Magnetically Coupling (Mutual) Circuits

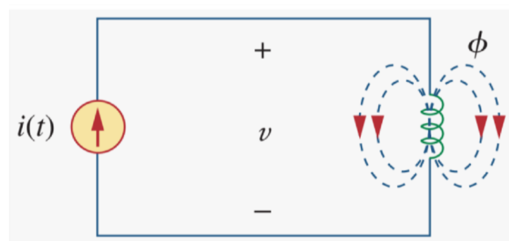
1- Mutual Inductance

When two loops with or without contacts between them affect each other through the magnetic field generated by one of them, they are said to be magnetically coupled.

The transformer is an electrical device designed on the basis of the concept of magnetic coupling. It uses magnetically coupled coils to transfer energy from one circuit to another. The transformers are used in electronic circuits such as radio and television receivers for such purposes as impedance matching, isolating one part of a circuit from another, and for stepping up or down ac voltages and currents.

When two inductors (or coils) are in a close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, thereby inducing voltage in the latter. This phenomenon is known as mutual inductance

A coil with N turns, when current i flows through the coil, a magnetic flux is produced around it. According to Faraday's law:



$$v = N\frac{d\phi}{dt}$$

Any change in ϕ is caused by a change in the current.

$$v = N\frac{d\phi}{di}\frac{di}{dt} \qquad L = N\frac{d\phi}{di}$$

L is called self-inductance, because it relates the voltage induced in a coil by a time-varying current in the same coil. When two coils with self-inductances L1 and L2 those are close to each other. Coil 1 has N1 turns, while coil 2 has N2 turns. Assume that the second inductor carries no current. The magnetic flux $\phi 1$ emanating from coil 1 has two components: One component $\phi 11$ links only coil 1, and another component $\phi 12$ links both coils.

$$\phi_1 = \phi_{11} + \phi_{12}$$

Entire flux $\phi 1$ links coil 1, so the voltage induced in coil 1 is

$$v_1 = N_1\frac{d\phi_1}{dt} \qquad \textbf{or} \qquad v_1 = N_1\frac{d\phi_1}{di_1}\frac{di_1}{dt} = L_1\frac{di_1}{dt}$$

Where $L_1=N_1\, d\phi_1/di_1$ is the self-inductance of coil 1

Only flux ϕ_{12} links coil 2, so the voltage induced in coil 2 is

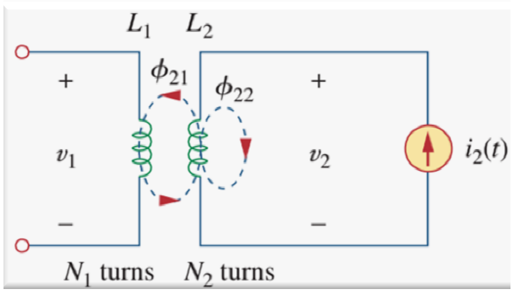
$$v_2 = N_2 \frac{d\phi_{12}}{dt} \quad \textbf{or} \quad v_2 = N_2 \frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21}\frac{di_1}{dt} \quad \textbf{Where} \quad M_{21} = N_2 \frac{d\phi_{12}}{di_1}$$

M_{21} is known as the mutual inductance of coil 2 with respect to coil 1. Subscript 21 indicates that the inductance relates the voltage induced in coil 2 to the current in coil 1. The open-circuit mutual voltage (or induced voltage) across coil 2 is

$$v_2 = M_{21}\frac{di_1}{dt}$$

Suppose the current flow in coil 2, while coil 1 carries no current. The magnetic flux ϕ_2 emanating from coil 2 comprises flux ϕ_{22} that links only coil 2 and flux ϕ_{21} that links both coils.

$$\phi_2 = \phi_{21} + \phi_{22}$$



The entire flux ϕ_2 links coil 2, so the voltage induced in coil 2 is

$$v_2 = N_2\frac{d\phi_2}{dt} = N_2\frac{d\phi_2}{di_2}\frac{di_2}{dt} = L_2\frac{di_2}{dt}$$

