Higher Order Differential Equations

Linear differential equation of order n in standard form

$$y^{(n)} + P_1(x)y^{(n-1)} + P_2(x)y^{(n-2)} + P_3(x)y^{(n-3)} + \cdots P_{n-1}(x)y' + P_n(x)y = R(x)$$

Linear differential equation of order n with constant coefficients

$$a_0y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + a_3y^{(n-3)} + \cdots a_{n-1}y' + a_ny = R(x)$$

Where

$$a_0, a_1, a_2, \dots \dots . a_n = constant$$

General Solution—Homogeneous Equations

Let $y_1, y_2, \dots y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation

$$a_0y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + a_3y^{(n-3)} + \cdots a_{n-1}y' + a_ny = 0$$

Then the general solution of the equation on the interval is

$$y = c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + \cdots . c_ny_n$$

General Solution—Nonhomogeneous Equations

the general solution of the equation on the interval is

$$y = y_c + y_p$$

$y_c = the\ solution\ of\ homogenouse\ DE(coplementary\ solution)$

$y_p = particular\ solution$

Example 1: Solve the $y''' - 6y'' + 11y' - 6y = 0$

Solution

$$y''' - 6y'' + 11y' - 6y = 0$$

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$m^2 - 5m + 6$$
$$m - 1 \overline{)\, m^3 - 6m^2 + 11m - 6}$$
$$m^3 - m^2$$
$$-5m^2 + 11m - 6$$
$$-5m^2 + 5m$$
$$6m - 6$$
$$6m - 6$$
$$0$$

By trial and error, we get $m_1 = 1$

We will divide the characteristics by (m-1) to find the other roots

hence

$m^3 - 6m^2 + 11m - 6 = (m-1)(m^2 - 5m + 6) = 0$

$m_1 = 1$
$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m_2 = 2$
$m_3 = 3$

$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

Example 2: Solve the $y''' - 6y'' + 11y' - 6y = 2e^x$

Solution

Complementary solution

$$y''' - 6y'' + 11y' - 6y = 0$$

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$m^3 - 6m^2 + 2m + 3 = (m-1)(m^2 - 5m + 6) = 0$$

$m_1 = 1$

$m_2 = 2$

$m_3 = 3$

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

particular solution

$$y_p = Axe^x$$

$$y'_p = Ae^x + Axe^x$$

$$y''_p = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$$

$$y'''_p = 2Ae^x + Ae^x + Axe^x = 3Ae^x + Axe^x$$

Substitute in the main equation

$$y''' - 6y'' + 11y' - 6y = 2e^x$$

$$(3Ae^x + Axe^x) - 6(2Ae^x + Axe^x) + 11(Ae^x + Axe^x) - 6(Axe^x) = 2e^x$$

$$3Ae^x - 12Ae^x + 11Ae^x = 2e^x$$

$$2Ae^x = 2e^x$$

$$A = 1$$

$$y_p = xe^x$$

$$y = y_c + y_p$$

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} + xe^x$$

Example 3: Solve the $y''' + y = 1$

Solution

Complementary solution

$$y''' + y = 0$$

$$m^3 + 1 = 0$$

$$m^3 + 1 = (m + 1)(m^2 - m + 1) = 0$$

$$m_1 = -1$$

Or

$$(m^2 - m + 1) = 0$$

$$m_1, m_2 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 1}$$

$$m_1, m_2 = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

$$y_c = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

particular solution

$$y_p = A$$

$$y'_p = 0$$

$$y''_p = 0$$

$$y'''_p = 0$$

Substitute in the main equation

$$y''' + y = 1$$

$$0 + A = 1 \qquad A = 1$$

$$y_p = 1$$

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + 1$$

The Euler-Cauchy differential equation

A linear differential equation of the form,

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + a_2 x^{n-2} y^{(n-2)} + a_{n-1} x y' + a_n y = 0$$

with $a_0, a_1, \dots a_n$ constants is called the homogeneous Euler-Cauchy equation of order n.

to solve this differential equation, we assume that

$$z = \ln|x|$$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{1}{x} = \frac{1}{x}.\frac{dy}{dz} \quad \dots\dots 1$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}.\frac{dz}{dx}.\frac{1}{x} + \frac{dy}{dz}\left(-\frac{1}{x^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}.\frac{1}{x}.\frac{1}{x} + \frac{dy}{dz}(-\frac{1}{x^2})$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}.\frac{1}{x^2} - \frac{dy}{dz}.\frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \ldots \ldots .2$$

For the second order homogeneous Euler-Cauchy differential equation

$$ax^2y'' + bxy' + cy = 0$$

Let

$$z = \ln|x|$$

$$ax^2\left(\frac{1}{x^2}(\frac{d^2y}{dz^2} - \frac{dy}{dz})\right) + bx\left(\frac{1}{x}.\frac{dy}{dz}\right) + c.y = 0$$

$$a\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + b\frac{dy}{dz} + c.y = 0$$

$$a(m^2 - m) + b(m) + c = 0$$

$$a.m(m-1) + b(m) + c = 0$$

So, the following relationships are used with the Euler-Cauchy differential equation

$$x^0y = 1$$

$$xy' = m$$

$$x^2y'' = m(m-1)$$

$$x^3y''' = m(m-1)(m-2)$$

$$x^4y'''' = m(m-1)(m-2)(m-3)$$

And so on

Example 1: Solve the $x^2y'' + xy' - y = x$

Complementary solution

$$x^2y'' + xy' - y = 0$$

Let

$$z = \ln|x|$$

$$m(m-1) + m - 1 = 0$$

$$m^2 - 1 = 0 \qquad m = \pm 1$$

$$y_c = c_1e^z + c_2e^{-z}$$

$$y_c = c_1e^{\ln x} + c_2e^{-\ln x}$$

$$y_c = c_1x + \frac{c_2}{x}$$

particular solution

$$x^2y'' + xy' - y = x$$

$$\frac{d^2y}{dz^2} - y = e^z$$

$$y_p = Aze^z$$

$$y_p' = Ae^z + Aze^z$$

$$y''_p = 2Ae^z + Aze^z$$

$$\frac{d^2y}{dz^2} - y = e^z$$

$$2Ae^z + Aze^z - Aze^z = e^z$$

$$2A = 1$$

$$A = \frac{1}{2}$$

$$y_p = Aze^z$$

$$y_p = \frac{1}{2}ze^z$$

$$y_p = \frac{1}{2}\ln|x|\, e^{\ln|x|}$$

$$y_p = \frac{x}{2}\ln|x|$$

$$y = y_c + y_p$$

$$y = c_1 x + \frac{c_2}{x} + \frac{x}{2}\ln|x|$$

Example 2: Solve the $x^3y''' + xy' - y = x^4$

$$z = \ln|x|$$

Complementary solution

$$x^3y''' + xy' - y = 0$$

$$m(m-1)(m-2) + m - 1 = 0$$

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0$$

$$m_1, m_2, m_3 = 1$$

$$y_c = c_1e^z + c_2ze^z + c_3z^2e^z$$

$$y_c = c_1e^{\ln x} + c_2\ln x\, e^{\ln x} + c_3(\ln x)^2e^{\ln x}$$

$$y_c = c_1x + c_2x\ln x + c_3x(\ln x)^2$$

particular solution

$$\frac{d^3y}{dz^3} - \frac{3d^2y}{dz^2} + \frac{3dy}{dz} - y = (e^z)^4$$

$$\frac{d^3y}{dz^3} - \frac{3d^2y}{dz^2} + \frac{3dy}{dz} - y = e^{4z}$$

$$y_p = Ae^{4z}$$

$$y'_p = 4Ae^{4z}$$

$$y''_p = 16Ae^{4z}$$

$$y'''_p = 64Ae^{4z}$$

Substitute in

$$\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 3\frac{dy}{dz} - y = e^{4z}$$

$$64Ae^{4z} - 3(16Ae^{4z}) + 3(4Ae^{4z}) - Ae^{4z} = e^{4z}$$

$$27Ae^{4z} = e^{4z} \qquad \rightarrow \qquad A = \frac{1}{27}$$

$$y_p = Ae^{4z}$$

$$y_p = \frac{1}{27}e^{4z} = \frac{1}{27}e^{4\ln x} = \frac{x^4}{27}$$

$$y = y_c + y_p$$

$$y = c_1x + c_2x\ln x + c_3x(\ln x)^2 + \frac{x^4}{27}$$

Applications of Second-Order Differential Equations

We consider the motion of an object with mass at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level surface (as in Figure 2). In Section 6.5 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) units from its natural length, then it exerts a force that is proportional to:

restoring force =- kx

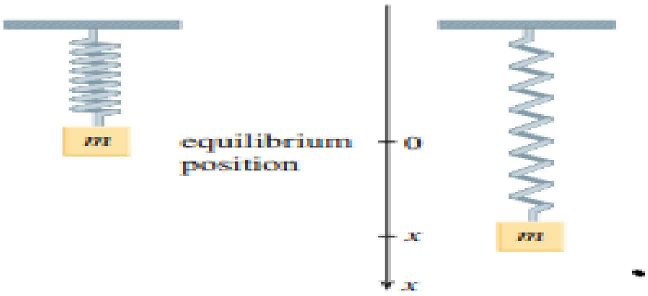


FIGURE 1

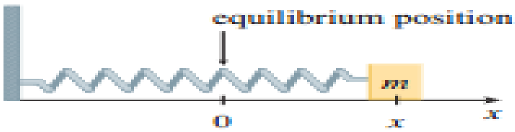


FIGURE 2

where $\boldsymbol{k}$ is a positive constant (called the spring constant). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$m\frac{d^2y}{dt^2} = -ky$$

$$m\frac{d^2y}{dt^2} + ky = 0$$

This is a second-order linear differential equation. Its auxiliary equation is

$$mr^2 + k = 0$$

$$r^2 + \frac{k}{m} = 0$$

$$r^2 + \omega_0^2 = 0$$

$$r = \pm\omega_0 i$$

Where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

here ω_0 is angular frequency of vibration

Thus, the general solution is

$$y = c_1 \cos\ \omega_0 t + c_2 \sin\ \omega_0 t$$

which can also be written as

$$y = A\cos\left(\omega_0 t - \phi\right)$$

or

$$y = A\sin\left(\omega_0 t + \phi\right)$$

Where

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{A=amplitude}$$

$$\arctan \phi = \frac{c_1}{c_2}$$

EXAMPLE 1 A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t.

