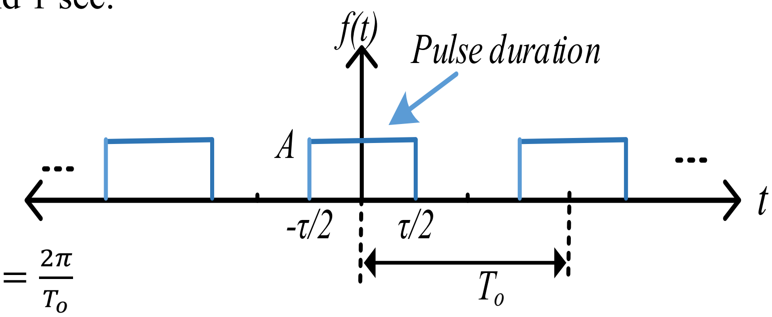
Ex 2-5:

Plot the double-sided amplitude and phase spectrum of the periodic signal shown below (rectangular pulse) when

a) τ=1, T_o=5, 10 and 20 sec.
b) T_o=20, τ=4, 2 and 1 sec.

Solution:



τ: pulse duration

T_o : period

$f_o : \frac{1}{T_o}$; $\omega = 2\pi f_o = \frac{2\pi}{T_o}$

$\frac{2\pi}{T_o} \leq 1$ "Duty cycle"

$$f(t) = \begin{cases} A, & |t| \leq \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < |t| < \frac{T_o}{2} \end{cases}$$

$$C_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_o t}dt = \frac{1}{T_o}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-jn\omega_o t}dt$$

$$= \frac{A}{-jn\omega_o T_o}(e^{-\frac{jn\omega_o\tau}{2}} - e^{+\frac{jn\omega_o\tau}{2}}) \qquad n\neq 0$$

$$= \frac{2A}{n\omega_o T_o}(\frac{e^{\frac{jn\omega_o\tau}{2}} - e^{-\frac{jn\omega_o\tau}{2}}}{2j}) \qquad n\neq 0$$

$$= \frac{2A}{n\omega_o T_o} sin(\frac{n\omega_o\tau}{2}) \quad *\{\tau/\tau\}$$

$$= \frac{A\tau}{T_o}\frac{\sin(\frac{n\omega_o\tau}{2})}{(\frac{n\omega_o\tau}{2})}$$

$$C_n = \frac{A\tau}{T_o} Sa(\frac{n\omega_o\tau}{2})$$

…(2-14) General for the rectangular pulses of amplitude A, duration τ, and period T_o.

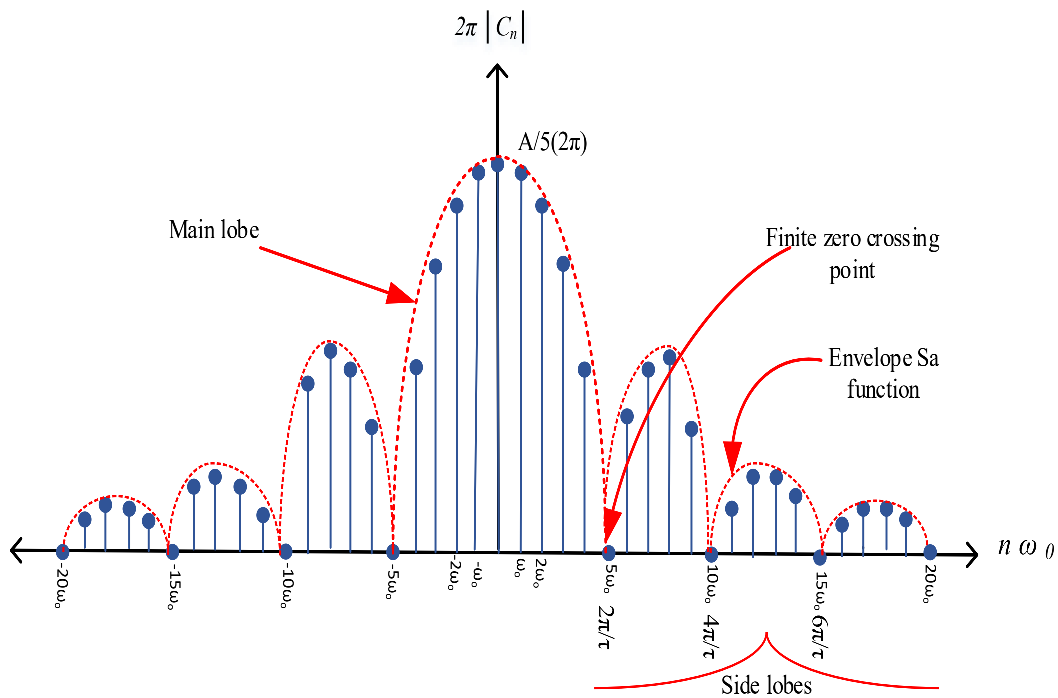
$$C_n = \frac{A\tau}{T_o} Sa(\frac{n\pi\tau}{T_o})$$