Where $L_2 = N_2 d\phi_2/di_2$ is the self-inductance of coil 2 Only flux ϕ_{21} links coil 1, so the voltage induced in coil 1 is

$$v_1 = N_1\frac{d\phi_{21}}{dt} = N_1\frac{d\phi_{21}}{di_2}\frac{di_2}{dt} = M_{12}\frac{di_2}{dt} \quad \text{and} \quad M_{12} = N_1\frac{d\phi_{21}}{di_2}$$

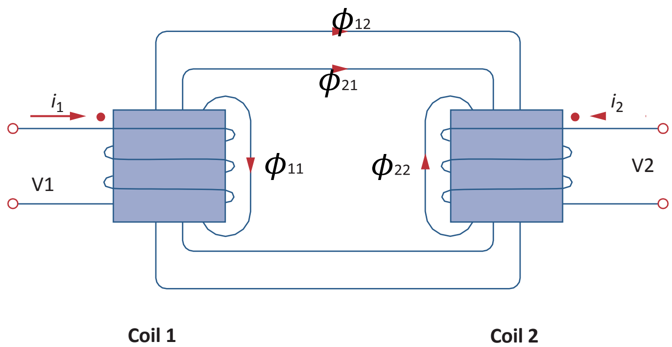
M_{12} is known as the mutual inductance of coil 1 with respect to coil 2. The open-circuit mutual voltage (or induced voltage) across coil 1 is :

$$v_1 = M_{12}\frac{di_2}{dt} \quad \text{and} \quad M_{12} = M_{21} = M$$

Mutual inductance is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

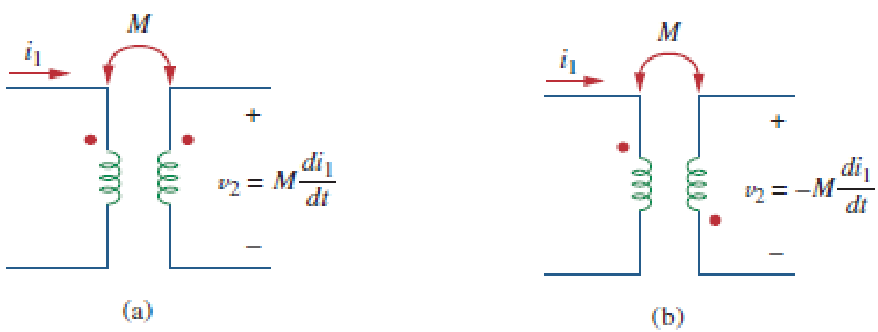
2- Analyze Circuits Involving Mutual Inductance

We apply the dot convention in circuit analysis. A dot is placed in the circuit at one of each of the two magnetically coupled coils to indicate the direction of magnetic flux of currents enters that dotted terminal of the coil, as shown in figure.

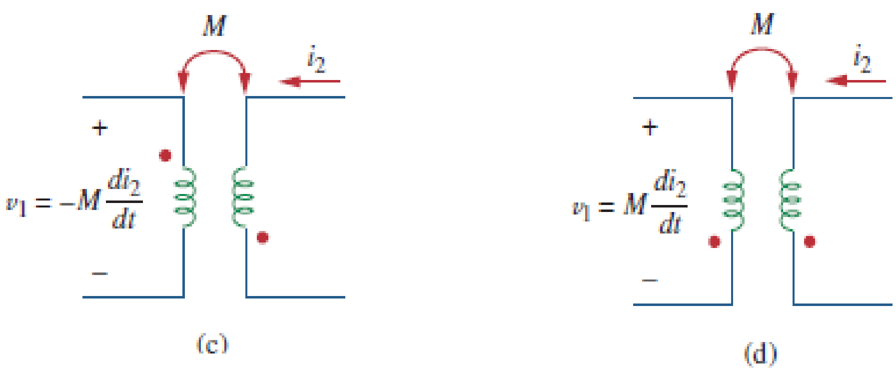


The dots are used along with the dot convention to determine the polarity of the mutual voltage. The dot convention is stated as follows:

a- If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.



b- If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.