SOLUTION From Hooke's Law, the force required to stretch the spring is

$$k(0.2) = 25.6$$

so $k = 25.6/0.2 = 128$. Using this value of the spring constant k, together with $m = 2$ in Equation 1, we have

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$2\frac{d^2y}{dt^2} + 128y = 0$$

$$\frac{d^2y}{dt^2} + 64y = 0$$

$$r^2 + 64 = 0$$

$$r = \pm 8i$$

$$y = c_1 \cos 8t + c_2 \sin 8t$$

$at\ t = 0 \quad y = 0.2\ m\ and\ v = y' = 0$

$$0.2 = c_1 \cos 8(0) + c_2 \sin 8(0)$$

$$c_1 = 0.2$$

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t$$

$$0 = -8c_1 \sin 8(0) + 8c_2 \cos 8(0)$$

$$c_2 = 0$$

$$y = 0.2 \cos 8t$$

Angular frequency $\omega_0 = 8$ rad/sec

$$Natural\ frequency = \frac{\omega_0}{2\pi} \quad (cycle/\sec = Hz)$$

$$Natural\ frequency = \frac{8}{2\pi} = 1.27\ Hz$$

Damped Vibrations

We next consider the motion of a spring that is subject to a frictional force (in the case of the horizontal spring of Figure 2) or a damping force (in the case where a vertical spring moves through a fluid as in Figure 3). An example is the damping force supplied by a shock absorber in a car or a bicycle.

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. (This has been confirmed, at least approximately, by some physical experiments.) Thus

$$damping\ force = -c\ \frac{dy}{dy}$$

where c is a positive constant, called the **damping constant**. Thus, in this case, Newton's Second Law gives

$$m\frac{d^2y}{dt^2} = restoring\ force + damping\ force = -ky - c\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

second-order linear differential equation and its auxiliary equation is

$$mr^2 + cr + k = 0$$

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

We need to discuss three cases.

CASE I ▫ $c^2 - 4mk > 0$ **(overdamping)**
In this case r_1 and r_2 are distinct real roots and

$$y = c_1e^{r_1t} + c_2e^{r_2t}$$

Since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < c$, so the roots r_1 and r_2 given by Equations 4 must both be negative. This shows that $x \to 0$ as $t \to \infty$. Typical graphs of x as a function of t are shown in Figure 4. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means that there is a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

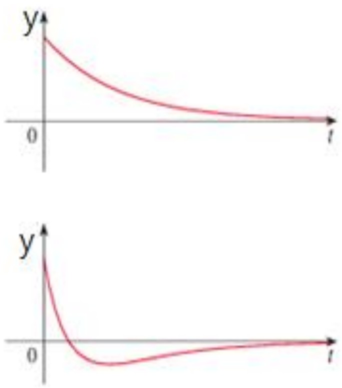


FIGURE 4
Overdamping

CASE II ▫ $c^2 - 4mk = 0$ **(critical damping)**
This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

And the solution is

$$y = c_1e^{-\frac{c}{2m}t} + c_2te^{-\frac{c}{2m}t}$$

$$y = (c_1 + c_2t)e^{-\frac{c}{2m}t}$$

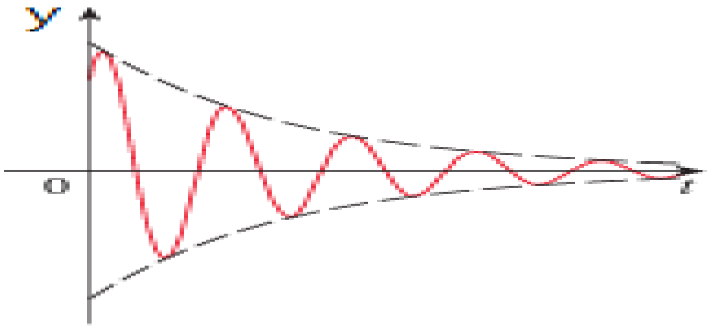
CASE III ▫ $c^2 - 4mk < 0$ **(underdamping)**
Here the roots are complex:

Since $c_2 = -c_1$, this gives $12c_1 = 0.6$ or $c_1 = 0.05$. Therefore

$$r_{1,2} = -\frac{c}{2m} \pm \frac{\sqrt{4km - c^2}}{2m}i$$

$$x = 0.05(e^{-4t} - e^{-16t})$$

$$y = e^{\left(-\frac{c}{2m}\right)t}\left[A\cos\frac{\sqrt{4km-c^2}}{2m}t + B\sin\frac{\sqrt{4km-c^2}}{2m}t\right]$$



EXAMPLE Suppose that the spring of Example 1 is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s.

SOLUTION From Example 1 the mass is $m = 2$ and the spring constant is $k = 128$, so the differential equation (3) becomes

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

$$2\frac{d^2y}{dt^2} + 40\frac{dy}{dt} + 128y = 0$$

$$\frac{d^2y}{dt^2} + 20\frac{dy}{dt} + 64y = 0$$

$$r^2 + 20r + 64 = 0$$

$$(r+4)(r+16) = 0$$

$$r = -4 \quad or \quad r = -16$$

$$y = c_1e^{-4t} + c_2e^{-16t}$$

$at\ t = 0 \quad y = 0 \quad and \quad v = y' = 0.6\ m/sec$

$$0 = c_1e^0 + c_2e^0$$

$$c_1 + c_2 = 0 \;\ldots\ldots\ldots\ldots..1$$

$$y' = -4c_1e^{-4t} - 16c_2e^{-16t}$$

$$0.6 = -4c_1e^0 - 16c_2e^0$$

$$0.6 = -4c_1 - 16c_2 \ldots\ldots\ldots.2$$

$$c_1 = 0.05$$

$$c_2 = -0.05$$

$$y = 0.05e^{-4t} - 0.05e^{-16t}$$

Forced Vibrations

Suppose that, in addition to the restoring force and the damping force, the motion of the spring is affected by an external force $F(t)$. Then Newton's Second Law gives

$$m\frac{d^2y}{dt^2} = \text{restoring force} + \text{damping force} + \text{external force}$$

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt} + F_0\cos\omega t$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F_0\cos\omega t$$

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_0}{m}\cos\omega t$$

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \omega_0^2 y = \frac{F_0}{m}\cos\omega t$$

If the $\omega = \omega_0$ *and* $c = 0$, the solution will be

$$\frac{d^2y}{dt^2} + \omega_0^2 y = \frac{F_0}{m}\cos\omega_0 t$$

$$y = y_c + y_p$$

$$\frac{d^2y}{dt^2} + \omega_0^2\, y = 0$$

$$r^2 + \omega_0^2 = 0$$

$$r_{1,2} = \pm\omega_0 i$$

$$y_c = A\cos\omega_0 t + B\sin\omega_0 t$$

Now we will find the particular solution

$$y_p = c_1 t\cos\omega_0 t + c_2 t\ \sin\omega_0 t$$

$$y'_p = c_1\cos\omega_0 t - \omega_0 t\sin\omega_0 t + c_2\ \sin\omega_0 t + \omega_0\, c_2 t\cos\omega t$$

$$y''_p = -\omega_0^2 c_1 t\cos\omega_0 t - \omega_0^2 c_2 t\sin\omega_0 t + 2\omega_0 c_2\cos\omega_0 t - 2\omega_0 c_1\sin\omega_0 t$$

Substitute in the main equation

$$\frac{d^2y}{dt^2} + \omega_0^2\, y = \frac{F_0\cos\omega_0\, t}{m}$$

$$-\omega_0^2 c_1 t\cos\omega_0 t - \omega_0^2 c_2 t\sin\omega_0 t + 2\omega_0\, c_2\cos\omega_0 t - 2\omega_0 c_1\sin\omega_0 t + \omega_0^2\,(c_1 t\cos\omega_0 t + c_2 t\ \sin\omega_0 t) = \frac{F_0\cos\omega_0\, t}{m}$$

$$2\omega_0 c_2\cos\omega_0 t - 2\omega_0 c_1\sin\omega_0 t = \frac{F_0\cos\omega_0 t}{m}$$

$$c_1 = 0$$

$$2\omega_0\, c_2 = \frac{F_0}{m} \qquad\qquad c_2 = \frac{F_0}{2\omega_0 m}$$

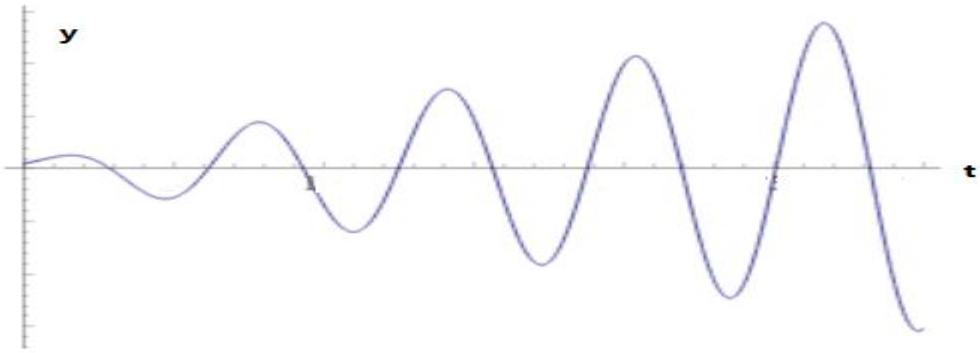
$$y_p = c_1 t\cos\omega_0 t + c_2 t\ \sin\omega_0 t$$

$$y_p = 0 * t\cos\omega_0 t + \frac{F_0}{2\omega_0 m} t\ \sin\omega_0 t$$

$$y_p = \frac{F_0}{2\omega_0 m} t\ \sin\omega_0 t$$

$$y = y_c + y_p$$

$$y = A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0}{2\omega m} t\ \sin\omega_0 t$$



If the $\omega \neq \omega_0\ and\ \ c = 0$, the solution will be

$$y = A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} t\ \sin\omega t$$

If the $\omega \neq \omega_0\ and\ \ c \neq 0$, the solution will be

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F_0\cos\omega t$$

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}\ y = \frac{F_0}{m}\cos\omega t$$

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \omega_0^2\ y = \frac{F_0}{m}\cos\omega t$$

The complementary solution

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \omega_0^2\ y = 0$$

$$r^2 + \frac{c}{m}r + \omega_0^2\ = 0$$

$$r_{1,2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \omega_0^2}$$

$$y_c = c_1y_1 + c_2y_2 \quad according\ to\ the\ three\ cases$$

The particular solution

$$y_p = A\cos\omega t + B\sin\omega t$$

$$y'_p = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$y''_p = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

Substitute in the main equation

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \omega_0^2\, y = \frac{F_0}{m}\cos\omega t$$

$$y'' + \frac{c}{m}y' + \omega_0^2\, y = \frac{F_0}{m}\cos\omega t$$

$$(-\omega^2 A\cos\omega t - \omega^2 B\sin\omega t) + \frac{c}{m}(-\omega A\sin\omega t + \omega B\cos\omega t)$$
$$+ \omega_0^2\,(A\cos\omega t + B\sin\omega t) = \frac{F_0}{m}\cos\omega t$$

$$A = \frac{\omega c F_0}{[k - m\omega^2]^2 + \omega^2 c^2}$$

$$B = \frac{[k - m\omega]F_0}{[k - m\omega^2]^2 + \omega^2 c^2}$$

Using a little trigonometry, we can rewrite this as

$$y_p = C\cos(\omega t - \phi)$$

$$C = \sqrt{A^2 + \mathrm{B}^2}$$

$$C = \frac{F_0}{\sqrt{[k - m\omega^2]^2 + \omega c^2}}$$

$$\cos\phi = \frac{A}{C} \qquad \sin\phi = \frac{B}{C}$$

$$y_p = \frac{F_0}{\sqrt{[k - m\omega^2]^2 + \omega^2 c^2}}\cos(\omega t - \phi)$$

$$f_n = natural\ frequency = \frac{\omega_0}{2m}$$

H.W

1. A spring with a 3-kg mass is held stretched 0.6 m beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s, find the position of the mass after t seconds.
2. A spring with a 4-kg mass has natural length 1 m and is maintained stretched to a length of 1.3 m by a force of 24.3 N. If the spring is compressed to a length of 0.8 m and then released with zero velocity, find the position of the mass at any time t.
3. A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t.
4. A spring with a mass of 3 kg has damping constant 30 and spring constant 123.
   (a) Find the position of the mass at time t if it starts at the equilibrium position with a velocity of 2 m/s.
   (b) Graph the position function of the mass.
5. For the spring in Exercise 3, find the mass that would produce critical damping.
6. For the spring in Exercise 4, find the damping constant that would produce critical damping.

