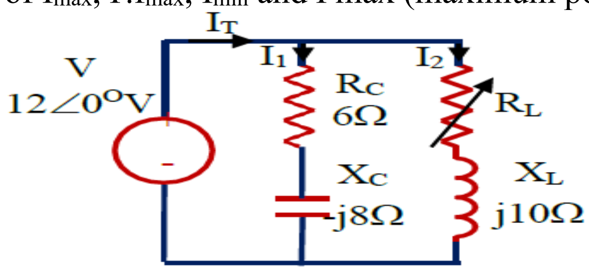
$$\therefore L_2 = \frac{X_{L2}}{\omega} = \frac{13.7}{5000} = 2.74\text{mH}$$



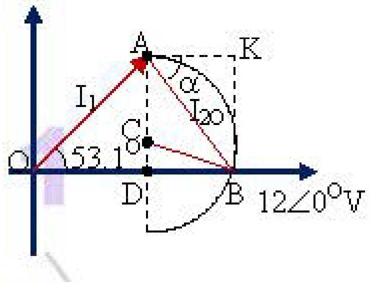


Example:

Find the value of "R_L" which result in parallel for the circuit shown besides using the I- Locus diagram. Also find the value of I_{max}, P.f_{max}, I_{min} and Pmax (maximum power dissipated).



Solution



1. $I_1 = \frac{12\angle 0^O}{6-j8} = 1.2\angle 53.1^O = 0.72 + j0.96 = OD + jAD$

   From the current locus we find that:

   Radius$=\frac{V}{2X_L} = \frac{12\angle 0^O}{2\times 10} = 0.6A = CB$

   And by using Pythagoras rule for triangle (CDB):

   $DB = \sqrt{(CB)^2 - (CD)^2} = \sqrt{(0.6)^2 - (0.36)^2} = 0.48A$

   Were: CD=AD-AC=0.96-0.6=0.36A

   Again using Pythagoras rule for triangle (ADB):

   $|I_{2O}| = \sqrt{(AD)^2 + (DB)^2} = \sqrt{(0.96)^2 + (0.48)^2} = 1.1A$

   $\alpha = tan^{-1}\frac{KB}{KA} = tan^{-1}\frac{0.96}{0.48} = -63.43^O$

   $\therefore I_{2O}=1.0733\angle -63.43^O A$ (is the current in the second branch at resonance)

   $Z_2 = R_L + j10 = \frac{V}{I_z} = \frac{12\angle 0^O}{1.1\angle -63.43^O} = 11.18\angle 63.43^O = (5 + j10)\Omega$

   $\therefore R_L=5\Omega$

2. $I_T = V \times \left[\left(\frac{6}{100} + \frac{R_L}{R_L^2+100}\right) + j\left(\frac{8}{100} - \frac{10}{R_L^2+100}\right)\right]$

   To get the maximum value condition the above equation must be derived with respect to the variable element in the equation, and equating it with zero.

   $\frac{dI_T}{dR} = V \times \left[\frac{-R_L^2+100}{(R_L^2+100)^2} + j\frac{20R_L}{(R_L^2+100)^2}\right] = 0 + j0$

   From real part $R_L=10\Omega$ (maximum) and from imaginary part $R_L=0\Omega$ (minimum)

   $I_{Tmax} = 12\angle 0^O \times \left[\left(\frac{6}{100} + \frac{10}{200}\right) + j\left(\frac{8}{100} - \frac{10}{200}\right)\right] = 1.368\angle 15.255^O A$

   $I_{Tmin} = 12\angle 0^O \times \left[\left(\frac{6}{100}\right) + j\left(\frac{8}{100} - \frac{10}{100}\right)\right] = 0.76\angle -18.4^O A$

   $P_{max}=V\times I_{Tmax}\times \cos\theta=12\times 1.368\times \cos(0-15.255)=15.84W$

   Or $P_{max}= (I_{Tmax})^2\times Re(Z_{Tmax})=(1.368)^2\times 8.463=15.84W$

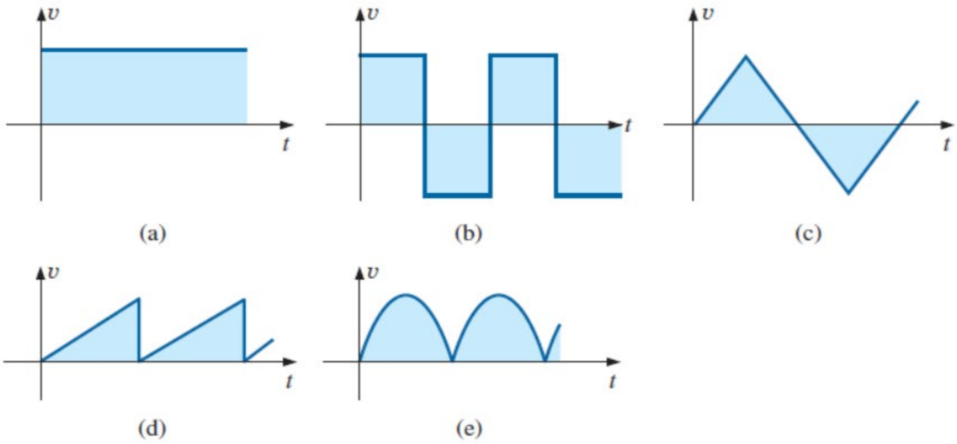
   $p.f_{max}=\cos\theta_{max}=\cos 0=1$

Periodic Non-Sinusoidal Signals

Periodic Non-Sinusoidal Signals

Any waveform that differs from the basic description of the sinusoidal waveform is referred to as **non-sinusoidal.** The most obvious and familiar are:

- D.C.
- Square wave.
- Triangular.
- Sawtooth.
- Rectified waveforms.



Common non sinusoidal waveforms: (a) dc; (b) square-wave; (c) triangular; (d) sawtooth; (e) rectified.

A periodic function can be represented by an infinite sum of **sine and cosine signals** (functions) which are **related harmonically**. The harmonic signals (waves) are generated from the fundamentals signals with frequency multiplied by an integer number.

Signal or wave with frequency f_1 (ω_1) is called fundamental frequency. The n_{th} harmonic wave has a frequency of $n\omega_1$.

- When n=2 it is called the second harmonic.
- When n=3 it is called the third harmonic.

Therefore, the non-sinusoidal wave can be represented by:

D.C.+ Fundamental wave+ Sum of harmonics.

that can be used to represent a non-sinusoidal periodic waveform. Or, in general representation the equation of the instantaneous value is:

$$e(t) = e_{dc} + e_1 + e_2 + \cdots \ldots \ldots + e_n$$

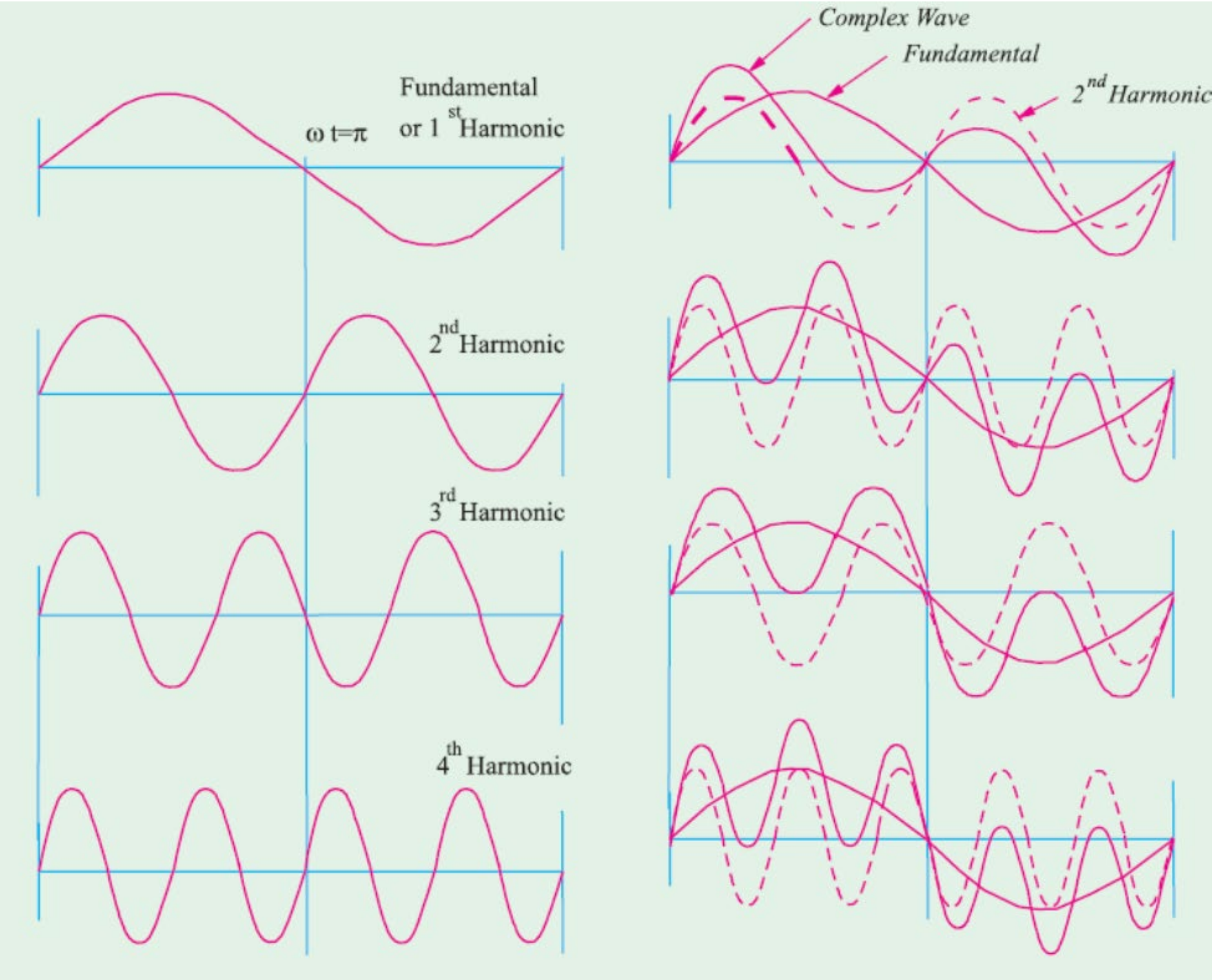
$$e(t) = V_{dc} + V_{m1}\sin(\omega_1 t \mp \theta) + V_{m2}\sin(2\omega_1 t \mp \emptyset) + V_{m3}\sin(3\omega_1 t \mp \beta) + \cdots \ldots + V_{mn}\sin(n\omega_1 t \mp \alpha)$$

And

$$i(t) = I_{dc} + I_{m1}\sin(\omega_1 t \mp \theta) + I_{m2}\sin(2\omega_1 t \mp \emptyset) + I_{m3}\sin(3\omega_1 t \mp \beta) + \cdots \ldots + I_{mn}\sin(n\omega_1 t \mp \alpha)$$

Depending on the waveform, many these terms may be required to approximate the waveform closely for the purpose of circuit analysis.