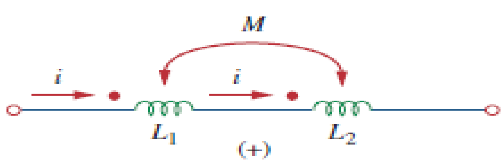


3- Connection of Mutual Inductance

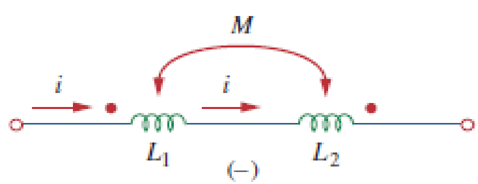
Mutually connected in series or parallel inductors can be classed as either “aiding” or “opposing” the total inductance can determine depends on the type of connections :

3-1 **Series connection**: When inductors are connected together in series so that the magnetic field of one links with the other, the effect of mutual inductance either increases or decreases the total inductance depending upon the amount of magnetic coupling. The effect of this mutual inductance depends upon the distance apart of the coils and their orientation to each other. The total inductance for coupled coils in series is:

$L = L_1 + L_2 + 2M$ (Series-aiding connection) $\boxed{L_T > L_1 + L_2}$



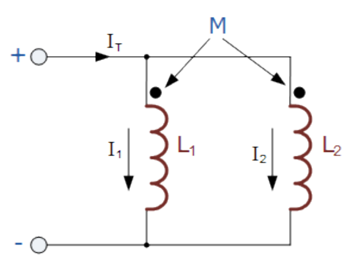
$L = L_1 + L_2 - 2M$ (Series-opposing connection) $\boxed{L_T < L_1 + L_2}$



3-2 **Parallel Connection**: When inductors are connected together in parallel so that the magnetic field of one links with the other, the effect of mutual inductance either increases or decreases the total inductance depending upon the amount of magnetic coupling that exists between the coils. The effect of this mutual inductance depends upon the distance apart of the coils and their orientation to each other.

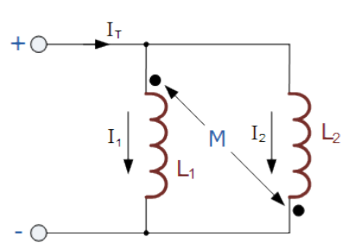
The total inductance for coupled coils in parallel is:

Parallel Aiding Inductors



$$L_T = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

Parallel opposing Inductors



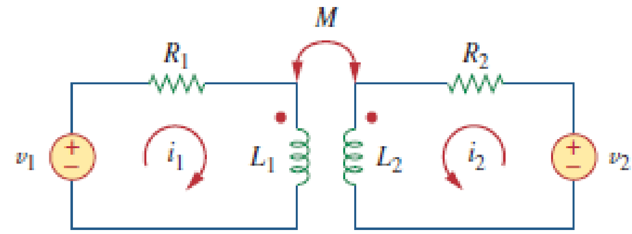
$$L_T = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

For the time domain circuit shown in fig.(a). Applying KVL to coil 1 gives

$$v_1 = i_1R_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

For coil 2, KVL gives

$$v_2 = i_2R_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt}$$



The above two equations can be written in the frequency domain as:

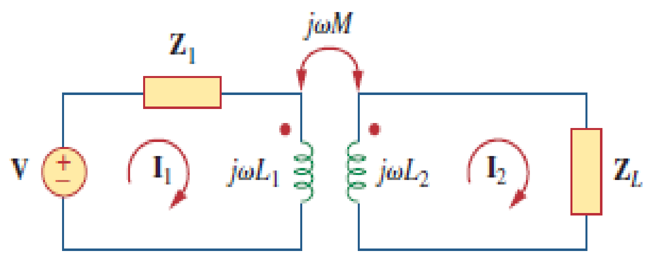
$$\mathbf{V}_1 = (R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M\mathbf{I}_2$$
$$\mathbf{V}_2 = j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2)\mathbf{I}_2$$