7. A spring has a mass of 1 kg and its spring constant is $k = 100$. The spring is released at a point 0.1 m above its equilibrium position. Graph the position function for the following values of the damping constant c: 10, 15, 20, 25, 30. What type of damping occurs in each case?

8. A spring has a mass of 1 kg and its damping constant is $c = 10$. The spring starts from its equilibrium position with a velocity of 1 m/s. Graph the position function for the following values of the spring constant k: 10, 20, 25, 30, 40. What type of damping occurs in each case?

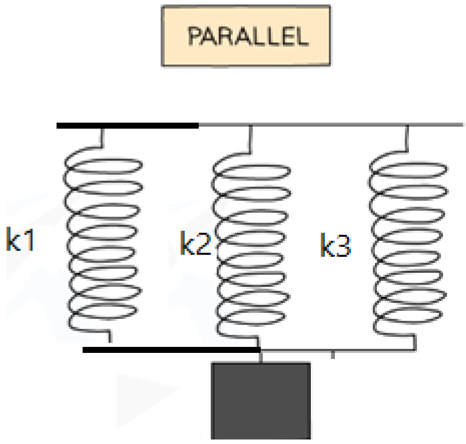
9. Suppose a spring has mass m and spring constant k and let $\omega = \sqrt{k/m}$. Suppose that the damping constant is so small that the damping force is negligible. If an external force $F(t) = F_0 \cos \omega_0 t$ is applied, where $\omega_0 \neq \omega$, use the method of undetermined coefficients to show that the motion of the mass is described by Equation 6.

10. As in Exercise 9, consider a spring with mass m, spring constant k, and damping constant $c = 0$, and let $\omega = \sqrt{k/m}$. If an external force $F(t) = F_0 \cos \omega t$ is applied (the applied frequency equals the natural frequency), use the method of undetermined coefficients to show that the motion of the mass is given by $x(t) = c_1 \cos \omega t + c_2 \sin \omega t + (F_0/(2m\omega))t \sin \omega t$.

11. Show that if $\omega_0 \neq \omega$, but ω/ω_0 is a rational number, then the motion described by Equation 6 is periodic.

Equivalent spring:

$$\frac{1}{K_{equ}} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$



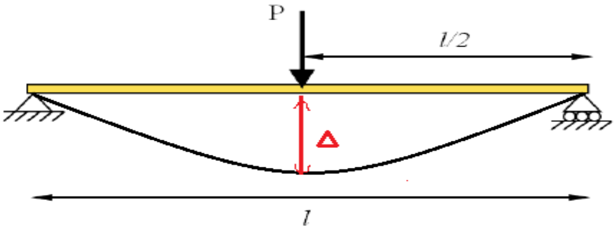
$$K_{equ} = K_1 + K_2 + K_3$$

Equivalent spring constant of beams

$$Spring\ Force = K * \Delta$$

$$P = K * \frac{PL^3}{48EI}$$

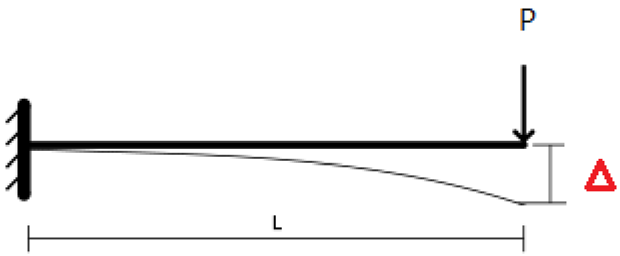
$$K = \frac{48EI}{L^3}$$



$$Spring\ Force = K * \Delta$$

$$P = K * \frac{PL^3}{3EI}$$

$$K = \frac{3EI}{L^3}$$

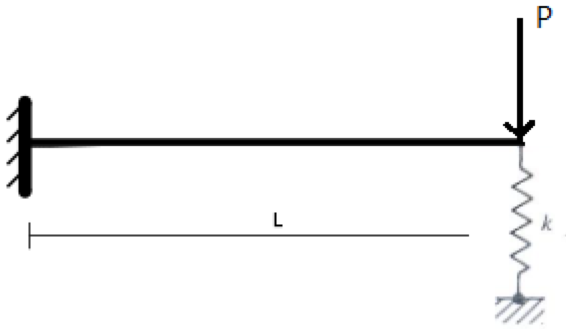


$$Spring\ Force = K * \Delta$$

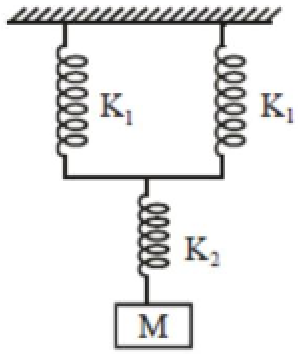
$$P = K_p * \frac{PL^3}{3EI}$$

$$K_p = \frac{3EI}{L^3}$$

$$K_{equ} = \frac{3EI}{L^3} + K$$



Example 1: Find the natural frequency of the system



$$\frac{1}{K_{equ}} = \frac{1}{2K_1} + \frac{1}{K_2}$$

$$\frac{1}{K_{equ}} = \frac{K_2 + 2K_1}{2K1K_2}$$

$$K_{equ} = \frac{2K_1K_2}{K_2 + 2K_1}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\omega_0 = \sqrt{\frac{\frac{2K_1K_2}{K_2 + 2K_1}}{m}} = \sqrt{\frac{2K1K2}{m(K_2 + 2K_1)}}$$

$$f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2K1K2}{m(K_2 + 2K_1)}}$$

Conservation of Energy

Newton's law is not always the most convenient way to setup the differential equation that describes the behavior of the mechanical system. Sometimes this is most easily done by using the principle of the conservation of the energy. The following are the ones most commonly encountered:

1. The kinetic energy of a body or a mass moving with velocity v is given by the following formula

$$KE = \frac{1}{2} mv^2 = \frac{1}{2} my'^2$$

2. The kinetic energy of a body of moment of inertia (I) rotating with angular velocity ω is given by the formula

$$KE = \frac{1}{2} I\, \omega^2 = \frac{1}{2} I\, \theta'^2$$

The kinetic energy of pulley is

$$KE = \frac{1}{2}\left(\frac{1}{2} mR^2\right) \theta'^2$$
$$KE = \frac{1}{4} mR^2\, \theta'^2$$

3. The potential energy (PE) stored in spring is

$$PE = \frac{1}{2} ky^2$$

4. The potential energy of a mass (m) when it is moved

$$PE = mgy$$

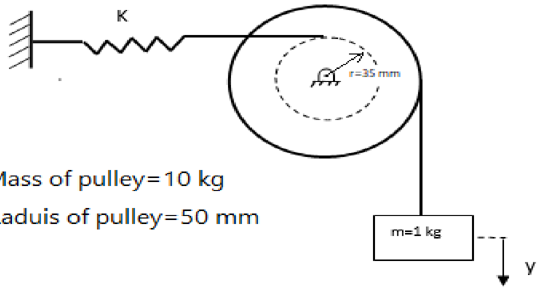
The energy method depends on the following fact

$$KE + PE = C$$

If differentiate both sides

$$\frac{d}{dt}(KE + PE) = 0$$

Example 1: A mass of 1 kg is suspended by a spring passing over the pulley as shown if fig. The system is supported horizontally by a spring of stiffness (k=1000 N/m). Determine the natural frequency of vibration of a system using the following data



$$\frac{d}{dt}(KE + PE) = 0$$

$$KE\ of\ mass = \frac{1}{2}my'^2$$

$$KE\ of\ pulley = \frac{1}{4}mR^2\theta'^2$$

$$PE\ of\ spring = \frac{1}{2}kx^2$$

$$\frac{d}{dt}\left(\frac{1}{2}my'^2 + \frac{1}{4}MR^2\theta'^2 + \frac{1}{2}kx^2\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}m(R\theta')^2 + \frac{1}{4}MR^2\theta'^2 + \frac{1}{2}k(r\theta)^2\right) = 0$$

$$mR^2\theta'\theta'' + \frac{1}{2}MR^2\theta'\theta'' + kr^2\theta\theta' = 0$$

$$mR^2\theta'' + \frac{1}{2}MR^2\theta'' + kr^2\theta = 0$$

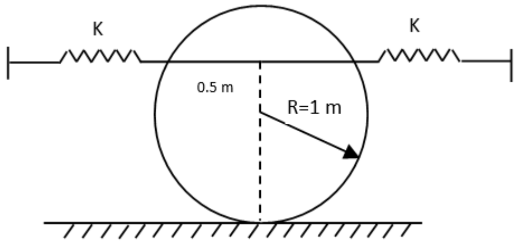
$$(mR^2 + \frac{1}{2}MR^2)\theta'' + kr^2\theta = 0$$

$$\theta'' + \frac{kr^2}{mR^2 + \frac{1}{2}MR^2}\theta = 0$$

$$\omega_0 = \sqrt{\frac{kr^2}{mR^2 + \frac{1}{2}MR^2}} = \frac{r}{R}\sqrt{\frac{k}{m + \frac{1}{2}M}}$$

$$f_n = \frac{\omega_0}{2\pi} = \frac{r}{2\pi R}\sqrt{\frac{k}{m + \frac{1}{2}M}} = \frac{0.35}{2\pi(0.5)}\sqrt{\frac{1000}{1 + \frac{1}{2} * 10}} = 1.44\ cycle/sec$$

Example 2: A cylinder of mass 1 kg and radius (1 m) connected by two springs at a height of (0.5 m) above the center as shown in fig. The cylinder rolls without slipping. If the spring constant is (30 kN/m) for each spring, find the natural frequency (f_n) of the system.