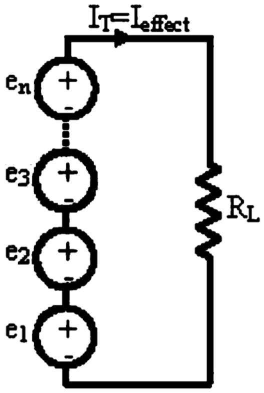
- The first term of the sine and cosine series is called the **Fundamental Component.** It represents the minimum frequency term required to represent a particular waveform, and it also has the same frequency as the waveform being represented.
- The other terms with higher order frequencies (integer multiples of the fundamental) are called the **harmonic terms.**



Circuit Response to a Non-Sinusoidal Input

The general representation of a non-sinusoidal input can be applied to a **linear network** using **the principle of superposition**. Recall that this theorem allowed us to consider the effects of each source of a circuit independently. If we replace the non-sinusoidal input with the general term representation necessary for practical considerations, we can use superposition to find the response of the network to each term.

- The total response of the system is then the algebraic sum of the values obtained for each term.
- The non-sinusoidal waves could be grouped into different frequency sine-wave sources connected in series to the load . Each source supplies the load by a power calculated using superposition theorem, and the total power is the sum of them.



$e_1 \rightarrow P_1$, $e_2 \rightarrow P_2$, $e_3 \rightarrow P_3$, ..., $e_n \rightarrow P_n$

The total power is $P_T = P_1 + P_2 + P_3 + \cdots + P_n$

$$P_T = \frac{V_{dc}^2}{R} + \frac{\left(V_{m1}/\sqrt{2}\right)^2}{R} + \frac{\left(V_{m2}/\sqrt{2}\right)^2}{R} + \cdots \ldots \ldots + \frac{\left(V_{mn}/\sqrt{2}\right)^2}{R}$$

$$\frac{V_{eff}^2}{R} = \frac{V_{dc}^2}{R} + \frac{V_{m1}^2}{2R} + \frac{V_{m2}^2}{2R} + \cdots \ldots \ldots \frac{V_{mn}^2}{2R}$$

$$V_{eff} = \sqrt{V_{dc}^2 + \frac{V_{m1}^2 + V_{m2}^2 + \cdots \ldots \ldots + V_{mn}^2}{2}}$$

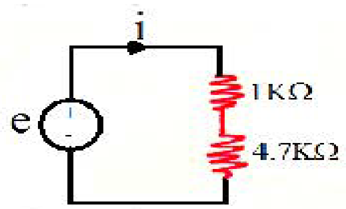
Similarly:

$$I_{eff} = \sqrt{I_{dc}^2 + \frac{I_{m1}^2 + I_{m2}^2 + \cdots \ldots \ldots + I_{mn}^2}{2}}$$

$$P_T = \frac{V_{eff}^2}{R_L} = I_{eff}^2 R_L$$

The power dissipated as heat in a resistance due to non-sinusoidal wave depends on constant current value. Assume this current as DC current give the same heat power at the same time in the same resistance, it is equal the effective current value (non-sinusoidal).

Example: non-sinusoidal wave source has a voltage equation ($e=0.45\sin\omega t+0.18\sin 2\omega t$) volt, applied on a series circuit contain (R_1=1KΩ) and (R_2=4.7KΩ). Find the voltage effective value on (4.7KΩ) resistor and the current effective value in the circuit?



Solution:

$R_T = 1\times10^3 + 4.7\times10^3 = 5.7\ K\Omega$

$i = \frac{0.45}{5.7\times10^3} sin\omega t + \frac{0.18}{5.7\times10^3} sin2\omega t$

$= 80 sin\omega t + 60 sin2\omega t\ \mu A$

$I_{eff} = \sqrt{\frac{(80\times10^{-6})^2+(30\times10^{-6})^2}{2}} = 60\ \mu A$

$V_{eff4.7K} = I_{eff} R_{4.7K} = 60\times10^{-6}\times4.7\times10^3 = 0.28\ V$

another way:

$V_{eff} = \sqrt{\frac{(0.45)^2+(0.18)^2}{2}} = 0.343\ V$ and $I_{eff} = \frac{V_{eff}}{R_T} = \frac{0.343}{5.7\times10^3} = 60\ \mu A$

using VDR: $V_{eff4.7K} = V_{eff}\times\frac{R_{4.7K}}{R_{4.7K}+R_{1K}} = 0.343\times\frac{4.7\times10^3}{4.7\times10^3+1\times10^3} = 0.28\ V$

Example: Find the active value of voltage and current, and the average power for an electrical circuit if the voltage supplied is: $e=200+100\cos(500t+30^o)+75\cos(1500t+60^o)$ V and the current is: $i=3.53\cos(500t+75^o)+3.55\cos(1500t+78.45^o)$ A.

Solution:

$I_{eff} = \sqrt{\frac{(3.53)^2+(3.55)^2}{2}} = 3.54\ A$

$V_{eff} = \sqrt{(200)^2+\frac{(100)^2+(75)^2}{2}} = 218.66\ V$

$P_{DC} = 200\times0 = 0\ W$

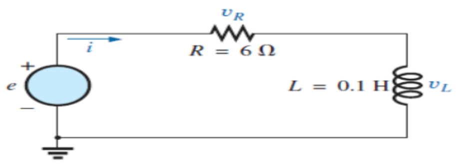
$P_1 = \frac{100}{\sqrt{2}}\times\frac{3.53}{\sqrt{2}}\cos(75^o-30^o) = 124.8\ W$

$P_3 = \frac{75}{\sqrt{2}}\times\frac{3.55}{\sqrt{2}}\cos(78.45^o-60^o) = 126.3\ W$

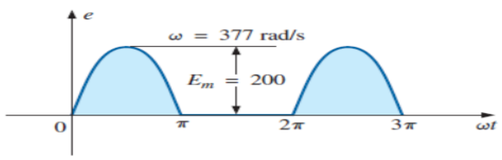
$P_T = P_{DC}+P_1+P_3 = 0+124.8+126.3 = 251.1\ W$

Example: Find the response of the circuit in Fig. below for first three terms of the input shown.

$$e = 0.318E_m + 0.500E_m\sin\omega t - 0.212E_m\cos2\omega t - 0.0424E_m\cos 4\omega t + \ldots$$



(a)

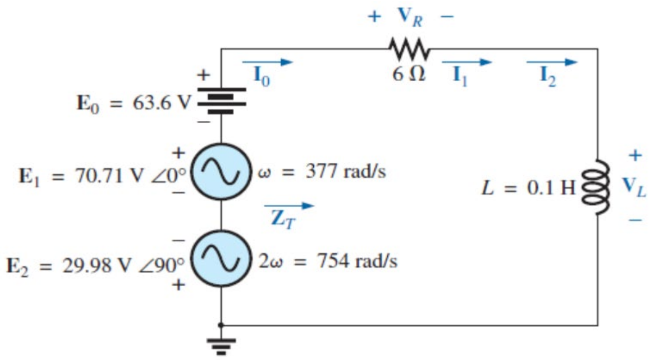


(b)

Solution: Converting the cosine terms to sine terms and substituting for E_m gives us:

$$e = 63.60 + 100.0\sin\omega t - 42.40\sin(2\omega t + 90^\circ)$$

Using phasor notation, the original circuit becomes like the one shown below:



$$I_0 = \frac{E_0}{R} = \frac{63.6\text{ V}}{6\ \Omega} = 10.60\text{ A}$$
$$V_{R_0} = I_0R = E_0 = 63.60\text{ V}$$
$$V_{L_0} = 0$$

The average power is

$$P_0 = I_0^2R = (10.60\text{ A})^2(6\ \Omega) = 674.2\text{ W}$$

For the fundamental term ($\mathbf{E}_1 = 70.71\text{ V}\ \angle 0°$, $\omega = 377$):

$$X_{L_1} = \omega L = (377\text{ rad/s})(0.1\text{ H}) = 37.7\ \Omega$$
$$\mathbf{Z}_{T_1} = 6\ \Omega + j\,37.7\ \Omega = 38.17\ \Omega\ \angle 80.96°$$
$$\mathbf{I}_1 = \frac{\mathbf{E}_1}{\mathbf{Z}_{T_1}} = \frac{70.71\text{ V}\ \angle 0°}{38.17\ \Omega\ \angle 80.96°} = 1.85\text{ A}\ \angle -80.96°$$
$$\mathbf{V}_{R_1} = (I_1\ \angle\theta)(R\ \angle 0°) = (1.85\text{ A}\ \angle -80.96°)(6\ \Omega\ \angle 0°)$$
$$= 11.10\text{ V}\ \angle -80.96°$$
$$\mathbf{V}_{L_1} = (I_1\ \angle\theta)(X_{L_1}\ \angle 90°) = (1.85\text{ A}\ \angle -80.96°)(37.7\ \Omega\ \angle 90°)$$
$$= 69.75\text{ V}\ \angle 9.04°$$

The average power is

$$P_1 = I_1^2R = (1.85\text{ A})^2(6\ \Omega) = 20.54\text{ W}$$

For the second harmonic ($\mathbf{E}_2 = 29.98\text{ V}\ \angle -90°$, $\omega = 754$): The phase angle of $\mathbf{E}_2$ was changed to $-90°$ to give it the same polarity as the input voltages $\mathbf{E}_0$ and $\mathbf{E}_1$.