Also applying KVL to coil 1 gives in figure below:

$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2$$

For coil 2, KVL yields

$$0 = -j\omega M\mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$$



Example 1 : Calculate the phasor currents I_1 and I_2 in the circuit below:

For loop 1, KVL gives

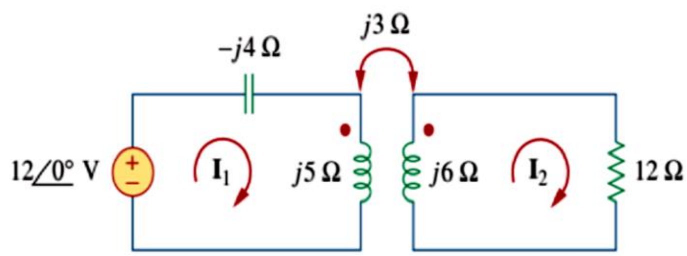
$$-12 + (-j4 + j5)\mathbf{I}_1 - j3\mathbf{I}_2 = 0$$

$$j\mathbf{I}_1 - j3\mathbf{I}_2 = 12 \qquad (1)$$

For loop 2, KVL gives

$$-j3\mathbf{I}_1 + (12 + j6)\mathbf{I}_2 = 0$$

$$\mathbf{I}_1 = \frac{(12 + j6)\mathbf{I}_2}{j3} = (2 - j4)\mathbf{I}_2 \qquad (2)$$

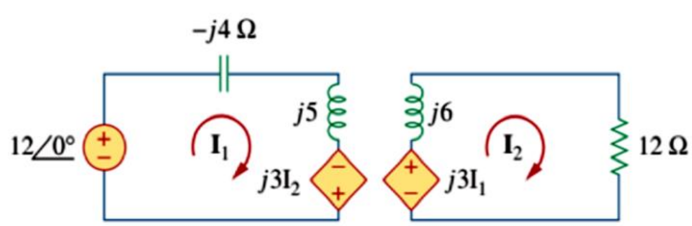


(a)

Substitute eq(1) in eq(2) gives

$$(j2 + 4 - j3)\mathbf{I}_2 = (4 - j)\mathbf{I}_2 = 12$$

$$\mathbf{I}_2 = \frac{12}{4 - j} = 2.91\underline{/14.04^\circ} \text{ A}$$



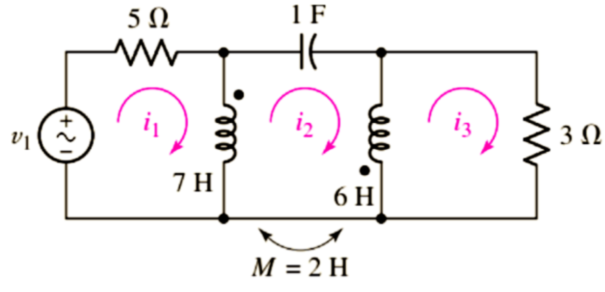
(b)

Substitute $\mathbf{I}_2$ in eq(2) gives

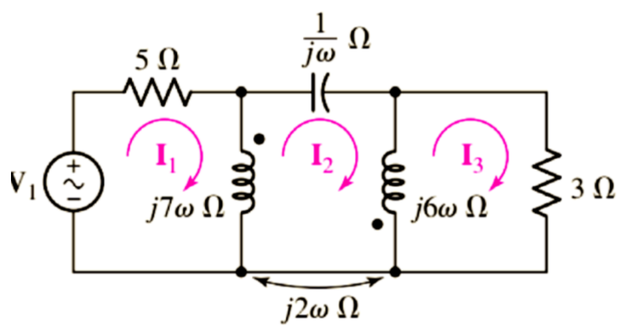
$$\mathbf{I}_1 = (2 - j4)\mathbf{I}_2 = (4.472\underline{/-63.43^\circ})(2.91\underline{/14.04^\circ}) = 13.01\underline{/-49.39^\circ} \text{ A}$$

Example2 : Write a complete set of phasor mesh equations for the circuit of fig. (a). and replace both the mutual inductance and the two self-inductances with their corresponding impedances as in fig. (b).