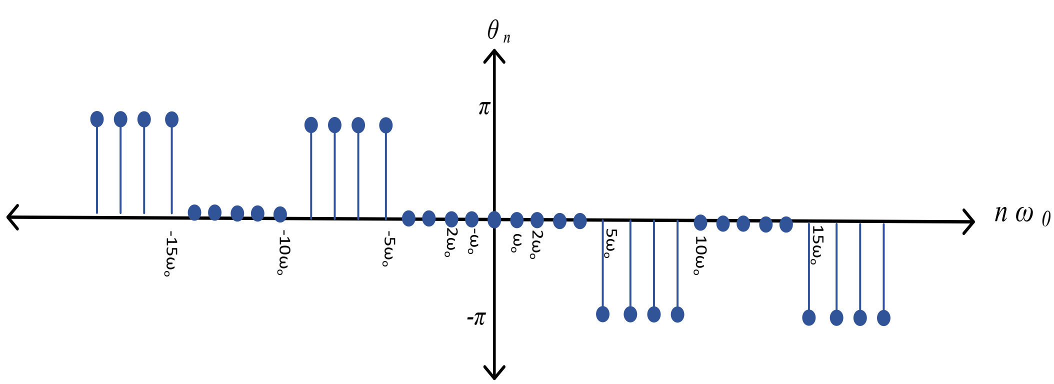
$$|C_n| = \frac{A\tau}{T_o}\left|Sa(\frac{n\pi\tau}{T_o})\right|$$

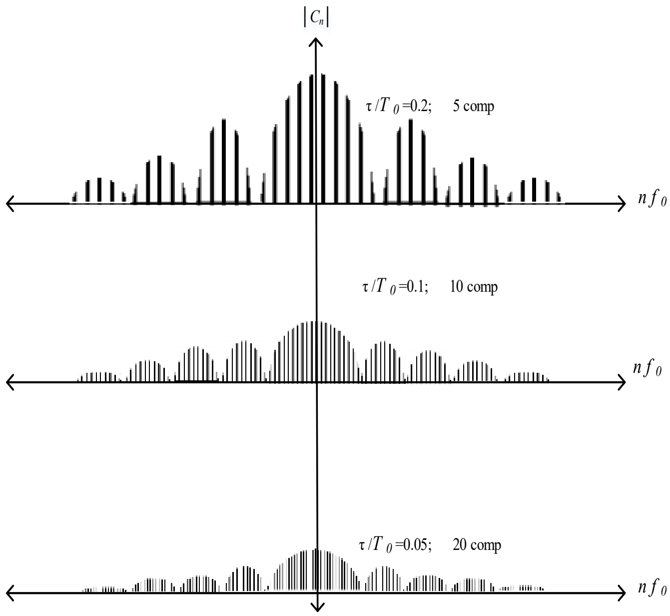
θ_n=0 or ±180°, (since there is no imaginary part)

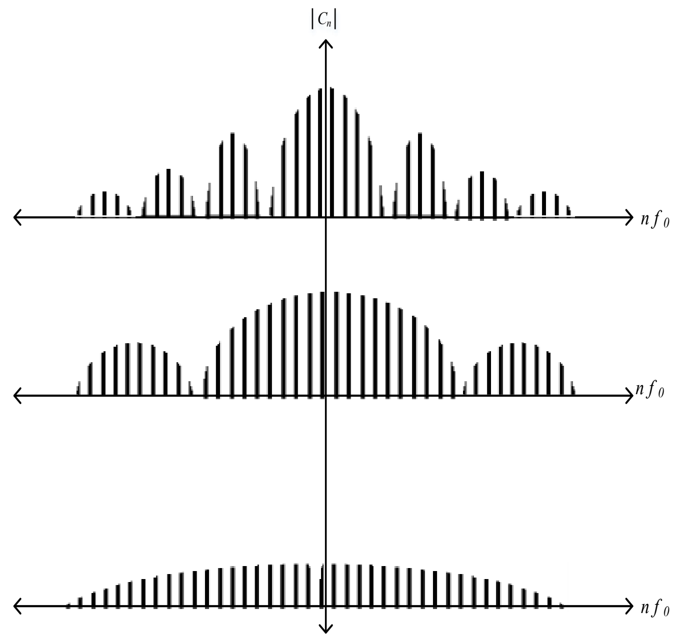
a) τ=1, T_o=5, $\Longrightarrow \frac{\tau}{T_o} = \frac{1}{5}$

$$|C_n| = \frac{A}{5}\left|Sa(\frac{n\pi}{5})\right|$$









τ: fixed
T_o: variable

T_o: fixed
τ: variable

When τ is fixed:

As T_o increases (1) The amplitude decreases as $1/T_o$.