$$X_{L_2} = \omega L = (754\text{ rad/s})(0.1\text{ H}) = 75.4\ \Omega$$
$$\mathbf{Z}_{T_2} = 6\ \Omega + j\,75.4\ \Omega = 75.64\ \Omega\ \angle 85.45°$$
$$\mathbf{I}_2 = \frac{\mathbf{E}_2}{\mathbf{Z}_{T_2}} = \frac{29.98\text{ V}\ \angle -90°}{75.64\ \Omega\ \angle 85.45°} = 0.396\text{ A}\ \angle -174.45°$$
$$\mathbf{V}_{R_2} = (I_2\ \angle\theta)(R\ \angle 0°) = (0.396\text{ A}\ \angle -174.45°)(6\ \Omega\ \angle 0°)$$
$$= 2.38\text{ V}\ \angle -174.45°$$
$$\mathbf{V}_{L_2} = (I_2\ \angle\theta)(X_{L_2}\ \angle 90°) = (0.396\text{ A}\ \angle -174.45°)(75.4\ \Omega\ \angle 90°)$$
$$= 29.9\text{ V}\ \angle -84.45°$$

The average power is

$$P_2 = I_2^2R = (0.396\text{ A})^2(6\ \Omega) = 0.941\text{ W}$$

$$i = \mathbf{10.6} + \sqrt{\mathbf{2}}(\mathbf{1.85})\ \mathbf{sin(377}t - \mathbf{80.96°)} + \sqrt{\mathbf{2}}(\mathbf{0.396})\ \mathbf{sin(754}t - \mathbf{174.45°)}$$

and

$$I_{\text{rms}} = \sqrt{(10.6\text{ A})^2 + (1.85\text{ A})^2 + (0.396\text{ A})^2} = \mathbf{10.77\ A}$$

$$v_R = 63.6 + \sqrt{2}(11.10)\sin(377t - 80.96^\circ) + \sqrt{2}(2.38)\sin(754t - 174.45^\circ)$$

and

$$V_{R_{rms}} = \sqrt{(63.6\text{ V})^2 + (11.10\text{ V})^2 + (2.38\text{ V})^2} = \mathbf{64.61\text{ V}}$$

$$v_L = \sqrt{2}(69.75)\sin(377t + 9.04^\circ) + \sqrt{2}(29.93)\sin(754t - 84.45^\circ)$$

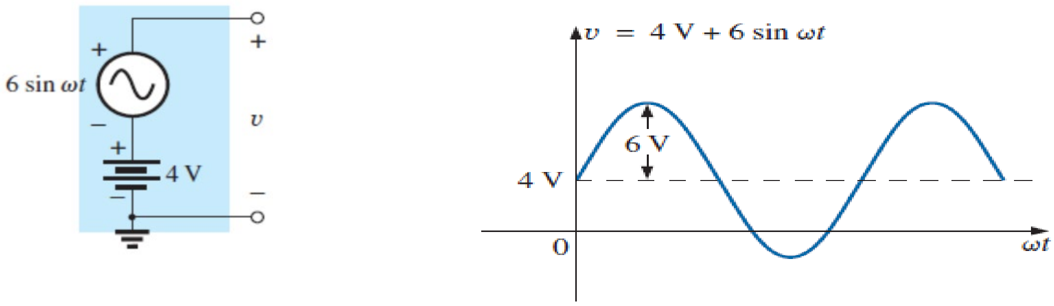
and $$V_{L_{rms}} = \sqrt{(69.75\text{ V})^2 + (29.93\text{ V})^2} = \mathbf{75.90\text{ V}}$$

The total average power is

$$P_T = I_{rms}^2 R = (10.77\text{ A})^2(6\ \Omega) = \mathbf{695.96\text{ W}} = P_0 + P_1 + P_2$$

Example:

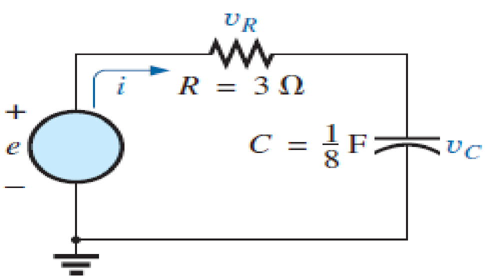
a. Sketch the input resulting from the combination of sources in Fig. below.
b. Determine the rms value of the input.



$$V_{rms} = \sqrt{V_0^2 + \frac{V_m^2}{2}}$$
$$= \sqrt{(4\text{ V})^2 + \frac{(6\text{ V})^2}{2}} = \sqrt{16 + \frac{36}{2}}\text{ V} = \sqrt{34}\text{ V}$$
$$= \mathbf{5.831\text{ V}}$$

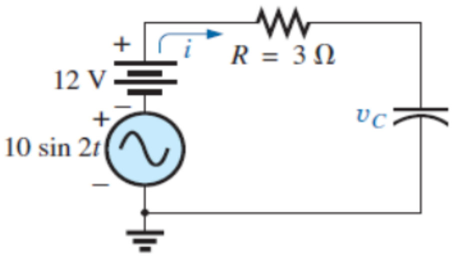
xample:

The input to the circuit in Fig. below is the following: $e = 12 + 10\sin 2t$
a. Find the current i and the voltages v_R and v_C.
b. Find the rms values of i, v_R, and v_C.
c. Find the power delivered to the circuit.



Solution:

a. Redraw the original circuit as shown, then apply superposition:



$$X_C = \frac{1}{\omega C} = \frac{1}{(2\ \text{rad/s})(\frac{1}{8}\text{F})} = 4\ \Omega$$

For the 12 V dc supply portion of the input, I = 0 since the capacitor is an open circuit to dc when v_C has reached its final (steady state) value. Therefore:

$$V_R = IR = 0\ \text{V} \quad \text{and} \quad V_C = 12\ \text{V}$$

For the a.c. supply:

$$\mathbf{Z} = 3\ \Omega - j\,4\ \Omega = 5\ \Omega \angle -53.13^\circ$$

$$\text{and} \quad \mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{\frac{10}{\sqrt{2}}\ \text{V} \angle 0^\circ}{5\ \Omega \angle -53.13^\circ} = \frac{2}{\sqrt{2}}\ \text{A} \angle +53.13^\circ$$

$$\mathbf{V}_R = (I \angle \theta)(R \angle 0^\circ) = \left(\frac{2}{\sqrt{2}}\ \text{A} \angle +53.13^\circ\right)(3\ \Omega \angle 0^\circ)$$

$$= \frac{6}{\sqrt{2}}\ \text{V} \angle +53.13^\circ$$

and

$$\mathbf{V}_C = (I \angle \theta)(X_C \angle -90^\circ) = \left(\frac{2}{\sqrt{2}}\ \text{A} \angle +53.13^\circ\right)(4\ \Omega \angle -90^\circ)$$

$$= \frac{8}{\sqrt{2}}\ \text{V} \angle -36.87^\circ$$

In the time domain,

$$i = \mathbf{0 + 2\sin(2t + 53.13^\circ)}$$

$$v_R = \mathbf{0 + 6\sin(2t + 53.13^\circ)}$$

$$v_C = \mathbf{12 + 8\sin(2t - 36.87^\circ)}$$

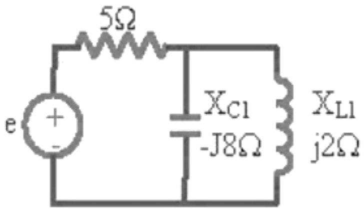
$$I_{\text{rms}} = \sqrt{(0)^2 + \frac{(2\ \text{A})^2}{2}} = \sqrt{2}\ \text{A} = \mathbf{1.414\ A}$$

$$V_{R_{\text{rms}}} = \sqrt{(0)^2 + \frac{(6\ \text{V})^2}{2}} = \sqrt{18}\ \text{V} = \mathbf{4.243\ V}$$

$$V_{C_{\text{rms}}} = \sqrt{(12\ \text{V})^2 + \frac{(8\ \text{V})^2}{2}} = \sqrt{176}\ \text{V} = \mathbf{13.267\ V}$$

$$P = I_{\text{rms}}^2 R = \left(\frac{2}{\sqrt{2}}\ \text{A}\right)^2 (3\ \Omega) = \mathbf{6\ W}$$

Example: For the circuit shown, find the total current equation if the supply voltage is: e=50+20sin500t+10sin1000t V.



Solution:

DC: $I_{DC} = \frac{50}{5} = 10\ A$ (L is s/c & C is o/c)

fundamental:

$Z_{T1} = 5 + (j2//(-j8)) = 5 + j2.667 = 5.667\angle 28^o\ \Omega$

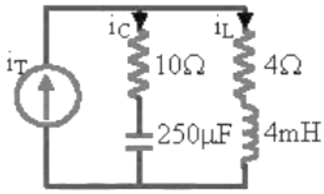
$i_1 = \frac{20\angle 0^o}{5.667\angle 28^o} = 3.53\angle -28^o\ A$

2nd harmonic: $X_{L2} = 2X_{L1} = 4\ \Omega$ & $X_{C2} = \frac{X_{C1}}{2} = 4\ \Omega$

$Z_{T2} = 5 + (j4//(-j4)) = \infty\ \Omega$ open circuit $i_2 = \frac{10\angle 0^o}{\infty} = 0\ A$

$\therefore i = 10 + 3.53\sin(500t - 28^o)\ A$

Example: Derive the current equation in each branch and calculate the consumed power in the circuit if the supplied current is: i=10+4sin500t+2sin1500t A.

