Sliding Mode Control

- Sliding mode control (SMC) is **a nonlinear control technique** featuring remarkable properties of accuracy, robustness, and easy tuning and implementation.
- SMC systems are designed to drive the system states onto **a particular surface in the state space, named sliding surface.** Once the sliding surface is reached, sliding mode control keeps the states on the close neighborhood of the sliding surface.
- Hence the sliding mode control is a **two part controller design**. The first part involves the design of a **sliding surface** so that the sliding motion satisfies design specifications. The second is concerned with the **selection of a control law** that will make the switching surface attractive to the system state.
- There are two main advantages of sliding mode control. First is that the dynamic behavior of the system may be tailored by the particular choice of the sliding function. **Secondly, the closed loop response becomes totally insensitive to some particular uncertainties.** This principle extends to model parameter uncertainties, disturbance and non- linearity that are bounded.
- From a practical point of view SMC allows for controlling nonlinear processes subject to external disturbances and heavy model uncertainties.

SMC brings with it several disadvantages associated with conventional designs:

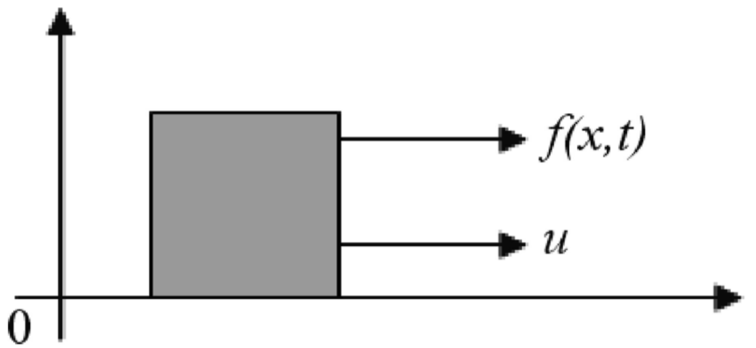
- For one, due to the discontinuous switching mechanism, **the undesired chattering** in the control input may excite high frequencies in system responses which are almost unbearable to operations of actuators and mechanical components. This phenomenon leads to deteriorations and potentially causes unpredictable instabilities of the closed-loop system.
- In addition, without knowing the information about the bounds of the uncertainties, it is hard, if not impossible, to design a robust SMC to ensure the robust stability of the closed-loop system.

These drawbacks to a large extent have restricted the applications of conventional SMC schemes in many practical circumstances. Therefore, the quest for developing novel intelligent control solutions to tackle these issues appears to be a demanding challenge of our times.

Main Concepts of Sliding Mode Control

Consider the single-dimensional motion of a unit mass (Fig. below) is considered. A state-variable description is easily obtained by introducing variables for the position (x_1) and the velocity (x_2).

$$\begin{cases} \dot{x}_1 = x_2 & x_1(0) = x_{10} \\ \dot{x}_2 = u + f(x_1, x_2, t) & x_2(0) = x_{20}, \end{cases} \qquad (1.1)$$



where u is the control force, and the disturbance term $f(x_1, x_2, t)$, which may comprise dry and viscous friction as well as any other unknown resistance forces, is assumed to be bounded, i.e.,

$|f(x_1, x_2, t)| <= L > 0$.

The problem is to design a feedback control law $u = u(x_1, x_2)$ that drives the mass to the origin (equilibrium point) asymptotically.

In other words, the control $u=u(x_1, x_2)$ is supposed to drive the state variables to zero. $\lim_{t \to \infty} x_1, x_2 = 0$

This apparently simple control problem is a challenging one, since asymptotic convergence is to be achieved in the presence of the unknown bounded disturbance $f(x_1, x_2, t)$. For instance, a linear state-feedback control law:

$$u = -k_1 x_1 - k_2 x_2, \quad k_1 > 0, \; k_2 > 0 \qquad (1.2)$$

provides asymptotic stability of the origin only for $f(x_1, x_2, t) = 0$ and typically only drives the states to a bounded domain $\Omega(k_1, k_2, L)$ for $|f(x_1, x_2, t)| \leq L > 0$.

Example

The results of the simulation of the system in Eqs. (1.1), (1.2) with $x_1(0)=1$, $x_2(0)=-2$, $k_1 = 3$, $k_2 = 4$, and $f(x_1, x_2, t)= \sin(2t)$, which illustrate this statement, are presented in Figs. 1.2 and 1.3.