$$KE\ of\ pulley = \frac{1}{2} I\theta'^2 = \frac{1}{2}\left(\frac{1}{2}mR^2 + mR^2\right)\theta'^2$$

$$KE\ of\ pulley = \frac{3}{4} mR^2\theta'^2$$

$$PE\ of\ spring = \frac{1}{2}kx^2 = 2 * \frac{1}{2}k(1.5\ \theta)^2 = 2.25k\theta^2$$

$$\frac{d}{dt}\left(\frac{3}{4}mR^2\theta'^2 + 2.25k\theta^2\right) = 0$$

$$2 * \frac{3}{4} mR^2\theta'\theta'' + 2 * 2.25k\theta\theta' = 0$$

$$\frac{3}{2}mR^2\theta'\theta'' + 4.5k\theta\theta' = 0$$

$$\frac{3}{2}(1)(1)^2\theta'' + 4.5 * 30000\theta = 0$$

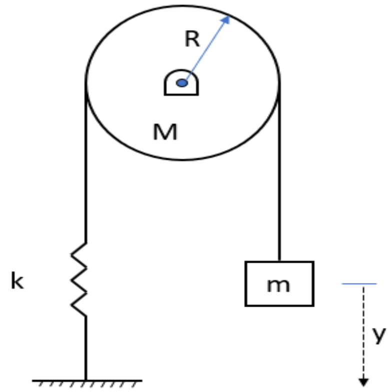
$$\theta'' + 90000\theta = 0$$

$$\omega_0^2 = 90000$$

$$\omega_0 = 300\ rad/sec$$

$$f_n = \frac{\omega_0}{2\pi} = \frac{300}{2\pi} = 47.7\ cycle/sec$$

Example 3: Find the natural frequency of the following system by using the energy method



$$\frac{d}{dt}(KE + PE) = 0$$

$$KE\ of\ mass = \frac{1}{2}my'^2$$

$$KE\ of\ pulley = \frac{1}{4}MR^2\theta'^2 = \frac{1}{4}\ MR^2\left(\frac{y'}{R}\right)^2 = \frac{1}{4}My'^2$$

$$PE\ of\ spring = \frac{1}{2}kx^2 = \frac{1}{2}k(y)^2 = \frac{1}{2}ky^2$$

$$\frac{d}{dt}\left(\frac{1}{2}my'^2+\frac{1}{4}My'^2+\frac{1}{2}ky^2\right)=0$$

$$my'y''+\frac{1}{2}My'y''+kyy'=0$$

$$my''+\frac{1}{2}My''+ky=0$$

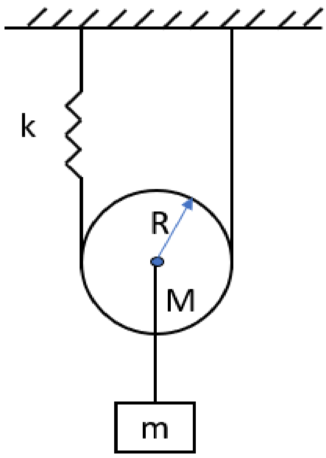
$$\left(m+\frac{1}{2}M\right)y''+ky=0$$

$$y''+\frac{k}{\left(m+\frac{1}{2}M\right)}y=0$$

$$\omega_0^2=\frac{k}{m+\frac{1}{2}M} \qquad \omega_0=\sqrt{\frac{k}{m+\frac{1}{2}M}}$$

$$f_n=\frac{\omega_0}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{k}{m+\frac{1}{2}M}}$$

Example 4: Find the natural frequency of the following system by using the energy method



$$KE\ of\ mass=\frac{1}{2}my'^2=\frac{1}{2}(m+M)y'^2$$

$$KE\ of\ pulley=\frac{1}{4}MR^2\theta'^2=\frac{1}{4}MR^2\left(\frac{y'}{R}\right)^2=\frac{1}{4}My'^2$$

$$PE\ of\ spring=\frac{1}{2}kx^2=\frac{1}{2}k(2y)^2=2ky^2$$

$$\frac{d}{dt}(KE+PE)=0$$

$$\frac{d}{dt}\left(\frac{1}{2}(m+M)y'^2+\frac{1}{4}My'^2+2ky^2\right)=0$$

$$(m+M)y'y''+\frac{1}{2}My'y''+4kyy'=0$$

$$(m+M)y''+\frac{1}{2}My''+4ky=0$$

$$\left(m+\frac{3}{2}M\right)y''+4ky=0$$

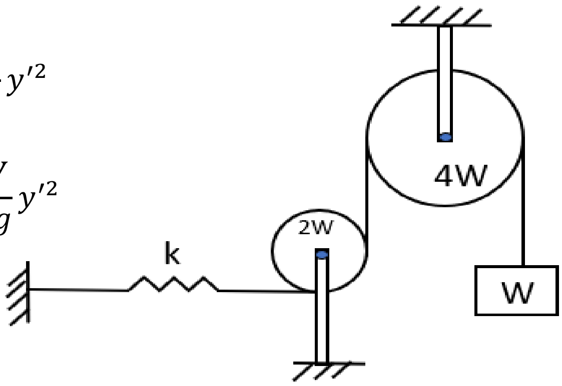
$$y''+\frac{4k}{\left(m+\frac{3}{2}M\right)}y=0$$

$$\omega_0^2=\frac{4k}{\left(m+\frac{3}{2}M\right)}=\frac{8k}{2m+3M}$$

$$\omega_0=\sqrt{\frac{8k}{2m+3M}}$$

$$f_n=\frac{\omega_0}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{8k}{2m+3M}}$$

Example 5: Find the natural frequency of the following system by using the energy method



$$KE\ of\ mass=\frac{1}{2}my'^2=\frac{1}{2}\frac{W}{g}y'^2$$

$$KE\ of\ pulley\ (4W)=\frac{1}{2}I\theta'^2=\frac{1}{2}\left(\frac{1}{2}\frac{4W}{g}R_1^2\right)\left(\frac{y'}{R}\right)^2=\frac{W}{g}y'^2$$

$$KE\ of\ pulley\ (2W)=\frac{1}{2}I\theta'^2=\frac{1}{2}\left(\frac{1}{2}\frac{2W}{g}R_2^2\right)\left(\frac{y'}{R_2}\right)^2=\frac{W}{2g}y'^2$$

$$PE\ of\ spring=\frac{1}{2}kx^2=\frac{1}{2}k(y)^2=\frac{1}{2}ky^2$$

$$\frac{d}{dt}(KE+PE)=0$$

$$\frac{d}{dt}\left(\frac{W}{g}y'^2+\frac{W}{2g}y'^2+\frac{1}{2}\frac{W}{g}y'^2+\frac{1}{2}ky^2\right)=0$$

$$2\frac{W}{g}y'y''+\frac{W}{g}y'y''+\frac{W}{g}y'y''+kyy'=0$$

$$2\frac{W}{g}y''+\frac{W}{g}y''+\frac{W}{g}y''+ky=0$$

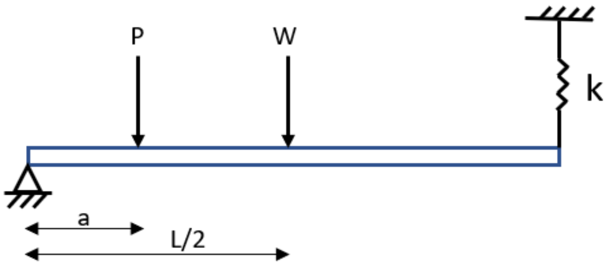
$$4\frac{W}{g}y''+ky=0$$

$$y''+\frac{g}{4W}ky=0$$

$$\omega_0^2 = \frac{gk}{4W} \qquad \omega_0 = \frac{1}{2}\sqrt{\frac{gk}{W}}$$

$$f_n = \frac{\omega_0}{2\pi} = \frac{1}{4\pi}\sqrt{\frac{gk}{W}}$$

Example 6: Find the natural frequency of the following system by using the energy method. The bar is of uniform cross section and of weight (W)



$$KE\ of\ mass = \frac{1}{2}my'^2 = 0$$

$$KE\ of\ \ bar = \frac{1}{2}I\theta'^2 = \frac{1}{2}\left(\frac{ML^2}{3}\right)\theta'^2 = \frac{WL^2}{6g}\theta'^2$$

$$KE\ of\ (P) = \frac{1}{2}I\theta'^2 = \frac{1}{2}(Ma^2)\theta'^2 = \frac{Pa^2}{2g}\theta'^2$$

$$PE\ of\ spring = \frac{1}{2}kx^2 = \frac{1}{2}k(L\theta)^2 = \frac{L^2}{2}k\theta^2$$

$$\frac{d}{dt}(KE + PE) = 0$$

$$\frac{d}{dt}\left(\frac{WL^2}{6g}\theta'^2 + \frac{Pa^2}{2g}\theta'^2 + \frac{L^2}{2}k\theta^2\right) = 0$$

$$\frac{WL^2}{3g}\theta'\theta'' + \frac{Pa^2}{g}\theta'\theta'' + L^2k\theta\theta' = 0$$

$$\frac{WL^2}{3g}\theta'' + \frac{Pa^2}{g}\theta'' + L^2k\theta = 0$$

$$(\frac{WL^2}{3g} + \frac{Pa^2}{g})\theta'' + L^2k\theta = 0$$

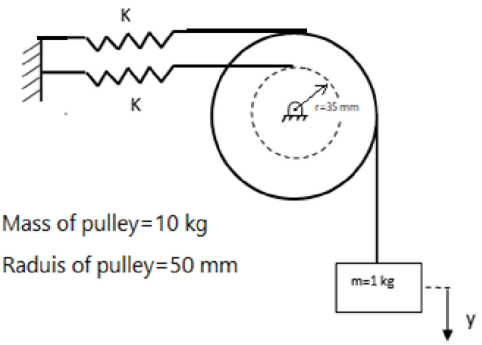
$$\theta'' + \frac{L^2k}{\frac{WL^2}{3g} + \frac{Pa^2}{g}}\theta = 0$$

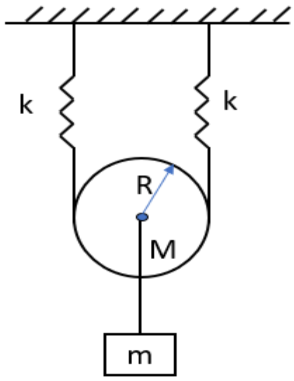
$$\omega_0^2 = \frac{L^2k}{\frac{WL^2}{3g} + \frac{Pa^2}{g}} \qquad \omega_0 = \sqrt{\frac{L^2k}{\frac{WL^2}{3g} + \frac{Pa^2}{g}}}$$

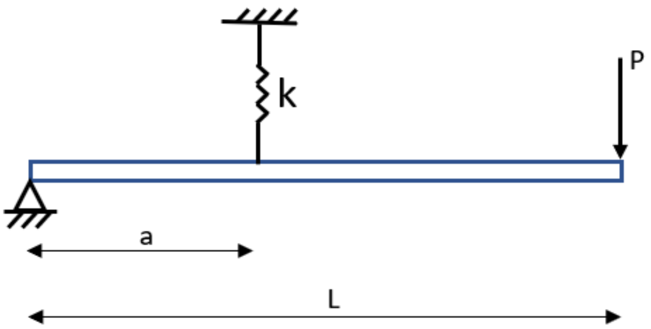
$$f_n = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{L^2 k}{\frac{WL^2}{3g} + \frac{Pa^2}{g}}}$$