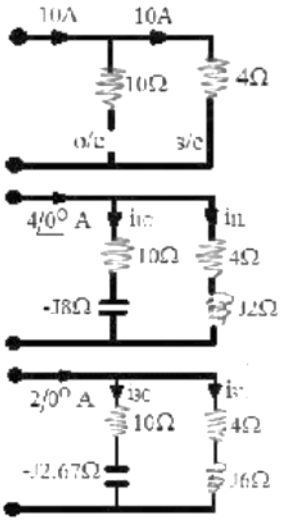


Solution:

DC: $I_{CDC} = 0\ A$ & $I_{LDC} = 10\ A$

fundamental: $X_{L1} = 500 \times 4 \times 10^{-3} = 2\ \Omega$

$X_{C1} = \frac{1}{500\times 250\times 10^{-6}} = 8\ \Omega$



$i_{1C} = 4\angle 0^o \times \frac{4+j2}{14-j6} = 1.2\angle 49.2^o\ A$

$i_{1L} = 4\angle 0^o \times \frac{10-j8}{14-j6} = 3.36\angle -15^o\ A$

3rd harmonic: $X_{L3} = 3X_{L1} = 6\ \Omega$ & $X_{C3} = \frac{X_{C1}}{3} = 2.67\ \Omega$

$i_{3C} = 2\angle 0^o \times \frac{4+j6}{14+j3.33} = 1\angle 42.9^o\ A$

$i_{3L} = 2\angle 0^o \times \frac{10-j2.67}{14+j3.33} = 1.4\angle -28.2^o\ A$

$i_C = 1.2\sin(500t + 49.2^o) + 1\sin(1500t + 42.9^o)\ A$

$i_L = 10 + 3.36\sin(500t - 15^o) + 1.4\sin(1500t - 28.2^o)\ A$

$I_{Ceff} = \sqrt{\frac{(1.2)^2+(1)^2}{2}} = 1.132\ A$

$I_{Leff} = \sqrt{(10)^2 + \frac{(3.36)^2+(1.4)^2}{2}} = 10.3\ A$

$P_T = P_C + P_L = I_{Ceff}^2 R_C + I_{Leff}^2 R_L = (1.132)^2 \times 10 + (10.3)^2 \times 4 = 436.1\ W$

The Transient Circuit Analysis

The Transient Circuit Analysis

Whenever a circuit is switched from one condition to another, either by a change in the applied source or a change in the circuit elements, there is a transitional period during which the branch currents and element voltages change from their former values to new ones. **This period is called the transient.**

After the transient has passed, the circuit is said to be in the **steady state**.

So, the general differential equation that describes the operation of any electrical circuit will contain two parts, one for transient and another for steady state:

$$i(t) = i_t(t) + i_{ss}(t)$$

Where:

$i(t)$ is the current as function of time.

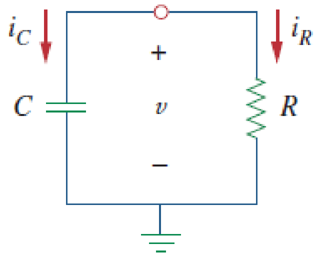
$i_t(t)$ is the transient current.

$i_{ss}(t)$ is the steady state current.

The Natural Response of R-L Circuit

- The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.
- A natural response or source free RC circuit occurs when its D.C. source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Consider a series combination of a resistor and an **initially charged capacitor**, as shown in Fig. below:



Since the capacitor is initially charged, we can assume that at time the initial voltage is $v(o) = V_0$

$$i_C = -\,i_R$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

This is a ***first-order differential equation***, since only the first derivative of v is involved. To solve it, we rearrange the terms as:

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

Integrating both sides, we get:

$$\ln v = -\frac{t}{RC} + \ln A$$

Where $\ln A$ is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC}$$

Taking powers of e produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v\,(0) = A = V_0$ Hence:

$$v(t) = V_0 e^{-t/RC}$$

$$v(t) = V_0 e^{-t/\tau}$$

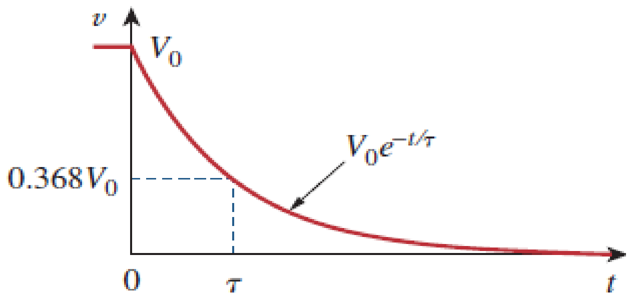
$$\tau = RC$$

The time constant (τ) of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.

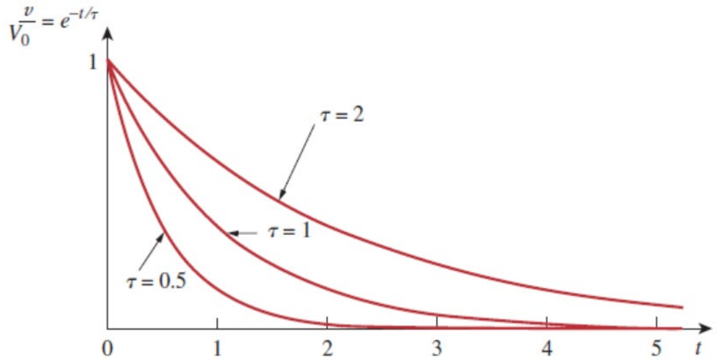
This implies that at $t=\tau$:

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$$

The voltage response of the *R-C* circuit is an exponential decay of the initial voltage.



- The rapidity with which the voltage decreases is expressed in terms of the *time constant*.
- Figure below shows the effect of different values of time constant on natural response of R-C circuit.



Also, we can find the current i(t):

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}$$

The power dissipated in the resistor is:

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

The energy absorbed by the resistor up to time *t* is:

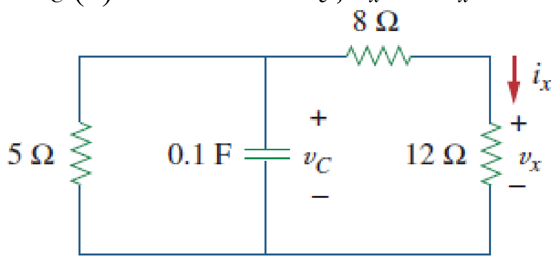
$$w = \int_0^t p\,dx = \int_0^t \frac{V_0^2}{R} e^{-2x/\tau}\,dx$$

$$= \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \quad t \geq 0.$$

For $t \to \infty, w_R(\infty) \to \frac{1}{2} C V_0^2,$

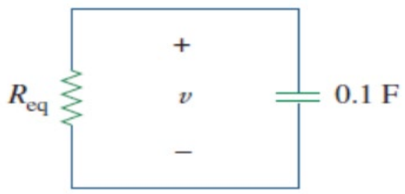
which is the same as the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

Example: In Fig. below, let $v_C(0) = 15$ V. Find v_c, v_x *and* i_x and for t>0.



Solution: The equivalent resistance or the Thevenin at the capacitor terminals.

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$



The time constant is:

$$\tau = R_{eq}C = 4(0.1) = 0.4 \text{ s}$$

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4} \text{ V}, \qquad v_C = v = 15e^{-2.5t} \text{ V}$$

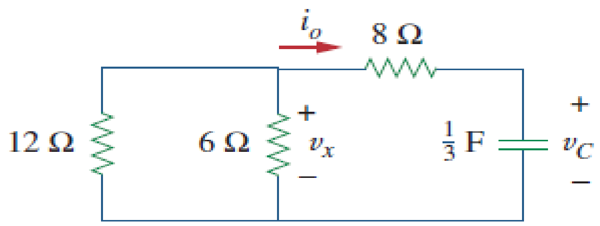
From Fig. 7.5, we can use voltage division to get v_x; so

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t} \text{ V}$$

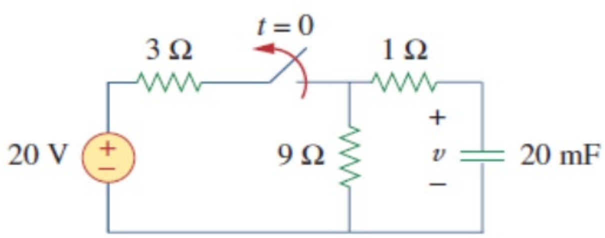
Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t} \text{ A}$$

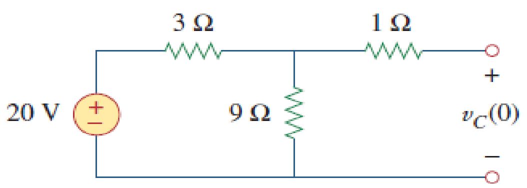
Homework: For the circuit in Fig. below. Let $v_C(0) = 60$V. Determine v_C, v_x and i_o for t≥0.



Example: The switch in the circuit in Fig. below has been closed for a long time, and it is opened at t=0 sec. Find *v(t)* for t≥0. Calculate the initial energy stored in the capacitor.



Solution: For t<0, the switch is closed; the capacitor is an open circuit to dc, as represented in Fig. below.



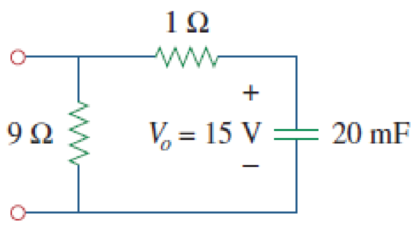
Using voltage divider rule:

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at t<0 is the same at t=0 or:

$$v_C(0) = V_0 = 15 \text{ V}$$

For t>0, the circuit will be as shown below:



$$R_{eq} = 1 + 9 = 10\ \Omega$$

The time constant is:

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2 \text{ s}$$

Thus, the voltage across the capacitor for t≥0 is:

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} \text{ V}$$

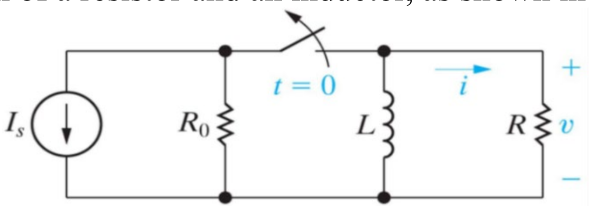
$$v(t) = 15e^{-5t} \text{ V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$$

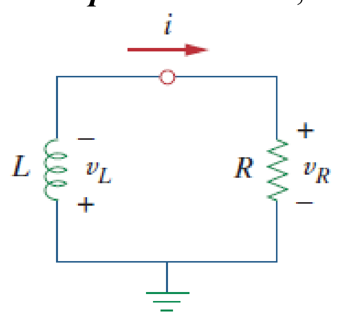
The Natural Response of R-L Circuit

Consider the connection of a resistor and an inductor, as shown in Fig. below.



- For $t < 0$, the inductor L is short and carries a current I_s, while R_0 and R carry no current.
- For $t > 0$, the inductor current decreases and the energy is dissipated via R.