(2) Spacing between lines decreases as $2\pi/T_o$.

When T_o is fixed:

As τ increases (1) The amplitude increases proportional to τ.

(2) The frequency content of the signal is compressed in narrower range.

Parseval's Power theorem:

The power of periodic signals can be computed in frequency domain rather than time domain using the spectrum function:

$$P_{av} = \sum_{-\infty}^{\infty} |C_n|^2 \quad \text{Watt} \qquad (2\text{-}15)$$

Note: if $|C_n|$ in volt $\Rightarrow P_{av} = \frac{1}{R}\sum_{-\infty}^{\infty}|C_n|^2$

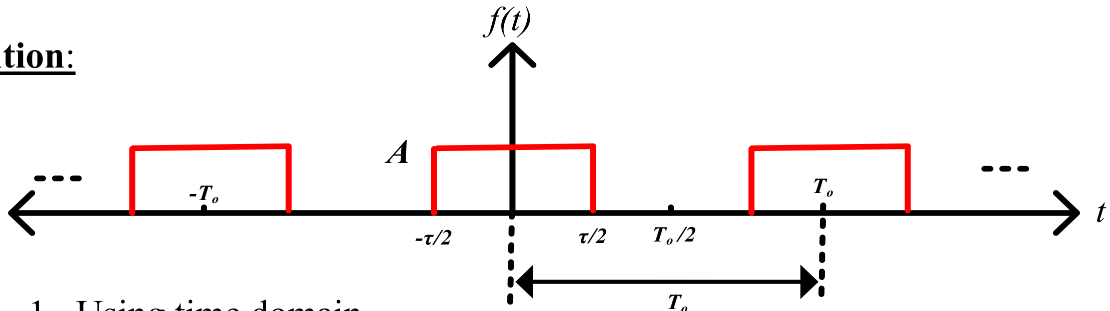
If $|C_n|$ in ampere $\Rightarrow P_{av} = R\sum_{-\infty}^{\infty}|C_n|^2$, For given R, if R=1, we use the form of equation (2-15).

Ex 2-6:

For the rectangular pulse below if $\frac{\tau}{T_o} = 0.25$ find;

1- The total average power.
2- The ratio of average power in the first three harmonics to the total average power.

Solution:



1- Using time domain

$$P_{av} = \frac{1}{T}\int_0^T |f(t)|^2 dt = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A^2 dt = A^2 \frac{\tau}{T_o} = 0.25\ A^2 watt$$

Using frequency domain

$A\prod(\frac{\tau}{T_o}) \Longrightarrow C_n = A\frac{\tau}{T_o}Sa(\frac{n\pi\tau}{T_o})$.. General formula for rectangular pulse

$$C_n = 0.25\ A\ Sa(0.25\ n\pi)$$

$$|C_n| = 0.25\ A|Sa(0.25\ n\pi)|$$

n	$\lvert C_n\rvert$
0	0.25 A
±1	0.225 A
±2	0.159 A
±3	0.075 A

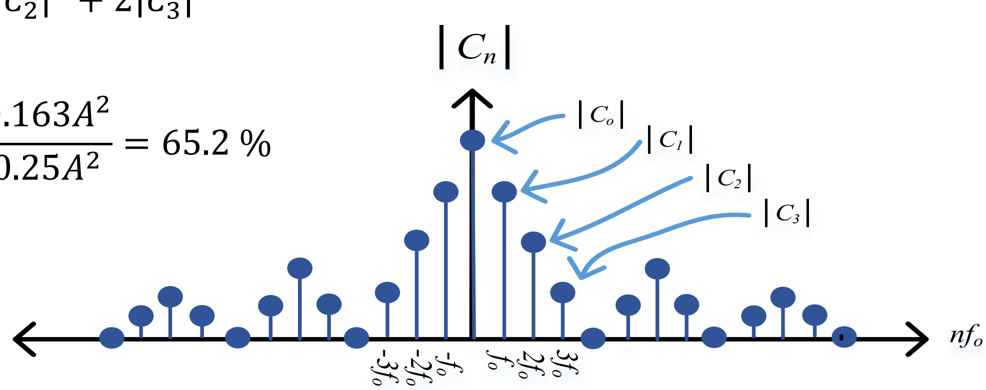
$$P_{av} = \sum_{-\infty}^{\infty} |C_n|^2$$