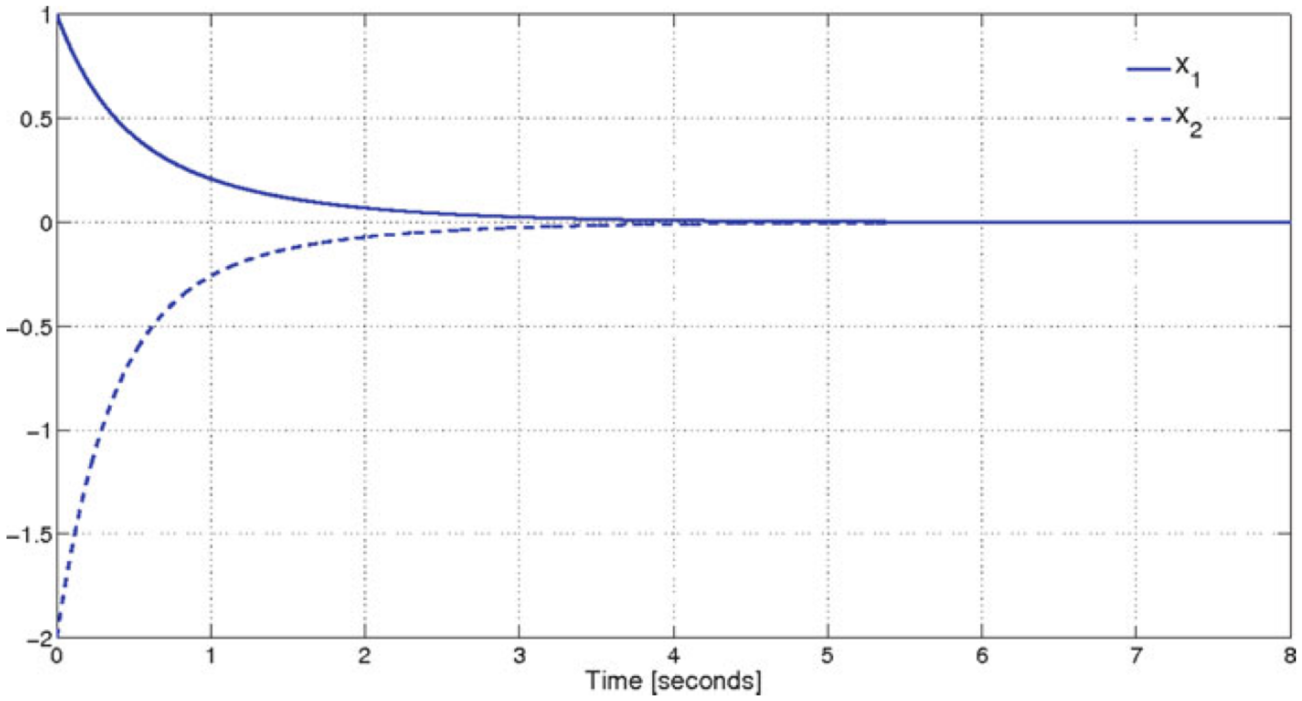


Fig. 1.2 Asymptotic convergence for $f(x_1, x_2, t) \equiv 0$

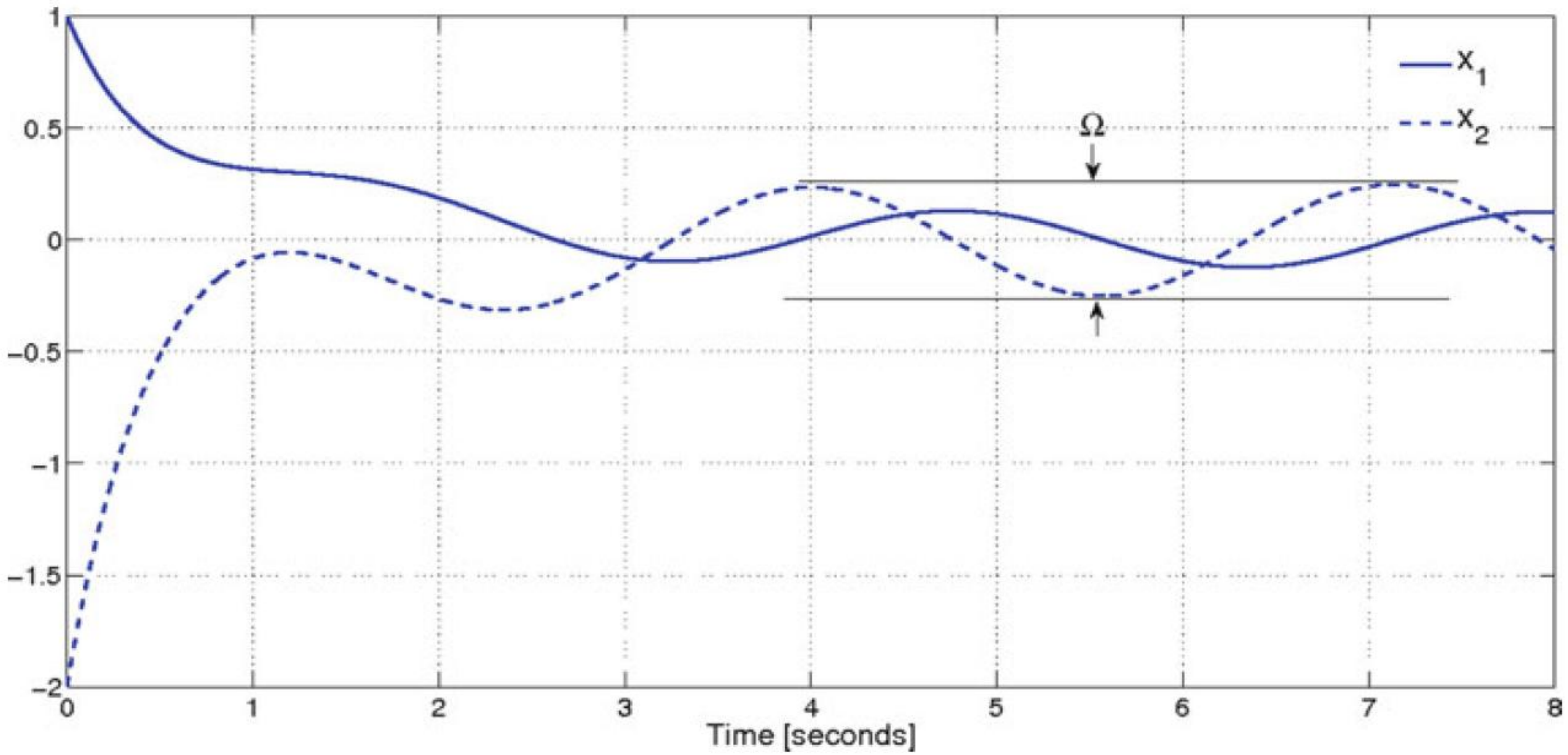


Fig. 1.3 Convergence to the domain Ω for $f(x_1, x_2, t) = \sin(2t)$

Main Concepts of Sliding Mode Control

Let us introduce desired compensated dynamics for system (1.1). A good candidate for these dynamics is the homogeneous linear time-invariant differential equation:

$$\dot{x}_1 + c x_1 = 0, \quad c > 0 \qquad (1.3)$$

Since: $x_2(t) = \dot{x}_1(t)$

a general solution of above equation and its derivative is given by:

$$\begin{aligned} x_1(t) &= x_1(0)\exp(-ct) \\ x_2(t) &= \dot{x}_1(t) = -c x_1(0)\exp(-ct) \end{aligned} \qquad (1.4)$$

both x_1(t) and x_2(t) converge to zero asymptotically. Note, no effect of the disturbance f (x_1, x_2,t) on the state compensated dynamics is observed.

How could these compensated dynamics be achieved? First, we introduce a new variable in the state space of the system in Eq. (1.1):

$$\sigma = \sigma(x_1, x_2) = x_2 + cx_1, \quad c > 0 \qquad (1.5)$$

In order to achieve asymptotic convergence of the state variables x_1, x_2 to zero, with a given convergence rate as in Eq. (1.4), in the presence of the bounded disturbance $f(x_1, x_2, t)$, we have to drive the variable σ in Eq. (1.5) to zero in finite time by **means of the control u**. This task can be achieved by applying Lyapunov function techniques to the σ dynamics that are derived using Eqs. (1.1) and (1.5):

$$\dot{\sigma} = cx_2 + f(x_1, x_2, t) + u, \quad \sigma(0) = \sigma_0 \qquad (1.6)$$

For the σ dynamics (1.6) a candidate Lyapunov function is introduced taking the form:

$$V = \frac{1}{2}\sigma^2 \qquad (1.7)$$

In order to provide the asymptotic stability of Eq. (1.6) about the equilibrium point σ=0, the following conditions must be satisfied:

(a) $\dot{V} < 0$ for $\sigma \neq 0$
(b) $\lim_{|\sigma|\to\infty} V = \infty$

From above two condition, a control law u that drives σ to zero in finite time:

$$u = -cx_2 - \rho\,\text{sign}(\sigma)$$

Basics of Sliding Mode Control Systems

The two-step procedure for SMC design is described as follows:

(i) A sliding surface is predefined in a way that desired system dynamics are achieved during sliding mode.