Our goal is to determine the ***circuit response*** for $t > 0$, so, the circuit reduces to:



The inductor has an initial current or $i(0) = I_0$. Applying KVL around the loop:

$$v_L + v_R = 0$$

But $v_L = L\,di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt$$

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Rightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

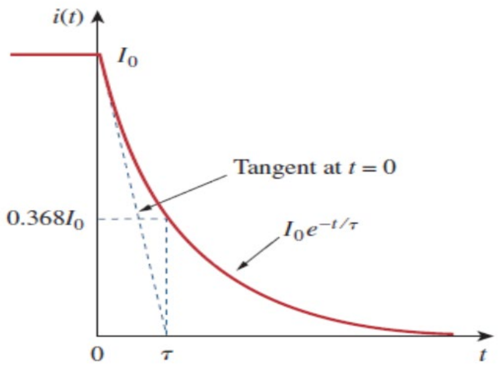
or

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L}$$

This shows that the natural response of the *R-L* circuit is an exponential decay of the initial current. The current response is shown in Fig. below.



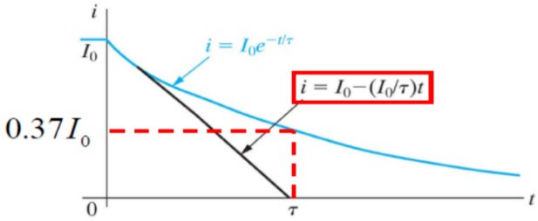
The time constant for the *R-L* circuit is:

$$\tau = \frac{L}{R}$$

The time constant of a circuit is the time required for the response to decay to a factor of $1/e$ or 36.8 percent of its initial value with τ again having the unit of seconds.

$$i(t) = I_0 e^{-t/\tau}$$

The loop current $i(t)$ will drop to e^{-1} (37%) of its initial value I_0 within one time constant τ. It will be $<0.01I_0$ after elapsing 5τ. If $i(t)$ is approximated by a linear function, it will vanish in one time constant.



We can find the voltage across the resistor as:

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$

The power dissipated in the resistor is:

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

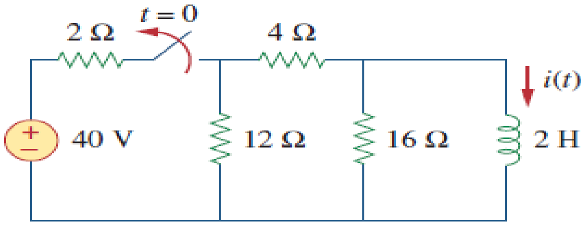
The energy delivered to the resistor during any interval of time after the switch has been opened is:

$$w = \int_0^t p\,dx = \int_0^t I_0^2 R e^{-2(R/L)x}\,dx$$

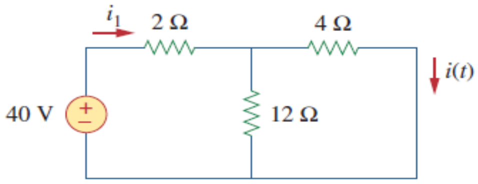
$$= \frac{1}{2(R/L)} I_0^2 R(1 - e^{-2(R/L)t})$$

$$= \frac{1}{2} L I_0^2 (1 - e^{-2(R/L)t}), \qquad t \geq 0.$$

Example: The switch in the circuit of Fig. below has been closed for a long time. At t=0, the switch is opened. Calculate i(t) for t>0.



Solution: When the switch is closed, and the inductor acts as a short circuit to dc. The resistor is short-circuited; the resulting circuit is shown in Fig. below.



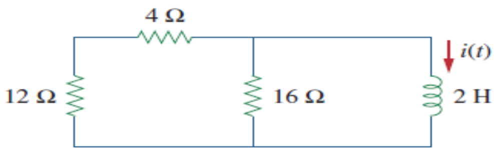
$$\frac{4 \times 12}{4 + 12} = 3\ \Omega$$

$$i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

$$i(t) = \frac{12}{12 + 4} i_1 = 6\ \text{A}, \qquad t < 0$$

$$i(0) = i(0^-) = 6\ \text{A}$$

When t > 0:



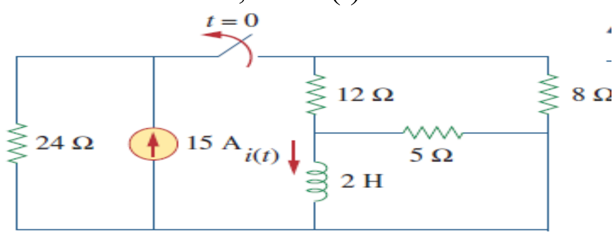
$$R_{eq} = (12 + 4) \parallel 16 = 8\ \Omega$$

The time constant is:

$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4}\ \text{s}$$

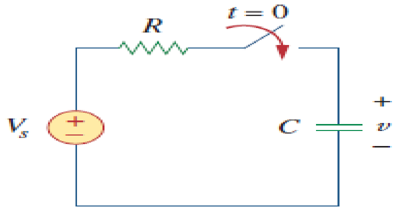
$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\ \text{A}$$

Homework: For the circuit shown below, find i(t) for t>0.



R-C Transients: Storage Cycle

Consider the *R-C* circuit in Fig. below:



We assume an initial voltage V_0 on the capacitor. Since the voltage of a capacitor cannot change instantaneously:

$$v(0^-) = v(0^+) = V_0$$

$$C\frac{dv}{dt} + \frac{v - V_s}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

or

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

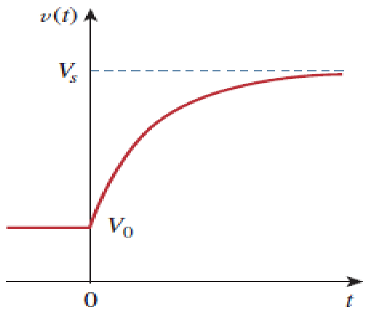
$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

Thus,

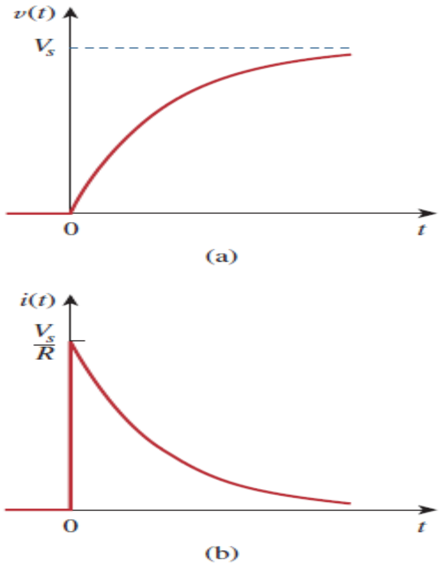
$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$



If V_0=0 volt, then:

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

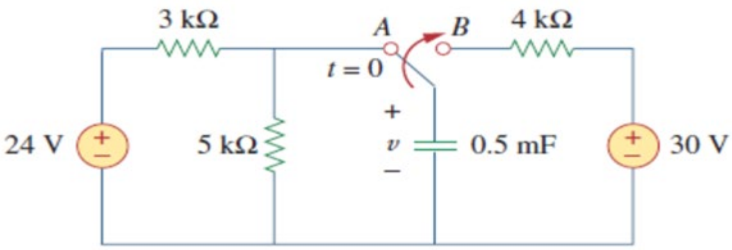
$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad \tau = RC, \qquad t > 0$$



Response of an R-C circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Example:

The switch in Fig. below has been in position *A* for a long time. At the switch moves to *B*. Determine v(t) for t>0 and calculate its value at t =1 s and 4 s.



Solution: For the switch is at position *A*. The capacitor acts like an open circuit to dc, but *v* is the same as the voltage across the resistor. Hence, the voltage across the capacitor just before is obtained by voltage division as:

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$

For the switch is in position *B*. The Thevenin resistance connected to the capacitor is and the time constant is:

$$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V}$$

At $t = 1$,

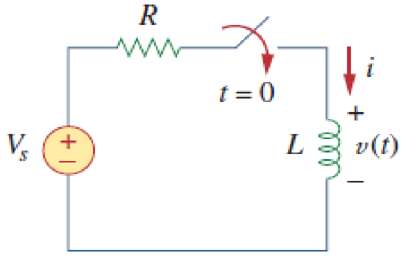
$$v(1) = 30 - 15e^{-0.5} = 20.9 \text{ V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97 \text{ V}$$

R-L Transients: Storage Cycle

The changing voltages and current that result during the storing of energy in the form of a magnetic field by an inductor in a dc circuit can best be described using the circuit shown below:



$$i = i_t + i_{ss}$$

$$V_s = Ri + L\frac{di}{dt},$$

$$\frac{di}{dt} = \frac{-Ri + V_s}{L} = \frac{-R}{L}\left(i - \frac{V_s}{R}\right).$$

$$\frac{di}{dt}dt = \frac{-R}{L}\left(i - \frac{V_s}{R}\right)dt,$$

$$di = \frac{-R}{L}\left(i - \frac{V_s}{R}\right)dt.$$

$$\frac{di}{i - (V_s/R)} = \frac{-R}{L}dt,$$

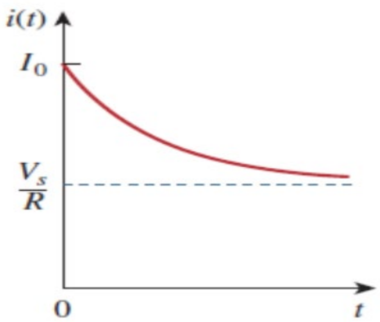
$$\int_{I_0}^{i(t)} \frac{dx}{x - (V_s/R)} = \frac{-R}{L}\int_0^t dy,$$

Where I_0 is the current at t=0 and i(t) is the current at any t>0.

$$\ln\frac{i(t) - (V_s/R)}{I_0 - (V_s/R)} = \frac{-R}{L}t,$$

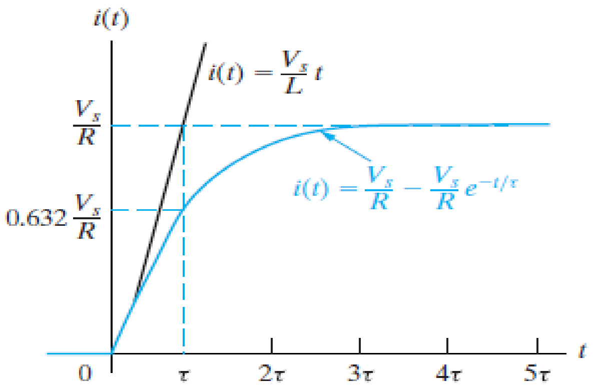
$$\frac{i(t) - (V_s/R)}{I_0 - (V_s/R)} = e^{-(R/L)t},$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t}.$$



When the initial energy in the inductor I_0 is zero, is zero.

$$i(t) = \frac{V_s}{R} - \frac{V_s}{R}e^{-(R/L)t}.$$



$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

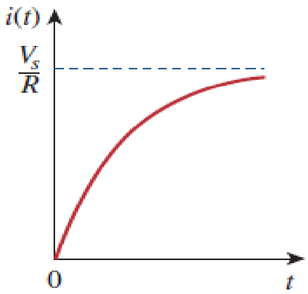
where $i(0)$ and $i(\infty)$ are the initial and final values of i, respectively.