2- $|C_n| = 0.25\ A|Sa(0.25\ n\pi)|$

$P_3 = 2|C_1|^2 + 2|C_2|^2 + 2|C_3|^2$

$= 0.163\ A^2$

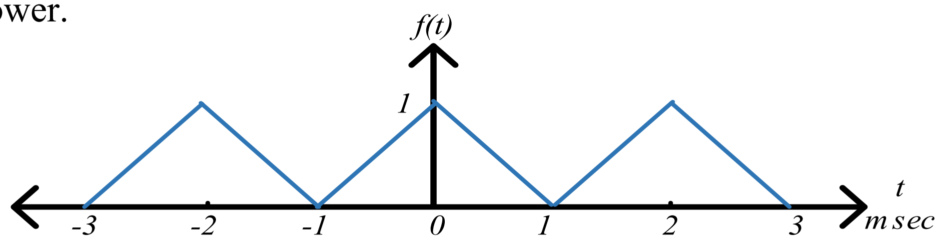
$$\frac{P_3}{P_T} = \frac{0.163A^2}{0.25A^2} = 65.2\ \%$$



H.W:

For the signal shown, find:

1- The total average power.
2- The average power in the fundamental frequency.
3- The average power in the first five harmonics.
4- The dc power.
5- The ratio of average power in the frequency range (0→ 3kHz) to the total average power.



3- Spectrum of non-periodic signals

The spectrum of any non-periodic signals (double sided) can be obtained by plotting |F(ω)| verses ω and θ(ω) versus ω, where:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad = Re + j\,Im \qquad (2\text{-}16)a$$

$$\theta(\omega) = tan^{-1}\frac{Im}{Re} \qquad \text{Phase spectrum} \qquad (2\text{-}16)b$$

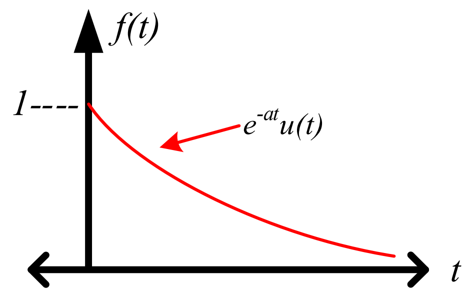
$$|F(\omega)| = \sqrt{Re^2 + Im^2} \qquad \text{Amplitude spectrum} \qquad (2\text{-}16)c$$

F(ω) is called Fourier Transform (F.T.). F(ω) can be transformed back to time domain (f(t)) using Inverse F.T. (I.F.T) given by:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad (2\text{-}17)$$

Ex 2-7:

Plot the double-sided amplitude and phase spectrum of the signal shown below:



Solution:

$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

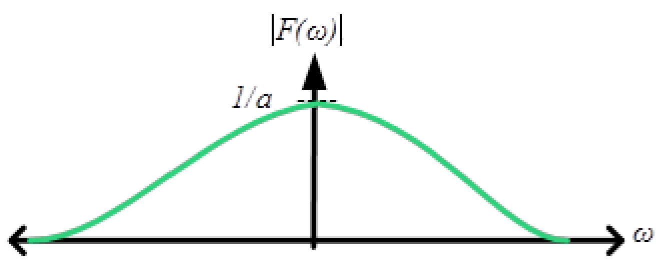
$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$

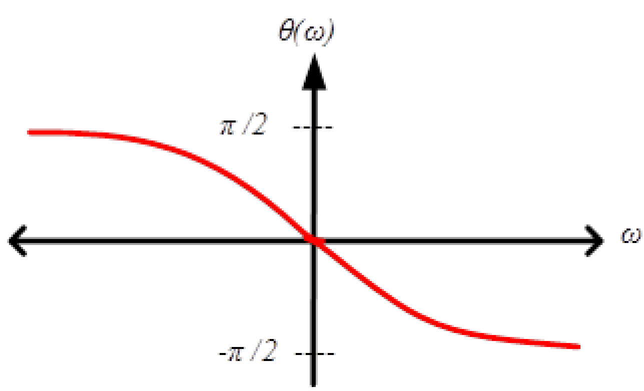
$F(\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \frac{1}{a+j\omega}; a > 0$

$= \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$

$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$

$\theta(\omega) = -tan^{-1}(\frac{\omega}{a})$





The spectrum of non-periodic signals is continuous.

Some Fourier Transform Properties:

Property	$f(t)$	$F(\omega)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
Delay	$f(t - t_o)$	$e^{-j\omega t_o} F(\omega)$
Frequency translation	$e^{j\omega_o t} f(t)$	$F(\omega - \omega_o)$
Amplitude translation	$f(t) \cos \omega_o t$	$\frac{1}{2} F(\omega + \omega_o) + \frac{1}{2} F(\omega - \omega_o)$
Time convolution	$\int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau)\, d\tau$	$F_1(\omega) F_2(\omega)$
Frequency convolution	$f_1(t) f_2(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(u) F_1(\omega - u) du$
Duality	$F(t)$	$2\pi f(-\omega)$
Time differentiation	$\frac{d^n}{(dt)^n} f(t)$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(t) d\tau$	$\frac{1}{j\omega} + \pi F(0) \delta(\omega)$