(ii) A controller is then designed to drive the closed-loop dynamics to reach and be retained on the sliding surface.

System Model and Sliding Mode Surface Design

Without loss of generality, we consider the following linear time-invariant (LTI) system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.7}$$

where $x \in R^n$ is the system state vector, $u \in R^m$ is the control input vector, $A \in R^{n\times n}$ and $B \in R^{n\times m}$ are constant system matrices.

It is assumed that n > m , B is of full rank m , the pair (A ,B) is completely controllable, that is, the controllability matrix $[B\ AB\ A^2B \ldots A^{n-1}B]$ has full rank m.

Define a sliding variable vector $s(t) \in R^m$ passing through the state space origin

$$s(t) = Cx(t) \quad (2.8)$$

where $C \in R^{m \times n}$ is the sliding mode parameter vector and $\|CB\| \neq 0$.

The system (2.7) is said to attain a sliding mode surface when the state variable vector reaches and remains on the intersection of the m switching plane variables.

The method of equivalent control is a way to determine the system motion restricted to the sliding mode surface $s(x) = 0$. On the sliding mode surface, $s(x) = 0$ and $\dot{s}(x) = 0$, using expressions (2.7) and (2.8), we have

$$\dot{s}(t) = C\dot{x}(t) = 0 \quad (2.9)$$

$$C\left(Ax(t) + Bu_{eq}(t)\right) = 0 \quad (2.10)$$

where $u_{eq}(t)$ is viewed as equivalent control.

From expression (2.10), the equivalent control can be expressed as

$$u_{eq}(t) = -(CB)^{-1}CAx(t) \qquad (2.11)$$

Substituting (2.11) into (2.7) yields the following differential equation

$$\dot{x}(t) = [I - B(CB)^{-1}C]Ax(t) \qquad (2.12)$$

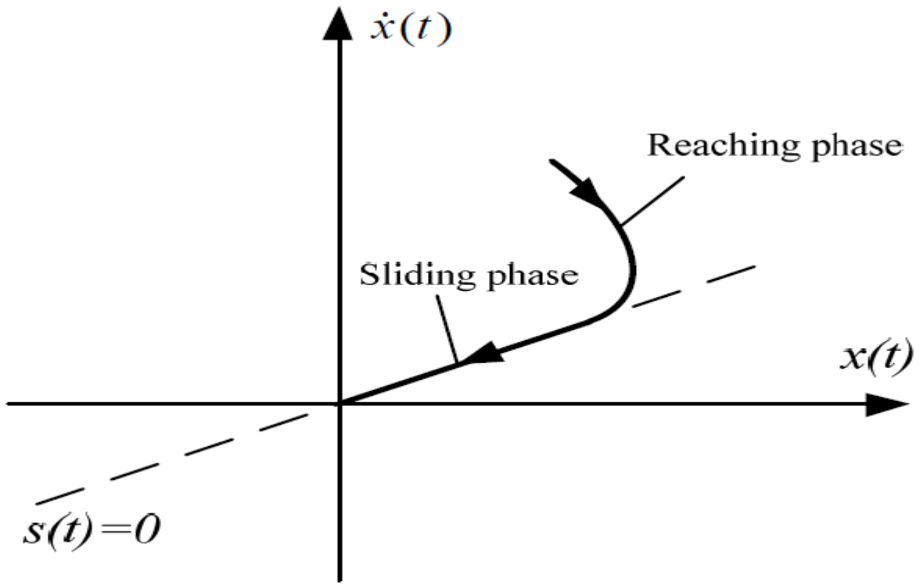
The system (2.12) is called the equivalent system which describes the dynamic motion of the system (2.7) on the sliding mode surface. The characteristics of the equivalent system can be summarised as below:

1. The dynamical behavior of the equivalent system is independent of the control input. Thus, the determination of the matrix C may be completed without prior knowledge of the form of control input. **Generally, the sliding parameter C is designed in a manner that the system response confined on the sliding mode surface (2.12) has a desired behavior such as asymptotic stability and prescribed transient response**. According to the linear control theory, in order to guarantee the solution of the differential equation in (2.12) to **be asymptotically stable, the sliding mode parameter vector C should be chosen, such that all the eigenvalues of the differential equation (2.12) have negative real parts**. What's more, though the sliding surface (2.8) is linear, it indeed could be any other forms with nonlinearity to ensure a finite time convergence of system dynamics in sliding mode.

2. The equivalent system (2.12) is an $(n - m)^{th}$ order system, i.e., the system dynamic is simplified on the sliding mode surface.

Reaching Phase

SMC design includes **Reaching phase and Sliding phase**. The reaching phase is crucial in the sense that the system dynamics are guaranteed to reach the sliding surface and be retained on it thereafter. For a case in point, the idea of a sliding mode of **a second order system** can be depicted in Figure.



The next important problem is how to design a controller to guarantee the reachability of the sliding variable to the sliding mode surface. Therefore, the task of the sliding mode controller **is to drive the sliding variable S to converge to zero**, and then the desired system dynamics prescribed in (2.12) will be obtained.

Reaching Condition

In fact, the condition for the **switching plane variables to reach the sliding mode surface** is a **convergence problem**. Therefore, the Lyapunov's direct method has been widely used in SMC designs as a stability condition to ensure the convergence of the sliding mode variable onto the sliding surface during the reaching phase. All too often, the following Lyapunov function candidate is used in the sliding mode controller design:

$$V(t) = \frac{1}{2} s^T(t) s(t) \qquad (2.13)$$

In order to guarantee the asymptotic stability of the system (2.7) about the equilibrium point x(t) = 0, the following reaching condition must be satisfied:

$$\dot{V}(t) = s^T(t)\dot{s}(t) < 0 \text{ for } s(t) \neq 0 \qquad (2.14)$$

Remark: The condition (2.14) indeed acts **as a sufficient condition to ensure the existence of the sliding mode**. It is worth noting that most of the sliding mode controllers are designed based on the **reachability condition** in (2.14) to ensure the sliding mode controller can drive the sliding variable s(t) to asymptotically converge to zero.

Reaching Laws

SMC can be designed based on reaching laws to guarantee the existence of the sliding mode. In general, reaching law can be generalized in the following form:

$$\dot{s} = -\varepsilon \operatorname{sign}(s) - f(s), \quad \varepsilon > 0 \qquad (2.15)$$

where $f(0) = 0$ and $sf(s) > 0$ when $s \neq 0$.