Solution



(a)



(b)

For mesh 1, KVL gives

$$5\mathbf{I}_1 + 7j\omega(\mathbf{I}_1 - \mathbf{I}_2) + 2j\omega(\mathbf{I}_3 - \mathbf{I}_2) = \mathbf{V}_1$$
$$(5 + 7j\omega)\mathbf{I}_1 - 9j\omega\mathbf{I}_2 + 2j\omega\mathbf{I}_3 = \mathbf{V}_1$$

For mesh 2, KVL gives

$$7j\omega(\mathbf{I}_2 - \mathbf{I}_1) + 2j\omega(\mathbf{I}_2 - \mathbf{I}_3) + \frac{1}{j\omega}\mathbf{I}_2 + 6j\omega(\mathbf{I}_2 - \mathbf{I}_3) + 2j\omega(\mathbf{I}_2 - \mathbf{I}_1) = 0$$
$$-9j\omega\mathbf{I}_1 + \left(17j\omega + \frac{1}{j\omega}\right)\mathbf{I}_2 - 8j\omega\mathbf{I}_3 = 0$$

For mesh 3, KVL gives

$$6j\omega(\mathbf{I}_3 - \mathbf{I}_2) + 2j\omega(\mathbf{I}_1 - \mathbf{I}_2) + 3\mathbf{I}_3 = 0$$
$$2j\omega\mathbf{I}_1 - 8j\omega\mathbf{I}_2 + (3 + 6j\omega)\mathbf{I}_3 = 0$$

4- The Coupling Coefficient

Coupling coefficient (k) is used to describe the degree of coupling between coils.

$$\boxed{M = k\sqrt{L_1 L_2}}$$

where $0 \le k \le 1$ or equivalently $0 \le M \le \sqrt{L_1 L_2}$

- *If the entire flux produced by one coil links another coil, then k=1 and we have 100 percent coupling, or the coils are said to be perfectly coupled.*
- *For k<0.5, the coils are said to be loosely coupled.*
- *For k>0.5, the coils are said to be tightly coupled*

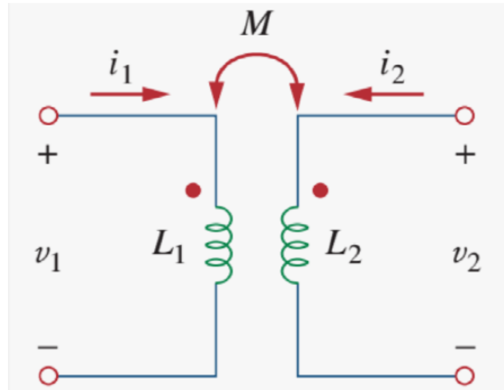
5- Energy in a Coupled Circuit

The energy stored in an inductor is given by: $w = \frac{1}{2}Li^2$

The total energy stored in the coils when both i1 and i2 have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$$

If we reverse the order by which the currents reach their final values, that is, if we first increase i2 from zero to I2 and later increase i1 from zero to I1, the total energy stored in the coils is:



$$w = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + M_{21}I_1I_2$$

So, $$M_{12} = M_{21} = M$$

$$w = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2$$

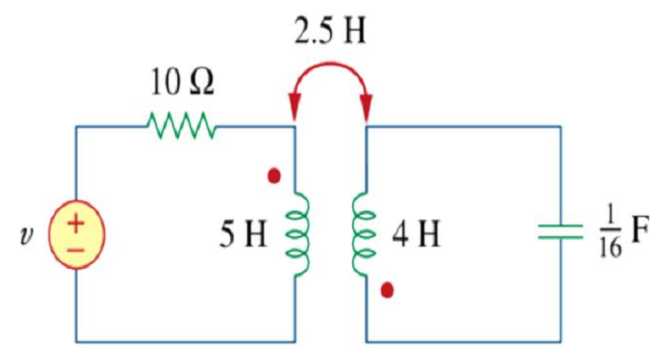
The above equation was derived based on the assumption that the coil currents both entered the dotted terminals.