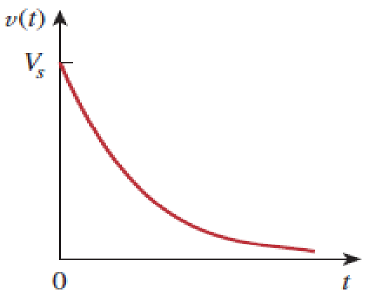
- If the switch is takes place at time $t=t_0$ instead of t=0, then $i(t)$ will be:

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

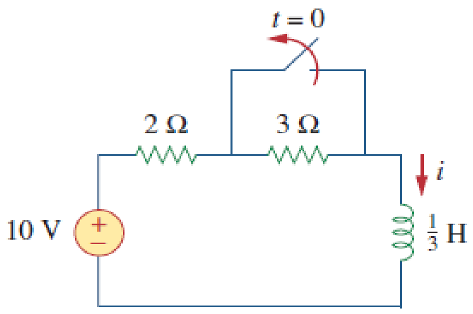
$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \qquad t > 0$$





Example:

Find $i(t)$ in the circuit of Fig. below for t>0. Assume that the switch has been closed for a long time.



Solution:

When t<0 the 3Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t=0^-$ (i.e., just before t=0) is:

$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$

Since the inductor current cannot change instantaneously,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When t>0 the switch is open. The 2Ω and 3Ω resistors are in series, so that:

$$i(\infty) = \frac{10}{2+3} = 2\text{ A}$$

The Thevenin resistance across the inductor terminals is:

$$R_{Th} = 2 + 3 = 5\ \Omega$$

For the time constant,

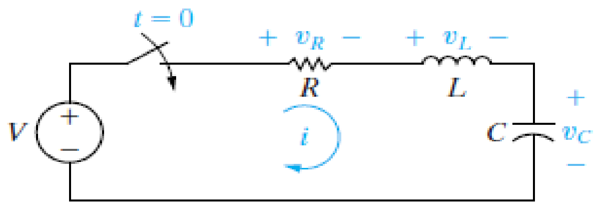
$$\tau = \frac{L}{R_{Th}} = \frac{\frac{1}{3}}{5} = \frac{1}{15}\text{ s}$$

Thus,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$
$$= 2 + (5 - 2)e^{-15t} = 2 + 3e^{-15t}\text{ A}, \qquad t > 0$$

Forced Response of Series R-L-C Circuit

The capacitor and inductor are initially uncharged and are in series with a resistor. When switch S is closed at t = 0, we can determine the complete solution for the current. Application of Kirchhoff's voltage law to the Transient Response of RLC Circuit results in the following differential equation.



$$V = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$$

By differentiating the above equation, we have

$$0 = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i$$

Let $D = \frac{d}{dt}$:

$$RDi + L\,D^2 i + \frac{i}{C} = 0$$

$$i(RD + L\,D^2 + \frac{1}{C}) = 0$$

Since i not equal to zero at transient interval, then:

$$LD^2 + RD + \frac{1}{C} = 0$$

The above equation is called the characteristics equation of the circuit. The solution of this second order equation is:

$$D1 = \frac{-R + \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

$$D2 = \frac{-R - \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

Since the circuits has two roots then the general solution of second order differential equation will be in the following form:

$$i(t) = Ae^{D1t} + Be^{D2t}$$

At t=0, *i(0)*=0 then:

$$0 = Ae^0 + Be^0$$

So, A=-B

The circuit response depends on values of D1 and D2(real, imaginary, or complex) and they depend on R^2 and $\frac{4L}{C}$.

<u>Case1:</u> if $R^2 > \frac{4L}{C}$ (Over Damped Case)

Here D1, D2 are real values.

$$\frac{di}{dt} = AD1e^{D1t} + BD2e^{D2t}$$

At t=0:

$$\frac{di}{dt} = \frac{E}{L}$$

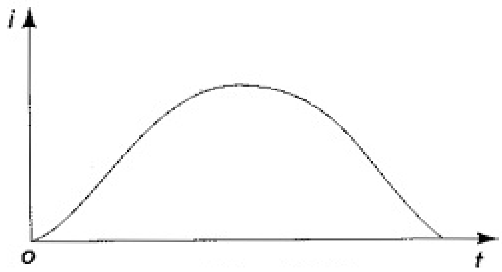
And A=-B

So, $\frac{E}{L} = AD1 - AD2$

$$A = \frac{E}{L(D1 - D2))}$$

$$B = -\frac{E}{L(D1 - D2))}$$

$$i(t) = \frac{E}{L(D1 - D2)}(e^{D1t} - e^{D2t})$$



<u>Case2:</u> if $R^2 = \frac{4L}{C}$ (Critically damped case)

$$D1 = D2 = D = \frac{-R}{2L}$$

$$i(t) = (A + Bt)e^{Dt}$$

At t=0 : i(0)= 0 and $\frac{di}{dt} = \frac{E}{L}$

$$0 = (A + 0)e^0$$

So:

$$\frac{di}{dt} = A\left(\frac{-R}{2L}\right) + B$$

$$B = \frac{E}{L}$$

Then:

$$i(t) = \frac{E}{L}te^{Dt}$$

<u>Case3:</u> if $R^2 < \frac{4L}{C}$ (Under damped case)

Here, D1 and D2 are complex numbers.

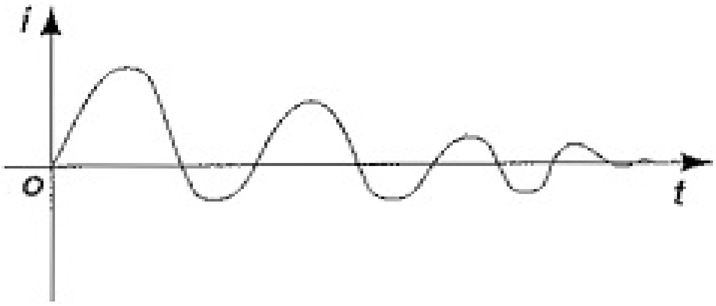
$$D1 = \frac{-R}{2L} + j\sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$D2 = \frac{-R}{2L} - j\sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Let: D1=α+jβ, D2=α-jβ

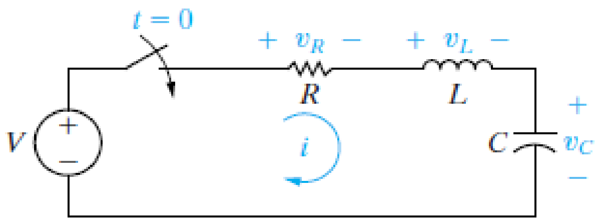
$\alpha = -\frac{R}{2L}$, $\beta = \sqrt{-\frac{R}{2L})^2 + \frac{1}{LC}}$

$$i(t) = Ae^{\alpha t}e^{j\beta t} + Be^{\alpha t}e^{-j\beta t}$$
$$i(t) = e^{\alpha t}(C1cos\beta t + C2sin\beta t)$$



Example:

For the circuit shown below:



If V=200 V, R=20Ω, L=10mH, C=100µF. Determine the time at which the current reach its maximum value.

Solution:

R^2=400, $\frac{4L}{C} = \frac{4*10*10^{-3}}{100*10^{-6}} = 400$

Then the circuit at critically damped case.

$$D1 = D2 = \frac{-R}{2L} = -\frac{20}{2*10^{-3}} = -1000$$
$$i(t) = (A + Bt)e^{Dt}$$

At t=0, i(0)=0, so A=0

$$i(t) = Bt\, e^{Dt}$$

Also, at t=0, $\frac{di}{dt} = \frac{E}{L}$

$$\frac{di}{dt} = Bte^{Dt}D + e^{Dt}B$$

$B = \frac{V}{L} = \frac{200}{10*10^{-3}} = 2*10^4$ Amp\sec.

$$i(t) = 2*10^4 t\, e^{-1000t}$$
$$\frac{di}{dt} = -2*10^4 t*1000e^{-1000t} + 2*10^4 e^{-1000t} = 0$$

t=1msec (the current at its maximum value)

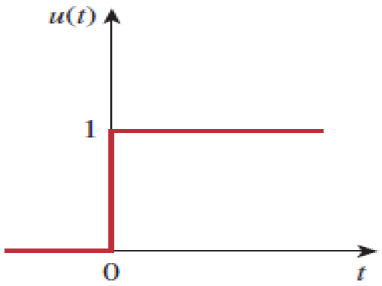
Switching Functions (Singularity Functions)

Switching functions are very useful in circuit analysis. They serve as good approximations to the switching signals that arise in circuits with switching operations. They are helpful in

the neat, compact description of some circuit phenomena, especially the step response of *R-C* or *R-L* circuits.
The three most widely used singularity functions in circuit analysis are:

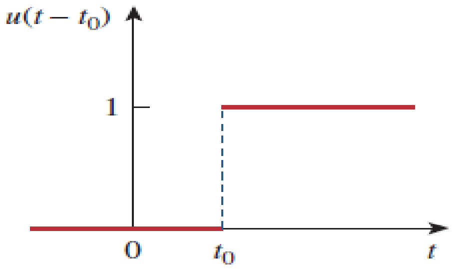
- *unit step*.
- *unit impulse*
- *unit ramp* function.