In practice, three special reaching laws commonly used can be derived from (2.15) as follows:

Constant rate reaching law:

$$\dot{s} = -\varepsilon \operatorname{sign}(s), \quad \varepsilon > 0 \qquad (2.16)$$

This law constrains the **switching variable** to reach the switching manifold at a constant rate ε. The merit of this reaching law is its simplicity. However, as ε is too small, the reaching time will be too long. On the other hand, too large ε will cause severe chattering.

Exponential rate reaching law

$$\dot{s} = -\varepsilon \mathrm{sign}(s) - ks, \quad \varepsilon > 0, k > 0 \qquad (2.17)$$

By adding the proportional rate term -ks , the states are forced to approach the switching manifold faster when s is large.

Power rate reaching law

$$\dot{s} = -k|s|^{\alpha}\mathrm{sign}(s), \quad 1 > \alpha > 0, \quad k > 0 \qquad (2.18)$$

This reaching law increases the reaching speed when the states are far away from the switching manifold. However, it reduces the rate when the states approach the manifold.

Equivalent Controller Design

The control input usually consists of two components as follows:

$$u(t) = u_{eq}(t) + u_s(t) \qquad (2.19)$$

where the linear component $u_{eq}(t)$ is defined as in (2.11) and the nonlinear signal incorporates the discontinuous component given below:

$$u_s(t) = -\eta(CB)^{-1}\text{sign}\big(s(t)\big) \qquad (2.20)$$

where $\eta>0$ is a constant control gain.

Substituting (2.11) and (2.20) into (2.14) leads to:

$$\dot{V}(t) = s^T(t)CAx(t) + s^T(t)CBu(t)$$
$$= -\eta|s(t)| < 0 \qquad (2.21)$$

From (2.21), we can conclude that the sliding mode variable is guaranteed to reach the sliding mode surface in finite time.

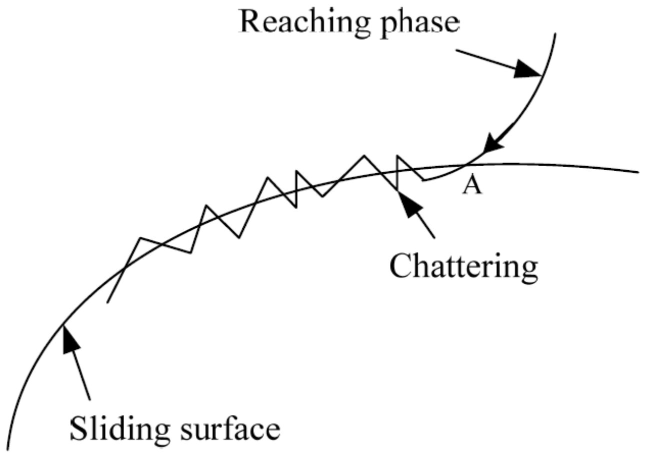
Remark: After the sliding variable vector s(t) is driven to zero, the closed-loop system dynamics are only determined by the desired dynamics in (2.12) and thus, the closed-loop system is insensitive to system uncertainties on the sliding mode surface. For this reason, SMC systems possess the property of robustness with respect to system uncertainties, that SMC becomes a powerful tool in the control of uncertain systems and significantly motivates the subsequent researchers in the area. However, it should be noted that the system remains affected by the perturbations during the reaching phase, that is to say, before the sliding surface has been reached.

Chattering Phenomenon

Zig-Zag Motion

An ideal sliding mode does not exist in practice since it would imply that the control commutes at an infinite frequency. As imperfections in switching devices, SMC suffers from

chattering, the discontinuity in the feedback control produces a particular dynamic behavior in the vicinity of the sliding mode surface as shown in Figure.



In Figure 2.2, the system trajectory in the region s(t) > 0 heading toward the sliding surface s(t) = 0. It first hits the surface at point A. In ideal SMC the trajectory should start sliding on the surface from point A. However, due to a delay between the time the sign of s(t) changes and the time the control switches, the trajectory reverses its direction and heads again toward the surface. The repetition of this process creates the “zig-zag motion” which oscillating around the predefined sliding surface.

The chattering results in low control accuracy, high heat losses in electric power circuits and high wear of moving mechanical parts. It may excite unmodeled high- frequency dynamics, which degrades the performance of the system and may even lead to instability.

Boundary Layer Technique

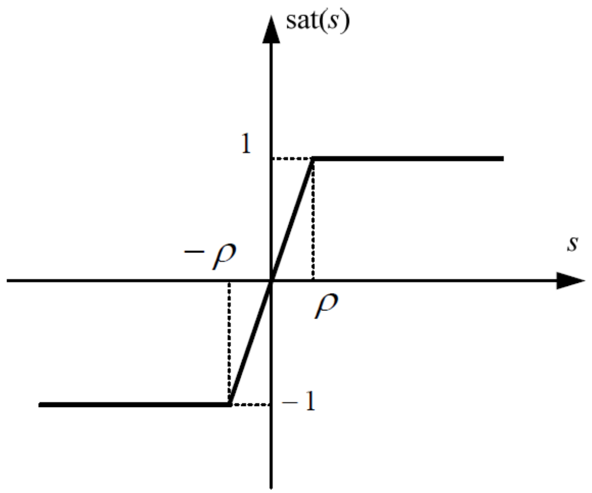
Various techniques have been proposed to reduce or eliminate the chattering. The boundary layer technique is one of the common approaches to eliminate the chattering. It is seen that the discontinuous or switched component of SMC controller is designed as in (2.20). The boundary layer technique can be used to eliminate the chattering by replacing the sign function in (2.20) with a saturation function shown in Figure 2.3 as follows:

$$u_s(t) = -\eta(CB)^{-1}\text{sat}(s(t)) \qquad (2.26)$$

where $\text{sat}(s)$ is the saturation function defined by

$$\text{sat}(s) = \begin{cases} \dfrac{s}{\rho}, & \text{for } |s| \le \rho \\ \text{sign}(s), & \text{for } |s| > \rho \end{cases} \qquad (2.27)$$

and a positive constant $\rho > 0$ should be chosen in simulation or experiment to guarantee that the chattering can be eliminated and a reasonable control performance can be obtained.



Conventional Sliding Mode Controller Design

Consider the nonlinear SISO system:

$$\dot{x} = f(x,t) + g(x,t)u \qquad (1)$$

$$y = h(x,t) \qquad (2)$$

where y and u denote the scalar output and input variable, $x \in R^n$ denotes the state vector.

The control aim is to make the output variable y to track a desired profile y_{DES}, that is, it is required that the output error variable $e = y - y_{DES}$ tends to some small vicinity of zero after a transient of acceptable duration.