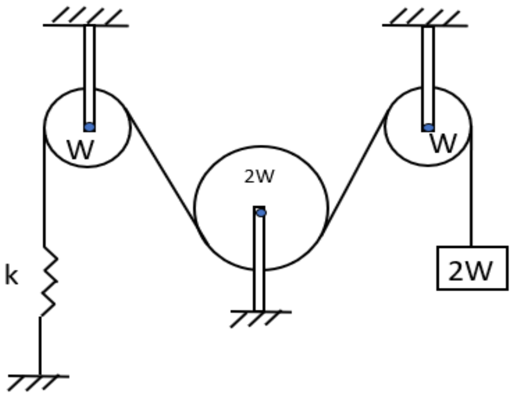
H.W

Find the natural frequency of the following system by using the energy method for the following systems



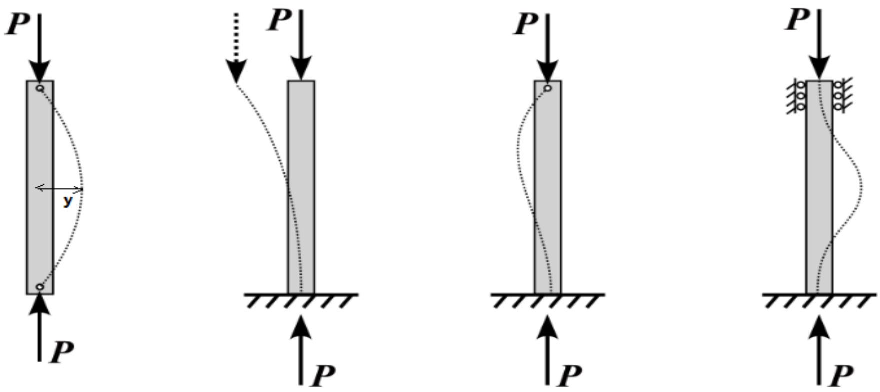






Column Buckling

A column is a straight bar is subjected to a compression axial load. A column can fail due to buckling(Large lateral deflection) before the compressive stress in the column reach its allowable (yield) value.



Consider a hinged-hinged column of length "L" and subjected to axial load "P", if we assume that the column became slightly bent.

$$M = -Py$$

$$EI\frac{d^2y}{dx^2} = -Py$$

$$EI\frac{d^2y}{dx^2} + Py = 0$$

A general solution for the differential equation is

$$EIm^2 + P = 0$$

$$m^2 + \frac{P}{EI} = 0 \qquad m = \pm\sqrt{\frac{P}{EI}}i$$

$$y = A\cos\sqrt{\frac{P}{EI}}x + B\sin\sqrt{\frac{P}{EI}}x$$

Use the boundary conditions

$$At\ \ x = 0\ \ y = 0$$

$$0 = A\cos\sqrt{\frac{P}{EI}}0 + B\sin\sqrt{\frac{P}{EI}}0 \qquad A = 0$$

$$y = B \sin \sqrt{\frac{P}{EI}} x$$

$At \;\; x = L \;\;\; y = 0$

$$0 = B \sin \sqrt{\frac{P}{EI}} L$$

The constant B cannot be zero because that is trivial solution which means that the column will not be buckling, thus

$$\sin \sqrt{\frac{P}{EI}} L = 0 \qquad \sqrt{\frac{P}{EI}} L = n\pi \qquad \sqrt{\frac{P}{EI}} = \frac{n\pi}{L}$$

$$\frac{P}{EI} = \left(\frac{n\pi}{L}\right)^2 \qquad P = \left(\frac{n\pi}{L}\right)^2 EI$$

The smallest critical load for the column is obtained when $n = 1$

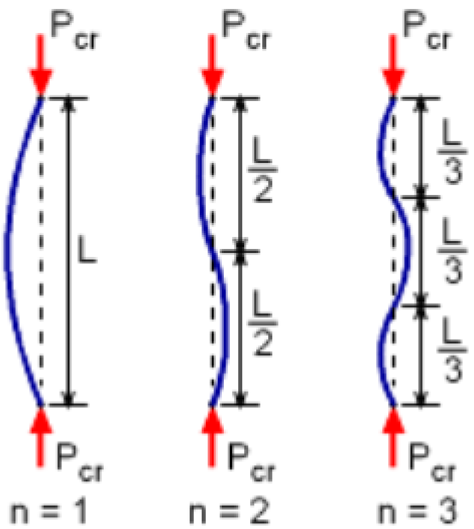
$$P_{cr} = \left(\frac{1\pi}{L}\right)^2 EI = \frac{\pi^2}{L^2} EI$$

when $n = 2$

$$P_{cr} = \left(\frac{2\pi}{L}\right)^2 EI = 4\left(\frac{\pi^2}{L^2}\right) EI$$

when $n = 3$

$$P_{cr} = \left(\frac{3\pi}{L}\right)^2 EI = 9\left(\frac{\pi^2}{L^2}\right) EI$$



If hinged-fixed column of length “L” and subjected to axial load “P”,

The differential equation is

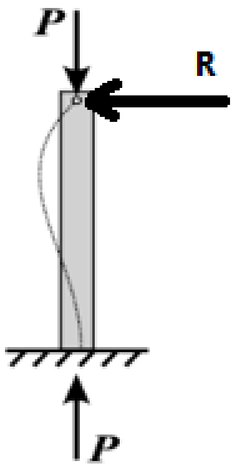
$$EI \frac{d^2y}{dx^2} + Py = -Rx$$

the solution is

complementary solution

$$EI\frac{d^2y}{dx^2} + Py = 0$$

$$y_c = A\cos\sqrt{\frac{P}{EI}}x + B\sin\sqrt{\frac{P}{EI}}x$$



Fixed-Pinned

Particular solution

$$y_p = c_1x + c_2$$

$$y'_p = c_1$$

$$y''_p = 0$$

Substitute in the main equation

$$EI\frac{d^2y}{dx^2} + Py = -Rx$$

$$EI(0) + P(c_1x + c_2) = -Rx$$

$$P(c_1x + c_2) = -Rx$$

$$c_0 = 0 \qquad\qquad Pc_1 = -R \qquad\qquad c_1 = -\frac{R}{P}$$

$$y_p = c_1x + c_2$$

$$y_p = -\frac{R}{P}x + 0$$

The general solution is

$$y = y_c + y_p$$

$$y = A\cos\sqrt{\frac{P}{EI}}x + B\sin\sqrt{\frac{P}{EI}}x - \frac{R}{P}x$$

Use the boundary conditions

$$At \;\; x = 0 \;\;\; y = 0$$

$$y = A\cos\sqrt{\frac{P}{EI}}x + B\sin\sqrt{\frac{P}{EI}}x - \frac{R}{P}x$$

$$0 = A\cos\sqrt{\frac{P}{EI}}0 + B\sin\sqrt{\frac{P}{EI}}0 - \frac{R}{P}0$$

$$A = 0$$

$At\ \ x = L\ \ \ y' = 0$

$$y = B\sin\sqrt{\frac{P}{EI}}x - \frac{R}{P}x$$

$$y' = B\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}x - \frac{R}{P}$$

$$0 = B\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}L - \frac{R}{P}$$

$$Let\ \ \lambda = \sqrt{\frac{P}{EI}}$$

$$0 = B\lambda\cos\lambda L - \frac{R}{P}$$

$$B = \frac{R}{P\lambda\cos\lambda L}$$

$$y = \frac{R}{P\lambda\cos\lambda L}\sin\sqrt{\frac{P}{EI}}x - \frac{R}{P}x$$

$At\ \ x = L\ \ \ y = 0$

$$0 = \frac{R}{P\lambda\cos\lambda L}\sin\sqrt{\frac{P}{EI}}L - \frac{R}{P}L$$

$$0 = \frac{R}{P\lambda\cos\lambda L}\sin\lambda L - \frac{R}{P}L$$

$$0 = \frac{R}{P\lambda\cos\lambda L}\sin\lambda L - \frac{R}{P}L$$

$$\frac{1}{\lambda\cos\lambda L}\sin\lambda L - L = 0$$

$$\tan\lambda L - \lambda L = 0$$

$$\lambda L = \tan\lambda L$$

by trial and error

$\lambda L = 4.493$

$$\lambda L = \sqrt{\frac{P}{EI}}L$$

$$4.493 = \sqrt{\frac{P}{EI}}L$$

$$(4.493)^2 = \frac{P}{EI}L^2$$

$$P = \frac{20.19L^2}{EI}$$

Simultaneous Linear Ordinary Differential Equations

1-Substitute Method:

Convert the system into a single linear differential equation and then solve it by using the methods developed in earlier lectures.

Example.1 Determine the general solution of y and z

$$2y + z' = e^x \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

$$y' - 2z = 1 + x \ldots\ldots\ldots\ldots\ldots 2$$

$$z = \frac{1}{2}(y' - x - 1) \ldots\ldots\ldots\ldots\ldots 3$$

$$z' = \frac{1}{2}(y'' - 1) \ldots\ldots\ldots\ldots\ldots 4$$

Substitute 4 in 1

$$2y + \frac{1}{2}(y'' - 1) = e^x$$

$$y'' + 4y = 2e^x + 1$$

Complementary solution

$$y'' + 4y = 0$$

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$y_c = A\cos 2x + B\sin 2x$$

Particular solution

$$y_p = C_1 e^x + C_2$$

$$y'_p = C_1 e^x$$

$$y''_p = C_1 e^x$$

$$y'' + 4y = 2e^x + 1$$

$$C_1 e^x + 4(C_1 e^x + C_2) = 2e^x + 1$$

$$5C_1 e^x + 4C_2 = 2e^x + 1$$

$$C_1 = \frac{2}{5} \qquad\qquad C_2 = \frac{1}{4}$$

$$y_p = \frac{2}{5}e^x + \frac{1}{4}$$

$$y = y_c + y_p$$

$$y = A\cos 2x + B\sin 2x + \frac{2}{5}e^x + \frac{1}{4}$$

Substitute the y' in equ.3 to get the solution of z

$$y' = -2A\sin 2x + 2B\cos 2x + \frac{2}{5}e^x$$

$$z = \frac{1}{2}(y' - x - 1)$$

$$z = \frac{1}{2}\left(-2A\sin 2x + 2B\cos 2x + \frac{2}{5}e^x - x - 1\right)$$

$$z = -A\sin 2x + B\cos 2x + \frac{1}{5}e^x - \frac{1}{2}x - \frac{1}{2}$$

2-D-Operator Method

Depending on Cramer method, we can solve the simultaneous linear differential equations:

Example.1 Determine the general solution of y and z

$$5y' - 2z' + 4y - z = e^{-x} \text{ ……….1}$$

$$y' + 8y - 3z = 5e^{-x} \text{ …………2}$$

By using the D-operator, we can rewrite the equations as follows

$$(5D+4)y+(-2D-1)z=e^{-x} \ldots\ldots\ldots.1$$

$$(D+8)y+(-3)z=e^{-x} \ldots\ldots\ldots.2$$

$$\begin{bmatrix}(5D+4) & (-2D-1)\\ (D+8) & (-3)\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}e^{-x}\\ 5e^{-x}\end{bmatrix}$$

$$y=\frac{\begin{vmatrix}e^{-x} & (-2D-1)\\ 5e^{-x} & (-3)\end{vmatrix}}{\begin{vmatrix}(5D+4) & (-2D-1)\\ (D+8) & (-3)\end{vmatrix}}=\frac{-3e^{-x}-(-2D-1)(5e^{-x})}{-3(5D+4)-(-2D-1)(D+8)}$$

$$y=\frac{-3e^{-x}+(2D+1)(5e^{-x})}{2D^2+2D-4}=\frac{-3e^{-x}+2D(5e^{-x})+(1)(5e^{-x})}{2D^2+2D-4}$$

$$y=\frac{-3e^{-x}-10e^{-x}+5e^{-x}}{2D^2+2D-4}=\frac{-8e^{-x}}{2D^2+2D-4}$$

$$(2D^2+2D-4)y=-8e^{-x}$$

$$2y''+2y'-4y=-8e^{-x}$$

$$y''+y'-2y=-4e^{-x}$$

Complementary solution

$$y''+y'-2y=0$$

$$m^2+m-2=0$$

$$(m+2)(m-1)=0$$

$$m_{1,2}=1,-2$$

$$y_c=c_1e^{x}+c_2e^{-2x}$$

particular solution

$$y_p=Ae^{-x}$$

$$y'_p = -Ae^{-x}$$

$$y''_p = Ae^{-x}$$

$$y'' + y' - 2y = -4e^{-x}$$

$$(Ae^{-x}) + (-Ae^{-x}) - 2(Ae^{-x}) = -4e^{-x}$$

$$-2(Ae^{-x}) = -4e^{-x}$$

$$-2A = -4 \qquad A = 2$$

$$y_p = 2e^{-x}$$

$$y = y_c + y_p$$

$$y = c_1e^{x} + c_2e^{-2x} + 2e^{-x}$$

$$z = \frac{\begin{vmatrix} (5D+4) & e^{-x} \\ (D+8) & 5e^{-x} \end{vmatrix}}{\begin{vmatrix} (5D+4) & (-2D-1) \\ (D+8) & (-3) \end{vmatrix}}$$

or we can find the solution of z by depending on the second equation

$$y' + 8y - 3z = 5e^{-x} \quad \dots\dots\dots\dots.2$$

$$z = \frac{1}{3}(y' + 8y - 5e^{-x})$$

$$y' = c_1e^{x} - 2c_2e^{-2x} - 2e^{-x}$$

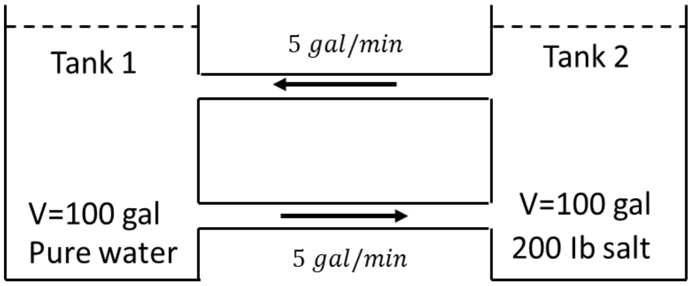
$$z = \frac{1}{3}(c_1e^{x} - 2c_2e^{-2x} - 2e^{-x} + 8(c_1e^{x} + c_2e^{-2x} + 2e^{-x}) - 5e^{-x})$$

$$z = \frac{1}{3}(c_1e^{x} - 2c_2e^{-2x} - 2e^{-x} + 8c_1e^{x} + 8c_2e^{-2x} + 16e^{-x} - 5e^{-x})$$

$$z = 3c_1e^{x} + 2c_2e^{-2x} + 3e^{-x}$$

Applications of Simultaneous Linear Differential Equations

Example.1 Two tanks are connected as shown in Fig.1. The first tank contains (100 gal) of pure water, the second contains (100 gal) of brine containing (2 Ib) of salt per gallon. Liquid circulates through the tanks at a constant rate of (5 gal/min). If the brine in each tank is kept uniform by stirring, find the amount of salt in each tank as a function of time.



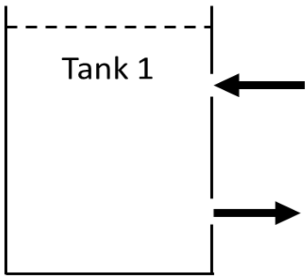
Solution

Tank 1

$$\frac{dy}{dt} = R_{in}K_{in} - R_{out}K_{out}$$

$$\frac{dy}{dt} = 5\left(\frac{z}{100}\right) - 5\left(\frac{y}{100}\right)$$

$$y' = \frac{z}{20} - \frac{y}{20} \dots\dots 1$$

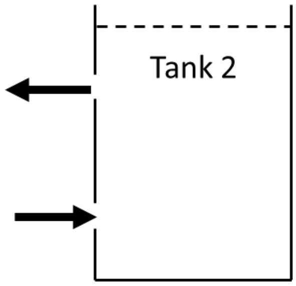


Tank 2

$$\frac{dz}{dt} = R_{in}K_{in} - R_{out}K_{out}$$

$$\frac{dz}{dt} = 5\left(\frac{y}{100}\right) - 5\left(\frac{z}{100}\right)$$

$$z' = \frac{y}{20} - \frac{z}{20} \dots\dots 2$$



$$y' = \frac{z}{20} - \frac{y}{20} \ldots\ldots 1$$

$$z' = \frac{y}{20} - \frac{z}{20} \ldots\ldots 2$$

From equation 1 $\quad y' = \frac{z}{20} - \frac{y}{20} \quad \rightarrow \quad z = 20y' + y \quad \rightarrow \quad z' = 20y'' + y'$

$$z' = \frac{y}{20} - \frac{z}{20}$$

$$20y'' + y' = \frac{y}{20} - \frac{20y' + y}{20}$$

$$400y'' + 40y' = 0$$

$$y'' + \frac{1}{10}y' = 0$$

$$m^2 + \frac{1}{10}m = 0$$

$$m\left(m + \frac{1}{10}\right) = 0$$

$$m_{1,2} = 0, -\frac{1}{10}$$

$$y = c_1 e^0 + c_2 e^{-\frac{1}{10}t}$$

$$y = c_1 + c_2 e^{-\frac{1}{10}t}$$

To find the solution of z

$$z = 20y' + y$$

$$z = 20\left(-\frac{1}{10}c_2 e^{-\frac{1}{10}t}\right) + c_1 + c_2 e^{-\frac{1}{10}t}$$

$$z = c_1 - c_2 e^{-\frac{1}{10}t}$$

$at\ t = 0\ \ z = 200 \qquad y = 0$

$y = c_1 + c_2 e^{-\frac{1}{10}t}$

$0 = c_1 + c_2 e^{-\frac{1}{10}(0)} \qquad\qquad c_1 + c_2 = 0 \ \ldots\ldots\ldots 1$

$$z = c_1 - c_2 e^{-\frac{1}{10}t}$$

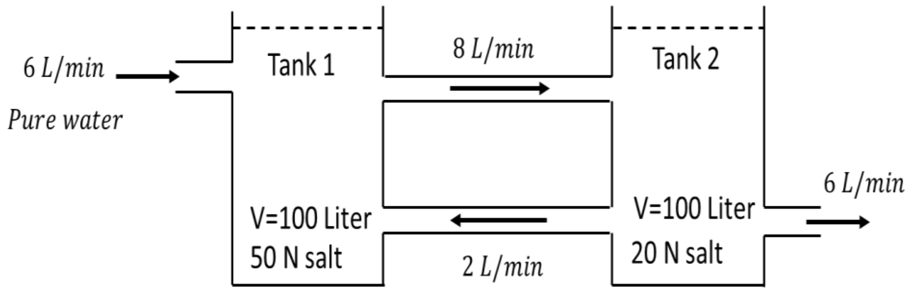
$$200 = c_1 - c_2 e^{-\frac{1}{10}(0)} \qquad c_1 - c_2 = 200 \quad \ldots\ldots 2$$

$$c_1 = 100 \qquad c_2 = -100$$

$$y = 100(1 - e^{-\frac{1}{10}t})$$

$$z = 100(1 - e^{-\frac{1}{10}t})$$

Example.2 Two tanks are connected as shown in Fig. below. The first tank contains initially (100 Liter) of brine contently (50 N) salt while the second tank contains (100 Liter) brine in which (20 N) salt is dissolved. Find the amount of salt in each tank as a function of time.



Solution

Tank 1

$$\frac{dy}{dt} = R_{in}K_{in} - R_{out}K_{out}$$

$$\frac{dy}{dt} = 8(0) + 2\left(\frac{z}{100}\right) - 8\left(\frac{y}{100}\right)$$

$$y' = \frac{z}{50} - \frac{2y}{25} \quad \ldots\ldots\ldots 1$$

Tank 1

$$\frac{dz}{dt} = R_{in}K_{in} - R_{out}K_{out}$$

$$\frac{dz}{dt} = 8\left(\frac{y}{100}\right) - 2\left(\frac{z}{100}\right) - 6\left(\frac{z}{100}\right)$$

$$\frac{dz}{dt} = 8\left(\frac{y}{100}\right) - 8\left(\frac{z}{100}\right)$$

$$z' = \frac{2}{25}y - \frac{2}{25}z \quad \ldots\ldots 2$$

from equation 1

$$z = 50y' + 4y \qquad z' = 50y'' + 4y'$$

Substitute in equation 2

$$z' = \frac{2}{25}y - \frac{2}{25}z \quad \ldots\ldots 2$$

$$50y'' + 4y' = \frac{2}{25}y - \frac{2}{25}(50y' + 4y)$$

$$50y'' + 4y' = \frac{2}{25}y - 4y' - \frac{8}{25}y$$

$$50y'' + 8y' + \frac{6}{25}y = 0$$

$$y'' + \frac{4}{25}y' + \frac{3}{625}y = 0$$

$$m^2 + \frac{4}{25}m + \frac{3}{625} = 0$$

$$\left(m + \frac{3}{25}\right)\left(m + \frac{1}{25}\right) = 0$$

$$m_{1,2} = -\frac{3}{25}, -\frac{1}{25}$$

$$y = c_1 e^{-\frac{1}{25}t} + c_2 e^{-\frac{3}{25}t}$$

To get the solution of z we use the equation

$$z = 50y' + 4y$$

$$y' = -\frac{1}{25}c_1 e^{-\frac{1}{25}t} - \frac{3}{25}c_2 e^{-\frac{3}{25}t}$$

$$z = 50\left(-\frac{1}{25}c_1 e^{-\frac{1}{25}t} - \frac{3}{25}c_2 e^{-\frac{3}{25}t}\right) + 4\left(c_1 e^{-\frac{1}{25}t} + c_2 e^{-\frac{3}{25}t}\right)$$

$$z = -2c_1 e^{-\frac{1}{25}t} - 6c_2 e^{-\frac{3}{25}t} + 4c_1 e^{-\frac{1}{25}t} + 4c_2 e^{-\frac{3}{25}t} = 2c_1 e^{-\frac{1}{25}t} - 2c_2 e^{-\frac{3}{25}t}$$

$$z = 2c_1 e^{-\frac{1}{25}t} - 2c_2 e^{-\frac{3}{25}t}$$

The use the boundary conditions to the c1 and c2

$$at\ t = 0 \ \ y = 50 \qquad z = 20$$

$$50 = c_1 e^{-\frac{1}{25}0} + c_2 e^{-\frac{3}{25}}$$

$$50 = c_1 + c_2 \ \ldots\ldots 1$$

$$20 = 2c_1 e^{-\frac{1}{25}0} - 2c_2 e^{-\frac{3}{25}0}$$

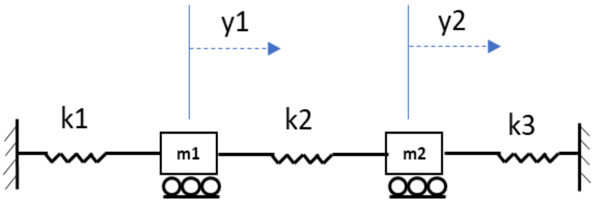
$20 = 2c_1 - 2c_2 \quad \dots \dots \dots 2$

$c_1 = 30 \quad c_2 = 20$

$y = 30e^{-\frac{1}{25}t} + 20e^{-\frac{3}{25}t}$

$z = 60e^{-\frac{1}{25}t} - 40e^{-\frac{3}{25}t}$

Example.3 The system with two degree of freedom shown in figure below. The following initial conditions are given.