If one current enters one dotted terminal while the other current leaves the other dotted terminal, the sign of the mutual energy term is reversed.

$$w = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 - MI_1I_2$$

Example 3 : Consider the circuit. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time t=1 s if v=60 cos(4t + 30o) V.

Solution



$$k = \frac{M}{\sqrt{L_1L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

$$60\cos(4t + 30°) \Rightarrow 60\angle 30°, \quad \omega = 4 \text{ rad/s}$$

$$5 \text{ H} \Rightarrow j\omega L_1 = j20\ \Omega$$

$$2.5 \text{ H} \Rightarrow j\omega M = j10\ \Omega$$

$$4 \text{ H} \Rightarrow j\omega L_2 = j16\ \Omega$$

$$\frac{1}{16}\text{F} \Rightarrow \frac{1}{j\omega C} = -j4\ \Omega$$

For mesh 1

$$(10 + j20)\mathbf{I}_1 + j10\mathbf{I}_2 = 60\angle 30° \qquad (1)$$

For mesh 2

$$j10\mathbf{I}_1 + (j16 - j4)\mathbf{I}_2 = 0$$

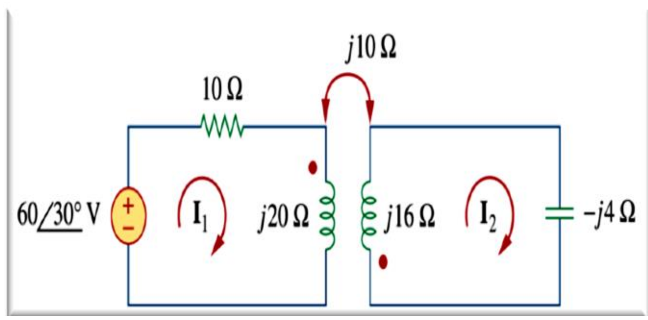
$$\mathbf{I}_1 = -1.2\mathbf{I}_2 \qquad (2)$$

Substitute (2) in (1)

$$\mathbf{I}_2(-12 - j14) = 60\angle 30° \Rightarrow \mathbf{I}_2 = 3.254\angle 160.6°$$

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 = 3.905\angle -19.4° \text{ A}$$

In time domain



$$i_1 = 3.905\cos(4t - 19.4^\circ), \qquad i_2 = 3.254\cos(4t + 160.6^\circ)$$

At time $t = 1$ s, $4t = 4$ rad $= 229.2^\circ$, and

$$i_1 = 3.905\cos(229.2^\circ - 19.4^\circ) = -3.389 \text{ A}$$

$$i_2 = 3.254\cos(229.2^\circ + 160.6^\circ) = 2.824 \text{ A}$$

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2 = \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73 \text{ J}$$

Examlpe 4: Two coils connected in series have a self-inductance of 20mH and 60mH respectively. The total inductance of the combination was found to be 100mH. Determine the amount of mutual inductance that exists between the two coils assuming that they are aiding each other.

Solution

$$L_T = L_1 + L_2 \pm 2M$$

$$100 = 20 + 60 + 2M$$

$$2M = 100 - 20 - 60$$

$$\therefore M = \frac{20}{2} = 10\text{mH}$$

Examlpe 5: Two coils having inductances of 5H and 4H respectively were wound uniformly onto a non-magnetic core; it was found that their mutual inductance was 1.5H. Calculate the coupling coefficient that exists between.

Solution

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{1.5}{\sqrt{5\times4}} = 0.335 = 33.5\%$$