The first phase is the definition of a **certain scalar function of the system state**, says:

$\sigma(x): R^n \rightarrow R$

Often, the sliding surface depends on the tracking error e_y together with a certain number of its derivatives:

$$\sigma = \sigma\left(e, \dot{e}, \ldots, e^{(k)}\right) \quad (3)$$

The function σ should be selected in such a way that its vanishing, $\sigma = 0$, gives rise to a "stable" differential equation any solution $e_y(t)$ of which will tend to zero eventually.

The most typical choice for the sliding manifold is a linear combination of the following type:

$$\sigma = \dot{e} + c_0 e \quad (4)$$

$$\sigma = \ddot{e} + c_1 \dot{e} + c_0 e \quad (5)$$

$$\sigma = e^{(k)} + \sum_{i=0}^{k-1} c_i e^{(i)} \quad (6)$$

The number of derivatives to be included (the “k” coefficient in (6)) should be k=r-1, where r is the input output relative degree of (1)-(2).

With properly selected c_i coefficients, if one steers to zero the variable, the exponential vanishing of the error and its derivatives is obtained.

If such property holds, then the control task is to provide for the finite time zeroing of σ ,“forgetting” any other aspects.

From a geometrical point of view, **the equation σ = 0 defines a surface in the error space**, **“sliding surface”.** The trajectories of the controlled system are forced onto the sliding surface, along which the system behavior meets the design specifications.

A typical form for the sliding surface is the following, which depends on just a single scalar parameter, p.

$$\sigma = \left(\frac{d}{dt} + p \right)^{k} e \qquad (7)$$

$$k=1 \qquad \sigma = \dot{e} + pe \qquad (8)$$

$$k=2 \qquad \sigma = \ddot{e} + 2p\dot{e} + p^{2}e \qquad (9)$$

The choice of the positive parameter p is almost arbitrary, and define the unique pole of the resulting "reduced dynamics" of the system when in sliding.

The integer parameter k is on the contrary rather critical, it must be equal to r-1, with r being the relative degree between y and u. This means that the relative degree of the σ variable is one.

The successive phase (PHASE 2) is finding a control action that steers the system trajectories onto the sliding manifold, that is, in other words, the control is able to steer the σ variable to zero in finite time.

There are several approaches based on the sliding mode control approach:

- standard (or first-order) sliding mode control.
- high-order sliding mode control.

Common feature of all sliding mode based techniques is that no precise information about the original system dynamics is requested, the controlled system being treated as a completely uncertain black object.

First Order Sliding Mode Control

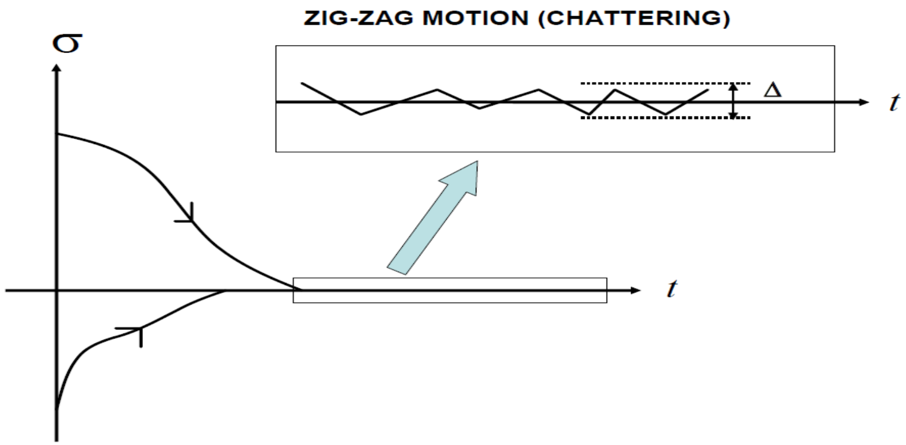
The control is discontinuous across the manifold $\sigma = 0$.

$$u = -U\,\mathrm{sgn}(\sigma)$$

that is

$$u = \begin{cases} -U & \sigma > 0 \\ U & \sigma < 0 \end{cases}$$

U is a sufficiently large positive constant.



Typical evolution of the σ variable starting from different intial conditions

In steady state the control variable u will commute at very high (theoretically infinite) frequency between the values u = U and u = −U(see Fig. 2).

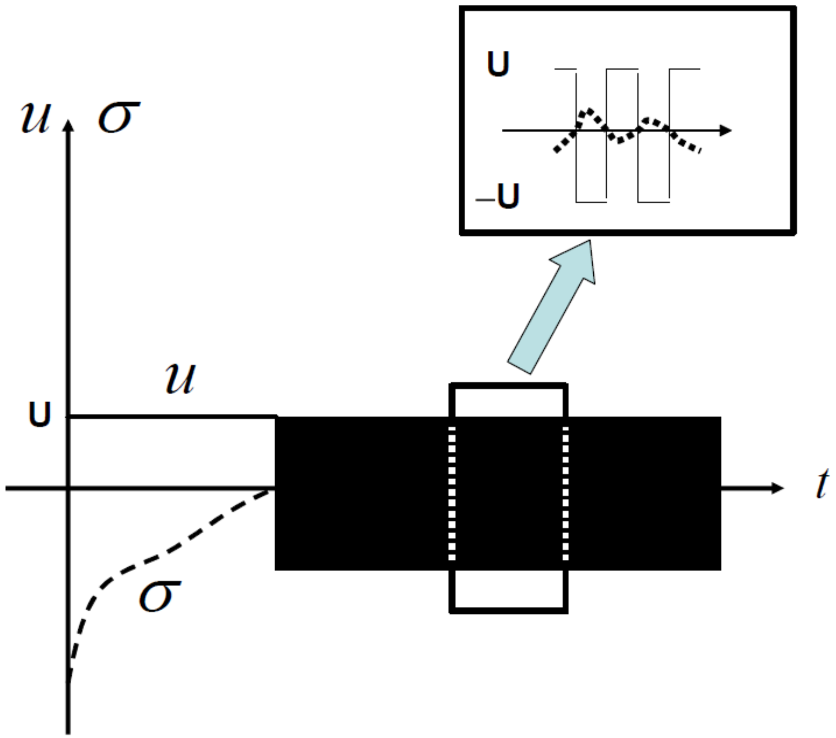


Fig. 2 Typical evolution of the control signal u (the dashed line represents σ)

The discontinuous high frequency switching control (Figure 2) is appropriate in applications (where PWM control signals are normally employed) but gives rise to oscillations and many problems in different areas like, e.g., the control of mechanical systems.