$y_1(0) = 1\ cm \qquad y_1'(0) = \sqrt{3k}\ \ cm/sec$

$y_2(0) = 1\ cm \qquad y_2'(0) = \sqrt{3k}\ \ cm/sec$

$m_1 = m_2 = 1\ kg$

$k_1 = k_2 = k_3 = k$

Solution

$m_1 y_1'' + (k_1 + k_2)y_1 - k_2 y_2 = 0$

$1y_1'' + (k + k)y_1 - ky_2 = 0$

$y_1'' + 2ky_1 - ky_2 = 0 \quad \dots \dots ..1$

$m_2 y_2'' + (k_2 + k_3)y_2 - k_1 y_1 = 0$

$1y_2'' + (k + k)y_2 - ky_1 = 0$

$y_2'' + 2ky_2 - ky_1 = 0 \quad \dots \dots \dots .2$

$(D^2 + 2k)y_1 - (k)y_2 = 0 \quad \dots \dots ..1$

$(-k)y_1 + (D^2 + 2k)y_2 = 0 \quad \dots \dots \dots .2$

$$y_1 = \frac{\begin{vmatrix} (D^2 + 2k) & 0 \\ (-k) & 0 \end{vmatrix}}{\begin{vmatrix} (D^2 + 2k) & (-k) \\ (-k) & (D^2 + 2k) \end{vmatrix}}$$

$$y_1 = \frac{0}{(D^2+2k)(D^2+2k)y_1 + k*k}$$

$$(D^2+2k)(D^2+2k)y_1 + k*ky_1 = 0$$

$$D^4y_1 + 4kD^2y_1 + 3k^2y_1 = 0$$

$$(D^4 + 4kD^2 + 3k^2)y_1 = 0 \qquad y_1'''' + 4ky_!'' + 3k^2y_1 = 0$$

$$m^4 + 4km^2 + 3k^2 = 0$$

$$(m^2+3k)(m^2+k) = 0$$

$$m^2 + 3k = 0 \qquad m_{1,2} = \pm\sqrt{3k}\, i$$

$$(m^2+k) = 0 \qquad m_{3,4} = \pm\sqrt{k}\, i$$

$$y_1 = c_1\cos\sqrt{3k}t + c_2\sin\sqrt{3k}t + c_3\cos\sqrt{k}t + c_4\sin\sqrt{k}\,t$$

$$at\ t=0 \qquad y_1 = 1 \quad and \quad y_1' = \sqrt{3k}$$

$$1 = c_1\cos\sqrt{3k}(0) + c_2\sin\sqrt{3k}(0) + c_3\cos\sqrt{k}(0) + c_4\sin\sqrt{k}\,(0)$$

$$1 = c_1 + c_3 \quad \rightarrow c_3 = 1 - c_1$$

$$y'_1 = -\sqrt{3k}c_1\sin\sqrt{3k}t + \sqrt{3k}c_2\cos\sqrt{3k}t - \sqrt{3k}c_3\sin\sqrt{k}t + \sqrt{3k}c_4\cos\sqrt{k}\,t$$

$$\sqrt{3k} = -\sqrt{3k}c_1\sin\sqrt{3k}(0) + \sqrt{3k}c_2\cos\sqrt{3k}(0) - \sqrt{3k}c_3\sin\sqrt{k}(0) + \sqrt{3k}c_4\cos\sqrt{k}\,(0)$$

$$\sqrt{3k} = \sqrt{3k}c_2 + \sqrt{3k}c_4$$

$$1 = c_2 + c_4 \qquad c_4 = 1 - c_2$$

$$y_1 = c_1\cos\sqrt{3k}t + c_2\sin\sqrt{3k}t + (1-c_1)\cos\sqrt{k}t + (1-c_2)\sin\sqrt{k}\,t$$

$$y_1 = c_1(\cos\sqrt{3k}t - \cos\sqrt{k}t) + c_2(\sin\sqrt{3k}t - \sin\sqrt{k}\,t)$$

$y_1'' + 2ky_1 - ky_2 = 0 \quad \ldots\ldots..1$

$y_2 = \frac{1}{k}(y_1'' + 2ky_1)$

$$y_1 = c_1(\cos\sqrt{3k}t - \cos\sqrt{k}t) + c_2(\sin\sqrt{3k}t - \sin\sqrt{k}\,t)$$

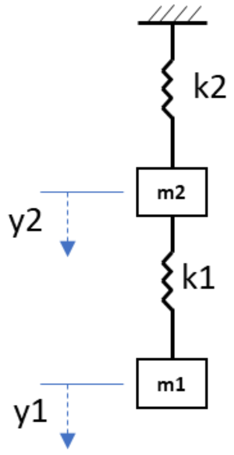
$$y'_1 = c_1(-\sqrt{3k}\sin\sqrt{3k}t + \sqrt{k}\sin\sqrt{k}t) + c_2(\sqrt{3k}\cos\sqrt{3k}t - \sqrt{k}\cos\sqrt{k}\,t)$$

$$y''_1 = c_1(-3k\cos\sqrt{3k}t + k\cos\sqrt{k}t) + c_2(-3k\sin\sqrt{3k}t + k\sin\sqrt{k}\,t)$$

$y_2 = \frac{1}{k}\Big(c_1(-3k\cos\sqrt{3k}t + k\cos\sqrt{k}t) + c_2(-3k\sin\sqrt{3k}t + k\sin\sqrt{k}\,t) + 2k\big(c_1(\cos\sqrt{3k}t - \cos\sqrt{k}t) + c_2(\sin\sqrt{3k}t - \sin\sqrt{k}\,t)\big)\Big)$

Then use the boundary conditions for y2 to find the c1 and c2

Example.4 Write the differential equations of the system in fig.1 below



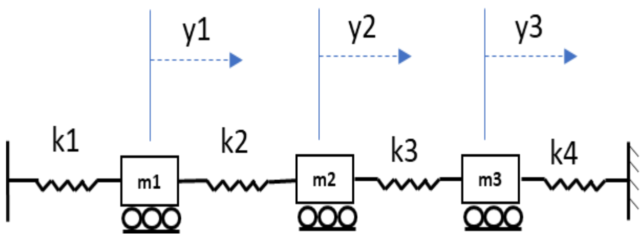
Solution

$$m_1 y_1'' + (k_1)y_1 - k_1 y_2 = 0$$

$$m_1 y_1'' + k_1 y_1 - k_1 y_2 = 0 \quad \ldots\ldots..1$$

$$m_2 y_2'' + (k_2 + k_3)y_2 - k_1 y_1 = 0 \quad \ldots\ldots..2$$

Example.5 Write the differential equations of the system in fig.1 below



Solution

$$m_1 y_1'' + (k_1 + k_2)y_1 - k_2 y_2 = 0 \quad \ldots\ldots.1$$

$$m_2 y_2'' + (k_2 + k_3)y_2 - k_2 y_1 - k_3 y_3 = 0 \quad \ldots\ldots..2$$

$$m_3 y_3'' + (k_3 + k_4)y_3 - k_3 y_2 = 0 \quad \ldots\ldots..3$$

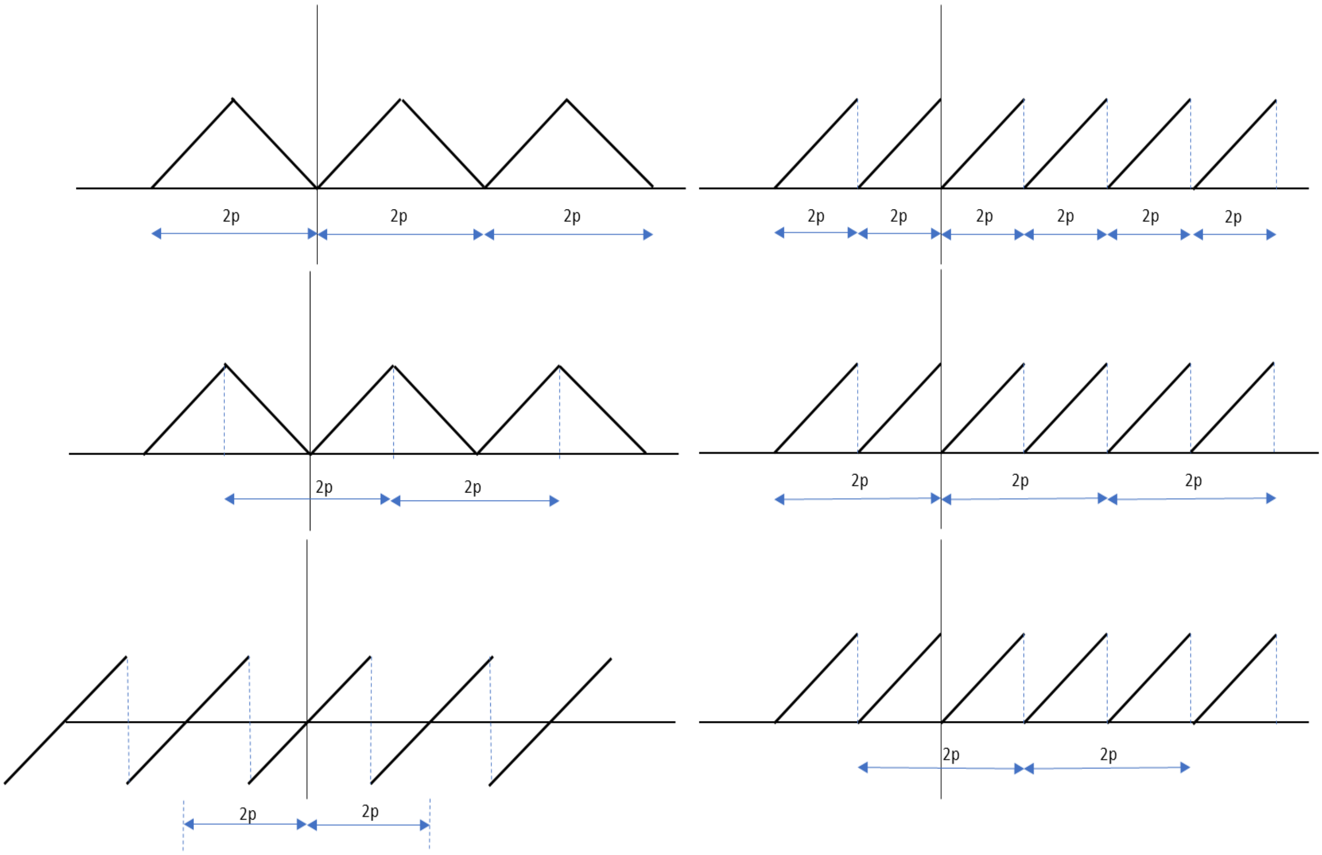
Fourier Series

Fourier series is a way to represent a function as the sum of simple sine wave (sinx, cosx)