The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t.

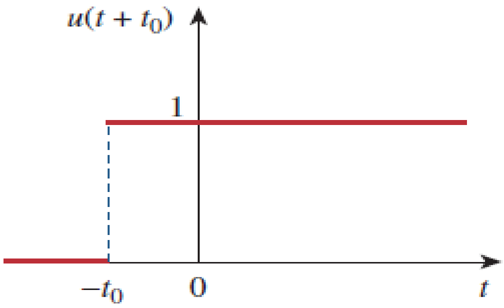


In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$



$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

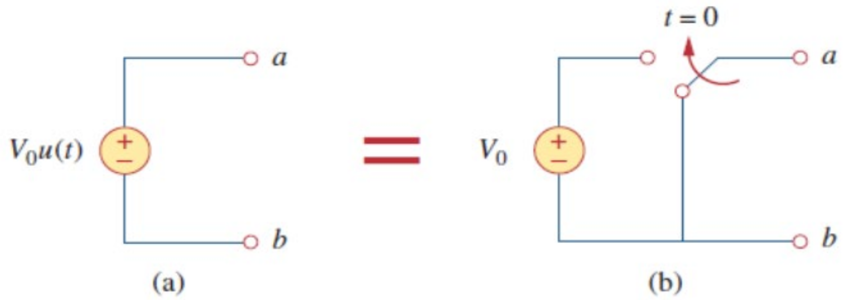
We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage:

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

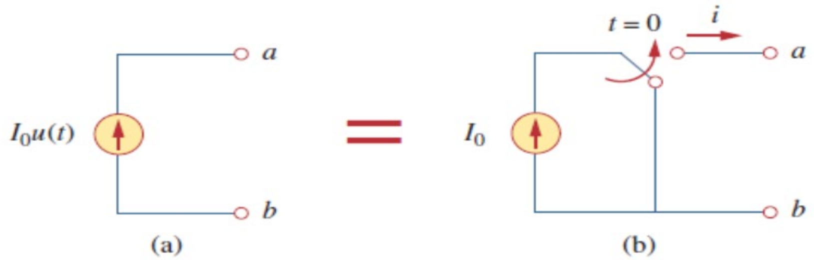
may be expressed in terms of the unit step function as:

$$v(t) = V_0 u(t - t_0)$$

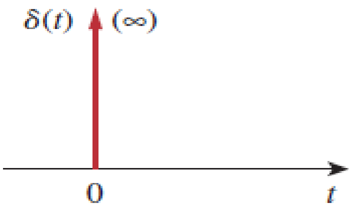
A voltage source ofV_0 u(t) is shown in Fig. (a); its equivalent circuit is shown in Fig. (b). It is evident in Fig. (b) that terminals *a-b* are short circuited (v =0) for $t < 0$ and that $v = V_0$ appears at the terminals for t>0.



For Similarly, a current source of I_0 u(t) is shown in Fig. (a) below, while its equivalent circuit is in Fig. (b) below. Notice that for t<0 there is an open circuit (i = 0), and that $i = I_0$ flows for $t > 0$.



The derivative of the unit step function u(t) is the *unit impulse Function δ(t)* which we write as:

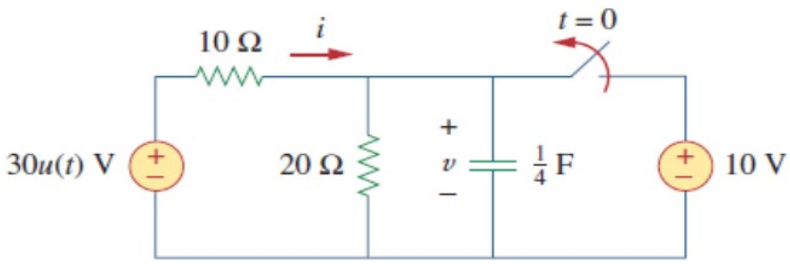


$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

The unit impulse function δ(t) is zero everywhere except at t= 0, where it is undefined.

Example:

In Fig. below, the switch has been closed for a long time and is opened at t = 0. Find $i(t)$ and $v(t)$ for all time.



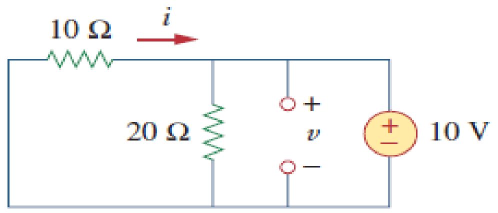
Solution:

The resistor current i can be discontinuous at t=0 while the capacitor Voltage v cannot. Hence, it is always better to find v and then obtain I from v.

Of the unit step function:

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

Then for t<0, the circuit will be as shown:

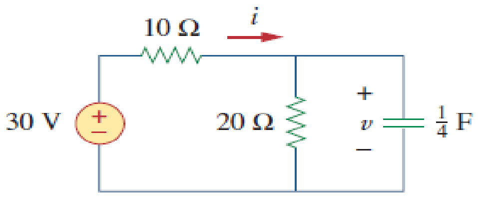


$$v = 10\text{ V}, \qquad i = -\frac{v}{10} = -1\text{ A}$$

Since the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = 10\text{ V}$$

For t>0:



$$v(\infty) = \frac{20}{20+10}(30) = 20\text{ V}$$

The Thevenin resistance at the capacitor terminals is

$$R_{Th} = 10 \parallel 20 = \frac{10 \times 20}{30} = \frac{20}{3}\,\Omega$$

and the time constant is:

$$\tau = R_{Th}C = \frac{20}{3}\cdot\frac{1}{4} = \frac{5}{3}\text{ s}$$

Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t})\text{ V}$$

$$i = \frac{v}{20} + C\frac{dv}{dt}$$
$$= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})\text{ A}$$

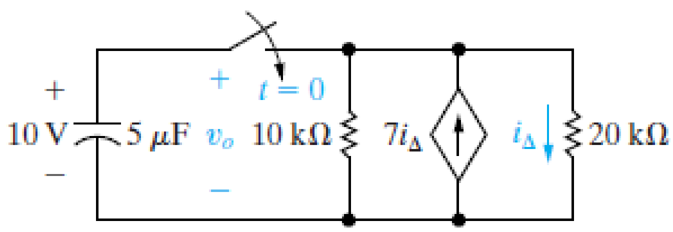
$$v = \begin{cases} 10\text{ V}, & t < 0 \\ (20 - 10e^{-0.6t})\text{ V}, & t \geq 0 \end{cases}$$

$$i = \begin{cases} -1\text{ A}, & t < 0 \\ (1 + e^{-0.6t})\text{ A}, & t > 0 \end{cases}$$

Homework:

a) When the switch is closed in the circuit shown in Fig. below, the voltage on the capacitor is 10 V. Find the expression for v_o for t ≥0.

b) Assume that the capacitor short-circuits when its terminal voltage reaches 150 V. How many milliseconds elapse before the capacitor short circuits?



The Use of Laplace Transform in Transient Analysis

R-L-C circuits are modeled by *differential equations* whose solutions describe the total response behavior of the circuits. We now introduce the powerful method of *Laplace transformation*, which involves turning **differential equations into *algebraic equations***, thus greatly facilitating the solution process.

When using phasors for the analysis of circuits, we transform the circuit from the **time domain** to the **frequency or phasor domain**. Once we obtain the phasor result, we transform it back to the time domain. We use the Laplace transformation to transform the circuit from the time domain to the frequency domain, obtain the solution, and apply **the inverse Laplace transform** to the result to transform it back to the time domain.

Given a function $f(t)$, its Laplace transform, denoted by $F(s)$ or is defined by:

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

where s is a complex variable given by:

$$s = \sigma + j\omega$$

- Since the argument st of the exponent e in above Eq. must be dimensionless, it follows that ***s* has the dimensions of frequency** and units of inverse seconds or "frequency."
- The Laplace transform is an integral transformation of a function $f(t)$ from the time domain into the complex frequency domain, giving $F(s)$.

A companion to the direct Laplace transforms in above Eq. is the ***inverse* Laplace transform** given by:

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds$$

Example:

Determine the Laplace transform of each of the following functions: (a) $u(t)$, (b) e^{-at} u(t), and (c) $\delta(t)$.

Solution: (a) For the unit step function u(t), shown in Fig.(a), the Laplace transform is:

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty}$$

$$= -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s}$$

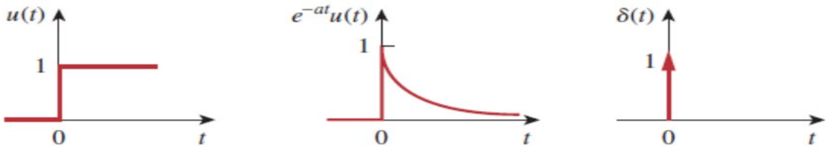
(b) For the exponential function, shown in Fig.(b), the Laplace transform is:

$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt$$

$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$

(a) For the unit impulse function, shown in Fig.(c):

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-0} = 1$$



(a) (b) (c)

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Some Properties of the Laplace Transform

Linearity

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace transforms of $f_1(t)$ and $f_2(t)$, then:

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

Scaling

If $F(s)$ is the Laplace transform of $f(t)$, then:

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Time Differentiation

Given that $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of its derivative is:

$$\mathcal{L}[f'(t)] = sF(s) - f(0^-)$$

Time Integration

If $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of its integral is:

$$\mathcal{L}\left[\int_0^t f(x)dx\right] = \frac{1}{s}F(s)$$

Initial and Final Values Theorems

The initial-value and final-value properties allow us to find the initial value and the final value of $f(t)$ directly from its Laplace transform $F(s)$.

$$f(0) = \lim_{s\to\infty} sF(s)$$

$$f(\infty) = \lim_{s\to 0} sF(s)$$

The Inverse Laplace Transform

Steps to Find the Inverse Laplace Transform:

1. Decompose $F(s)$ into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in above Table.

Applications of the Laplace Transform in Circuit Analysis

Steps in Applying the Laplace Transform in circuit analysis:

1. Transform the circuit from the time domain to the s-domain.
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique.
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

we transform a circuit in the time domain to the frequency or s-domain by Laplace transforming each term in the circuit.