In order to solve the above problem **(referred to as chattering phenomenon)** approximate (smoothed) implementations of sliding mode control techniques have been suggested where the discontinuous term is replaced by a continuous smooth approximation. Two examples follow:

SAT $$u = -U\,\text{sat}(\sigma;\varepsilon) \equiv -U\frac{\sigma}{|\sigma| + \varepsilon} \qquad \varepsilon > 0 \qquad \varepsilon \approx 0 \qquad (12)$$

TANH $$u = -U\,\tanh(\sigma/\varepsilon) \qquad \varepsilon > 0 \qquad \varepsilon \approx 0 \qquad (13)$$

Unfortunately this approach is effective only in specific case, the is when hard uncertainties are not present and the control action that counteract them can be set to zero in the sliding mode.

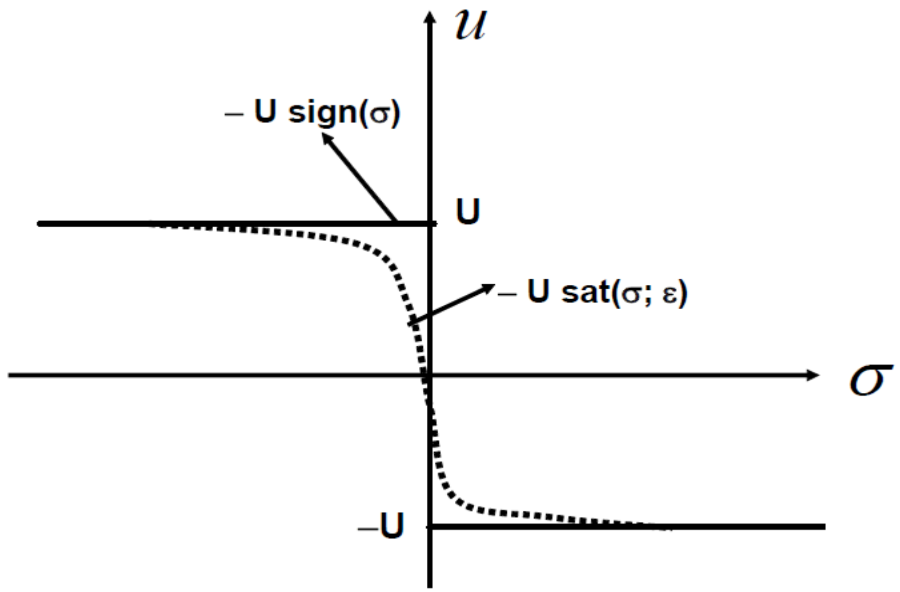


Fig. 3 Smooth approximations of sliding mode control

Second Order Sliding Mode Control

Using the above described smooth approximations, some problems are attenuated, at the price of a loss of robustness.

Second order sliding mode control algorithms are a powerful alternative that completely solves the chattering issue without compromising the robustness properties as well.

Super Twisting Algorithm

$$u = -\lambda\sqrt{|\sigma|}\ \mathrm{sgn}(\sigma) + w \qquad (14)$$

$$\dot{w} = -W\ \mathrm{sgn}(\sigma) \qquad (15)$$

A suitable way for tuning its parameters is the following pair of relationships

Super Twisting Algorithm

Under steady-state conditions, with an adequate control law, the sliding mode control forces the state of the system to meet the above conditions, i.e.,

$S = 0$, $S = 0$.

Usually, SMC law needs the measurement of the time derivative of the sliding function and provides a discontinuous form, which is sometimes inconvenient because of the chattering phenomenon.

$$\lambda = \sqrt{U} \qquad W = 1.1U \qquad (16)$$

where U is a positive constant to be taken sufficiently large. In practice, one has to progressively increase U until good performances are seen in the closed loop system. This kind of single-parameter “trial and error” tuning is particularly suited in practical implementation.

The super-twisting algorithm can be seen as a nonlinear version of the classical PI controller. This analogy is clearer by referring to the next Figure 4.

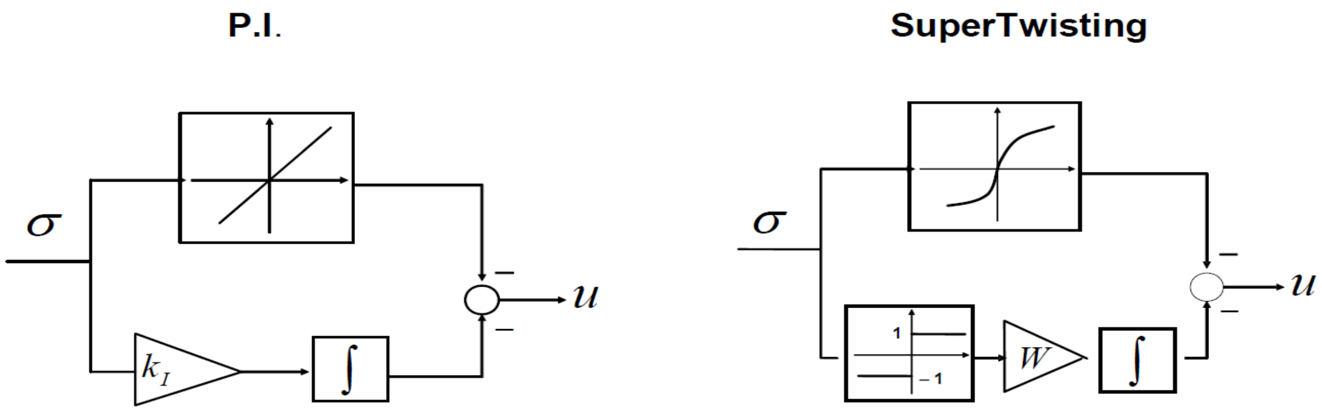


Fig 4. *Bloch scheme of PI (left) and Super-Twisting (right) controllers*

Example:

To investigate the main aspects of SMC design, let us consider a simple motion control problem, namely the position control for an uncertain mass-spring-damper subject to an uncertain time varying disturbance d(t).

$$M\ddot{x} + B\dot{x} + Kx = F + d(t)$$

Parameter values

M=2kg B=$5N/ms^2$ K=2 N/m

External disturbance

d(t)=2 + 2 sin (3t) + sin (5t)

Since the structure of the disturbance d(t) is unknown, no linear controller can completely reject it unlike in very special cases (e.g., d(t)=cost.).

Let us define the output as y = x. The desired position profile is y_{DES}=5 sin(2t).