For a resistor, the voltage-current relationship in the time domain is:

$$v(t) = Ri(t)$$

Taking the Laplace transform, we get

$$V(s) = RI(s)$$

For an inductor,

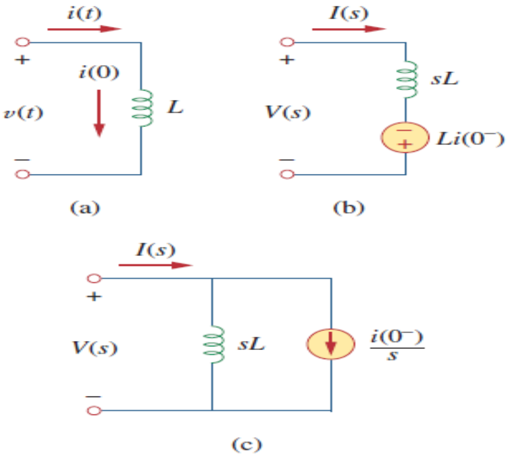
$$v(t) = L\frac{di(t)}{dt}$$

Taking the Laplace transform of both sides gives

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-)$$

or

$$I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}$$



For a capacitor,

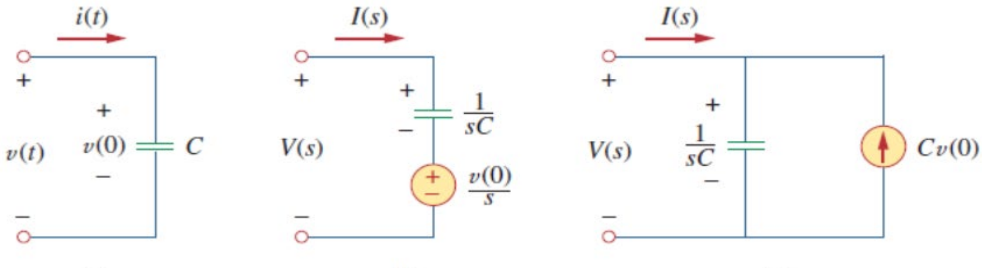
$$i(t) = C\frac{dv(t)}{dt}$$

which transforms into the s-domain as

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$

or

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$



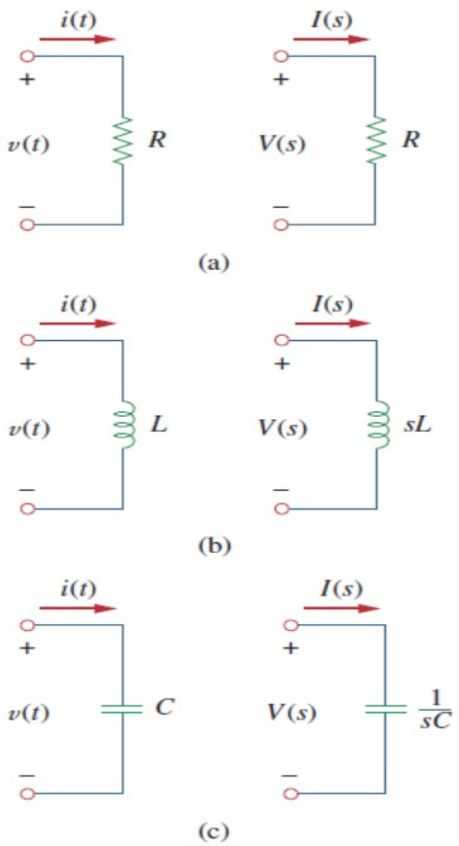
(a) (b) (c)

If we assume zero initial conditions for the inductor and the capacitor, the above equations reduce to:

$$\text{Resistor:} \quad V(s) = RI(s)$$

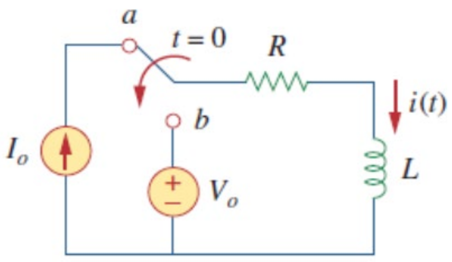
$$\text{Inductor:} \quad V(s) = sLI(s)$$

$$\text{Capacitor:} \quad V(s) = \frac{1}{sC}I(s)$$



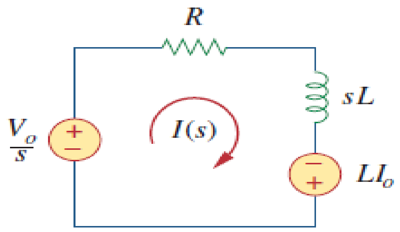
Example:

In the circuit of Fig. below, the switch moves from position *a* to position *b* at t=0. Find *i*(*t*) for t>0.



Solution:

The initial current through the inductor is $i(0)=I_0$. Using mesh analysis:



$$I(s)(R + sL) - LI_o - \frac{V_o}{s} = 0$$

or

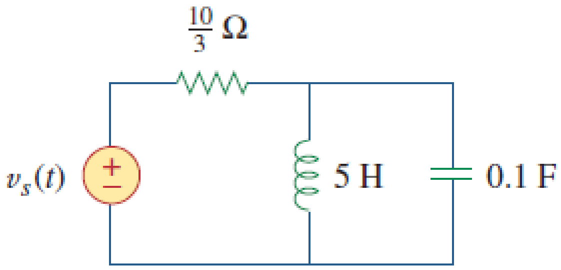
$$I(s) = \frac{LI_o}{R + sL} + \frac{V_o}{s(R + sL)} = \frac{I_o}{s + R/L} + \frac{V_o/L}{s(s + R/L)}$$

$$I(s) = \frac{I_o}{s + R/L} + \frac{V_o/R}{s} - \frac{V_o/R}{(s + R/L)}$$

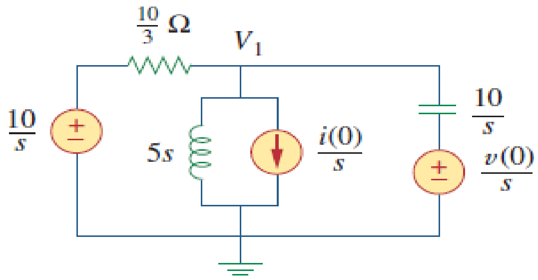
$$i(t) = \left(I_o - \frac{V_o}{R}\right)e^{-t/\tau} + \frac{V_o}{R}, \qquad t \geq 0$$

Example:

Consider the circuit in Fig. below. Find the value of the voltage across the capacitor if the value of $v_s(t)$=10u(t) and assume that at t=0, -1A flows through the inductor and is +5V across the capacitor.



Solution:



$$\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} + \frac{i(0)}{s} + \frac{V_1 - [v(0)/s]}{1/(0.1s)} = ($$

or

$$0.1\left(s + 3 + \frac{2}{s}\right) V_1 = \frac{3}{s} + \frac{1}{s} + 0.5$$

where $v(0) = 5$ V and $i(0) = -1$ A. Simplifying we get

$$(s^2 + 3s + 2)\, V_1 = 40 + 5s$$

or

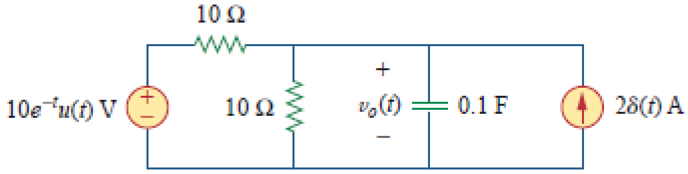
$$V_1 = \frac{40 + 5s}{(s + 1)(s + 2)} = \frac{35}{s + 1} - \frac{30}{s + 2}$$

Taking the inverse Laplace transform yields

$$v_1(t) = (35e^{-t} - 30e^{-2t})u(t)\ \text{V}$$

Example:

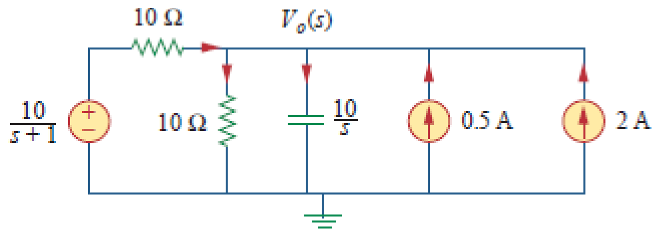
Find $v_o(t)$ in the circuit of Fig. below. Assume $v_o(0) = 5$ V.



Solution:

We transform the circuit to the s-domain as shown in Fig. below. The initial condition is included in the form of the current source:

$C\, v_o(0)$=0.1(5)=0.5A



We apply nodal analysis. At the top node:

$$\frac{10/(s + 1) - V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{10/s}$$

or

$$\frac{1}{s + 1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10} = \frac{1}{10}V_o(s + 2)$$

Multiplying through by 10,

$$\frac{10}{s + 1} + 25 = V_o(s + 2)$$

or

$$V_o = \frac{25s + 35}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}$$

where

$$A = (s+1)V_o(s)\big|_{s=-1} = \frac{25s+35}{(s+2)}\bigg|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s+2)V_o(s)\big|_{s=-2} = \frac{25s+35}{(s+1)}\bigg|_{s=-2} = \frac{-15}{-1} = 15$$

Thus,

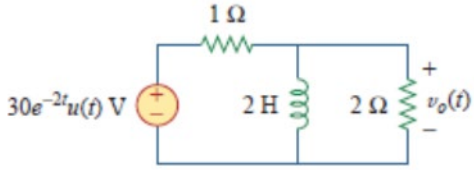
$$V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$v_o(t) = (10e^{-t} + 15e^{-2t})u(t)\text{ V}$$

Homework:

Find *vo(t)* in the circuit shown in Fig. below. Note that, since the voltage input is multiplied by $u(t)$, the voltage source is a short for t<0 all and $i_L(0)$=0.



Homework:

The initial energy in the circuit of Fig. below is zero at t=0. Assume that v_s=15u(t) V

(a) Find V_o(s) using the Thevenin theorem.

(b) Apply the initial- and final-value theorems to find $v_o(0)$ and $v_o(\infty)$.

(c) Obtain $v_o(t)$.