Define the tracking error as $e = y - y_{DES}$

PHASE 1. Sliding surface design

The relative degree between the output y(t) and the input F(t) is r=2. Thus, according to (8) (in this case k=r-1=1) define the sliding surface σ as follows:

$$\sigma = \dot{e} + pe = \dot{y} - \dot{y}_{DES} + p \cdot (y - y_{DES})$$

Let p = 1

$$\sigma = \dot{e} + e$$

PHASE 2. Control input design

Let us apply the three different suggested alternatives:

- First order SMC

$$F = -F^{*} sign(\sigma)$$

- First order “smoothed SMC:

$$F = -F^* sat(\sigma; \varepsilon)$$

- Super twisting 2-SMC

$$F = F_1 + F_2$$

$$F_1 = -\sqrt{F^*}\sqrt{|\sigma|}\,\mathrm{sgn}(\sigma)$$

$$\dot{F}_2 = -1.1F^*\,\mathrm{sgn}(\sigma)$$

F*=10 F*=20

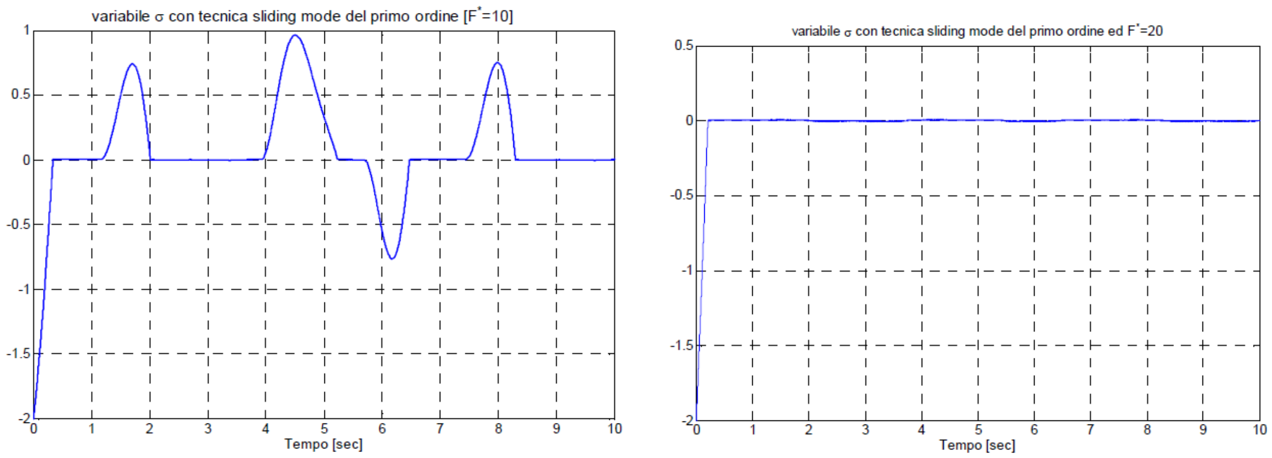


Fig. 5 The sliding variable σ with first order SMC Left: $F^*=10$. Right $F^*=20$.

In figure 5-left the control authority (i.e. the F* parameter) is too low, and, as a result, the sliding variable σ sometimes escapes from zero. In figure 5-right it has been increased enough to achieve good precision in keeping σ to zero.

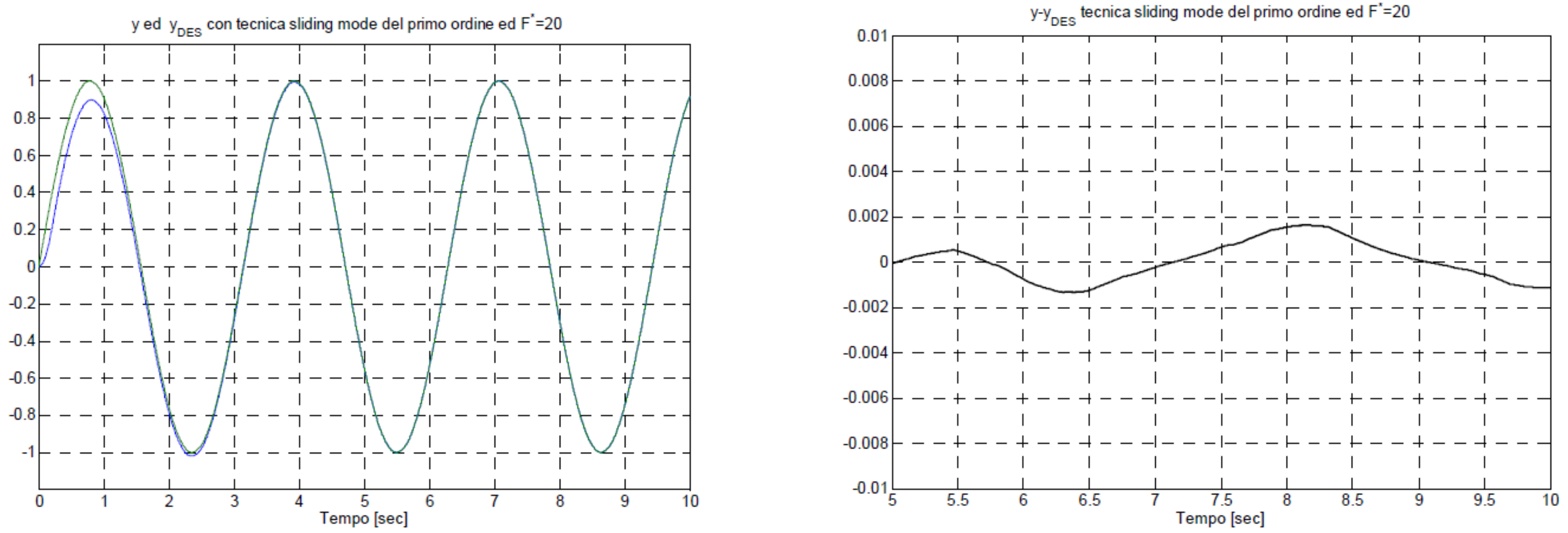


Fig. 6 First order SMC with F*=20. Left plot: y and y_{DES}. Right plot: e:= $y - y_{DES}$

The control input is depicted in the next plot. It is apparent the discontinuous high frequency nature of the control input. This behaviour is unacceptable for a physical signal like a mechanical force.

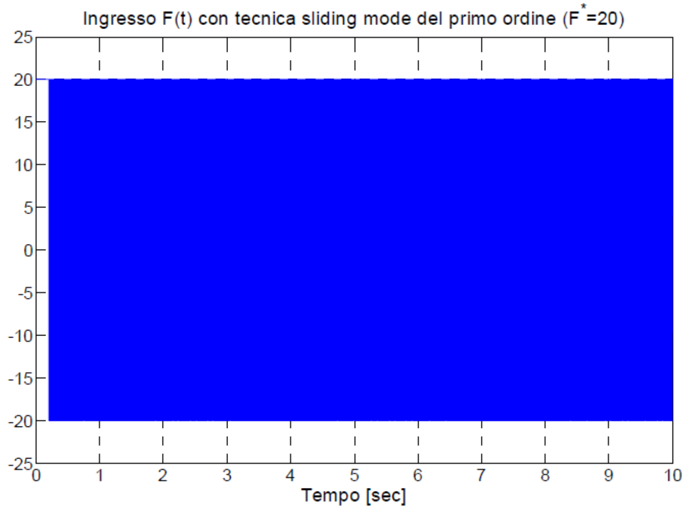


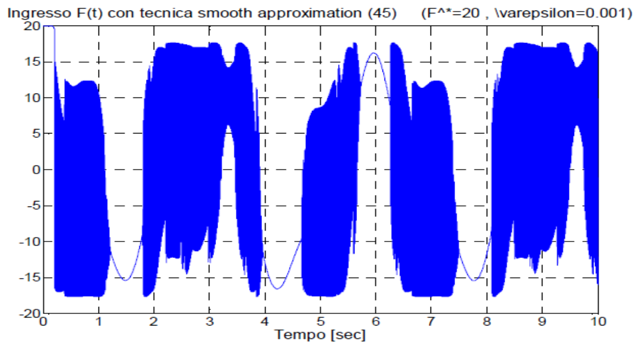
Fig. 7 First order SMC with F*=20. The control input F(t)

Smoothed first order SMC

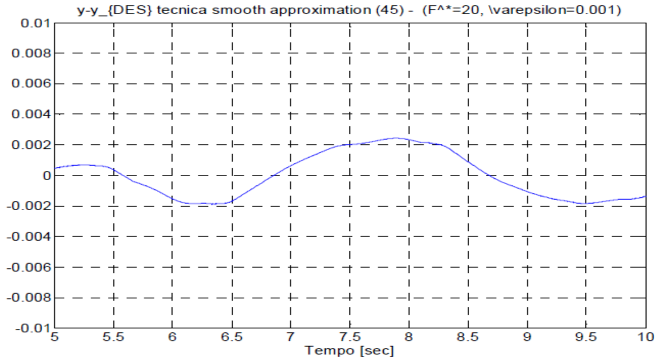
$$F = -F^{*}sat(\sigma;\,\varepsilon)$$

F* = 20 ed ε=0.001

Control input

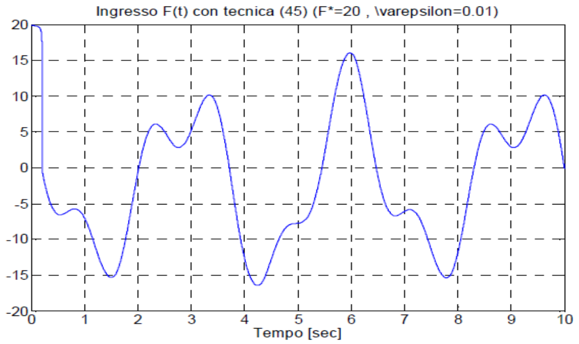


Tracking error

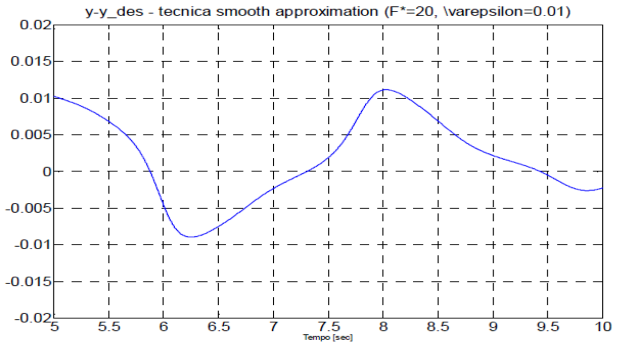


F* = 20 ed ε=0.01

Control input



Tracking error



The above two tests show that with small ε (ε=0.001) the smoothing effect on the control input is limited, but the control accuracy is retained, while with larger ε (ε=0.01) the smoothing effect is remarkable but there is a loss of accuracy. Therefore a good compromise must be found. This technique proved to be very effective and is of widespread use in many SMC implementation.

"Super-twisting" 2-SMC

The second order sliding mode control approach solves the chattering issue improving the control accuracy at the same time.

$$F = F_1 + F_2$$
$$F_1 = -\sqrt{F^*}\sqrt{|\sigma|}\,\mathrm{sgn}(\sigma)$$
$$\dot{F}_2 = -1.1F^*\,\mathrm{sgn}(\sigma)$$

The tuning parameter is set to F*=50

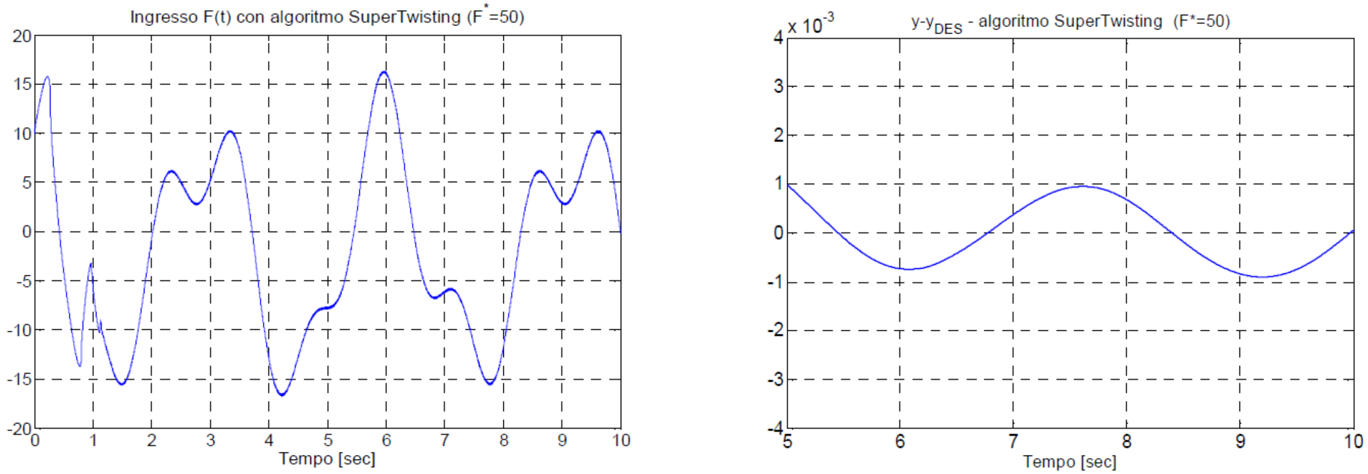


Fig. 8 *SuperTwisting with F^{*}=50 .Left: the control input. Right: the tracking error.*

It can be noted the high accuracy and simplicity of implementation of this class of techniques that, on the basis of practically no information about the plant dynamics, allows a very precise control.