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi}{p}x + a_2 \cos\frac{2\pi}{p}x + a_3 \cos\frac{3\pi}{p}x + .. + b_1 \sin\frac{\pi}{p}x + b_2 \sin\frac{2\pi}{p}x + b_3 \sin\frac{3\pi}{p}x + ..$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad n = 1,2,3, \ldots$$

Where $2p = period$



The Euler Coefficients

To obtain formulas for the coefficients a_n and b_n, we need the following integrals which are valid for all values of d

$$\int_{d}^{d+2p} \cos\frac{n\pi}{p}x\,dx = 0 \qquad n \neq 0$$

$$\int_{d}^{d+2p} \sin\frac{n\pi}{p}x\,dx = 0 \qquad n \neq 0$$

$$\int_{d}^{d+2p} \cos\frac{m\pi}{p}x\,\cos\frac{n\pi}{p}x\,dx = 0 \qquad m \neq n$$

$$\int_{d}^{d+2p} \cos^2 \frac{n\pi}{p} x \, dx = p \qquad n \neq 0$$

$$\int_{d}^{d+2p} \cos \frac{n\pi}{p} x \; \sin \frac{n\pi}{p} x \, dx = 0$$

$$\int_{d}^{d+2p} \sin \frac{m\pi}{p} x \; \sin \frac{n\pi}{p} x \, dx = 0 \qquad m \neq n$$

$$\int_{d}^{d+2p} \sin^2 \frac{n\pi}{p} x \, dx = p \qquad n \neq 0$$

to find the a_0

$$f(x) = \frac{a_0}{2} + a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \qquad n = 1,2,3,..$$

Integrate the both sides

$$\int_{d}^{d+2p} f(x)dx = \int_{d}^{d+2p} \frac{a_0}{2} dx + \int_{d}^{d+2p} a_n \cos \frac{n\pi}{p} xdx + \int_{d}^{d+2p} b_n \sin \frac{n\pi}{p} x \, dx$$

$$\int_{d}^{d+2p} f(x)dx = \frac{a_0}{2}(d + 2p - d) + 0 + 0$$

$$a_0 = \frac{1}{p}\int_{d}^{d+2p} f(x)dx$$

to find the a_n we will multiply by $(\cos\frac{m\pi}{p}x)$ and integrate the both sides

$$\int_{d}^{d+2p} f(x)\cos\frac{m\pi}{p}x\,dx = \int_{d}^{d+2p} \frac{a_0}{2}\cos\frac{m\pi}{p}x\,dx + \int_{d}^{d+2p} a_n\cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}xdx + \int_{d}^{d+2p} b_n\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx$$

$$\int_{d}^{d+2p} f(x)\cos\frac{m\pi}{p}x\,dx = 0 + \int_{d}^{d+2p} a_n\cos^2\frac{m\pi}{p}xdx + 0$$

$$\int_{d}^{d+2p} f(x)\cos\frac{m\pi}{p}x\,dx = \int_{d}^{d+2p} a_n\cos^2\frac{m\pi}{p}xdx \qquad let\ m = n$$

$$\int_{d}^{d+2p} f(x)\cos\frac{n\pi}{p}x\,dx = \int_{d}^{d+2p} a_n\cos^2\frac{n\pi}{p}xdx$$

$$\int_{d}^{d+2p} f(x)\cos\frac{n\pi}{p}x\,dx = \int_{d}^{d+2p} a_n(\frac{1}{2}+\frac{1}{2}\cos\frac{2n\pi}{p}x)dx$$

$$\int_{d}^{d+2p} f(x)\cos\frac{n\pi}{p}x\,dx = \int_{d}^{d+2p} a_n(\frac{1}{2}+\frac{1}{2}\cos\frac{2n\pi}{p}x)dx$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(x)\cos\frac{n\pi}{p}x\,dx$$

Now we will multiply by $(\sin\frac{m\pi}{p}x)$ and integrate it to get the b_n

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(x)\sin\frac{n\pi}{p}x\,dx$$

Hint:

For even function $(b_n = 0)$

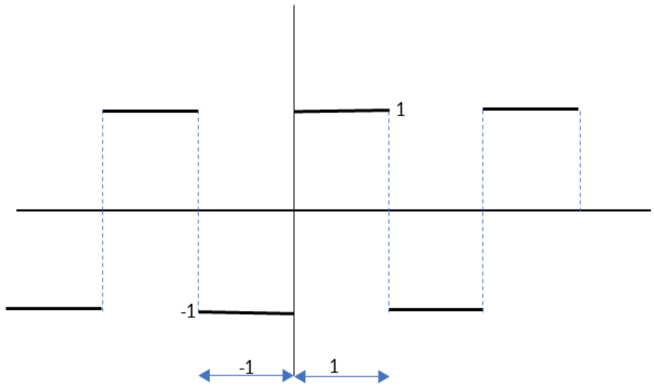
For odd function $(a_0 = 0, a_n = 0)$

Example.1 Find the Fourier series for the following periodic function whose defined in one period is as follows:

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

Solution

$2p = 2 \qquad p = 1$



$$a_0 = \frac{1}{p}\int_d^{d+2p} f(x)dx = \frac{1}{1}\int_{-1}^{1} f(x)dx$$

$$a_0 = \frac{1}{1}\int_{-1}^{0}(-1)dx + \frac{1}{1}\int_0^1 (1)dx = 0$$

$$a_0 = \frac{1}{2}(-x)|_{-1}^{0} + \frac{1}{2}(x)|_0^1$$

$$a_0 = \frac{1}{1}\big((-0) - (-(-1)\big) + \frac{1}{1}(1-0)$$

$$a_0 = 0$$

$$a_n = \frac{1}{p}\int_d^{d+2p} f(x)\cos\frac{n\pi}{p}x\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$a_n = \frac{1}{1}\int_{-1}^{0}(-1)\cos\frac{n\pi}{1}x\,dx + \frac{1}{1}\int_0^1 (1)\cos\frac{n\pi}{1}x\,dx$$

$$a_n = \int_{-1}^{0}(-1)\cos n\pi x\,dx + \int_0^1 (1)\cos n\pi x\,dx$$

$$a_n = (-\frac{1}{n\pi}\sin n\pi x)|_{-1}^{0} + (\frac{1}{n\pi}\sin n\pi x)|_0^1$$

$$a_n = \left(-\frac{1}{n\pi}\sin n\pi(0) - \left(-\frac{1}{n\pi}\sin n\pi(-1)\right)\right) + \left(\frac{1}{n\pi}\sin n\pi(1) - \frac{1}{n\pi}\sin n\pi(0)\right)$$

$$a_n = 0$$

$$b_n = \frac{1}{p}\int_d^{d+2p} f(x)\sin\frac{n\pi}{p}x\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{1}\int_{-1}^{0}(-1)\sin\frac{n\pi}{1}x\,dx + \frac{1}{1}\int_0^1 (1)\sin\frac{n\pi}{1}x\,dx$$

$$b_n = \int_{-1}^{0}(-1)\sin n\pi x\,dx + \int_0^1 (1)\sin n\pi x\,dx$$

$$b_n = \left|\frac{1}{n\pi}\cos n\pi x\right|_{-1}^{0} + \left|\frac{-1}{n\pi}\cos n\pi x\right|_0^1$$

$$b_n = \frac{1}{n\pi}(\cos n\pi 0 - \cos n\pi(-1)) + \frac{1}{n\pi}(-\cos n\pi 1 + \cos n\pi 0)$$

$$b_n = \frac{1}{n\pi}(2 - 2\cos n\pi)$$

$$b_n = \frac{4}{n\pi} \qquad n = 1,3,5, \dots..$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x)$$

$$f(x) = 0 + \sum_{n=1}^{\infty}(0 \cos\frac{n\pi}{p}x + \frac{4}{n\pi}\sin\frac{n\pi}{p}x)$$

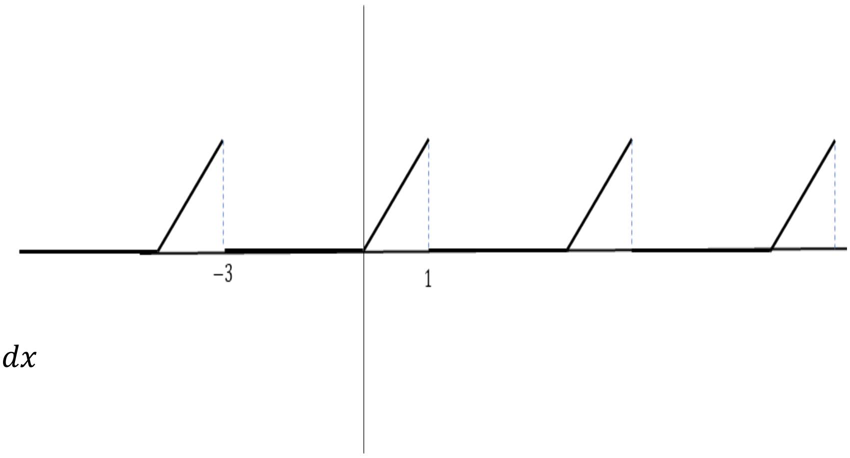
$$f(x) = \sum_{n=1,3,5,.}^{\infty}\frac{4}{n\pi}\sin\frac{n\pi}{1}x$$

$$f(x) = \sum_{n=1,3,5,..}^{\infty}\frac{4}{n\pi}\sin n\pi x$$

$$f(x) = \frac{4}{\pi}\sin \pi x + \frac{4}{3\pi}\sin 3\pi x + \frac{4}{5\pi}\sin 5\pi x + \frac{4}{7\pi}\sin 7\pi x + \frac{4}{9\pi}\sin 9\pi x + \cdots..$$

Example.2 Find the Fourier series for the following periodic function whose defined in one period is as follows:

$$f(x) = \begin{cases} 0 & -3 < x < 0 \\ x & 0 < x < 1 \end{cases}$$



Solution

$2p = 4 \qquad p = 2$

$$a_0 = \frac{1}{p}\int_d^{d+2p} f(x)dx = \frac{1}{2}\int_{-3}^{1} f(x)dx$$

$$a_0 = \frac{1}{2}\int_{-3}^{0} 0dx + \frac{1}{2}\int_0^1 xdx$$

$$a_0 = \frac{1}{2}\int_{-3}^{0} 0dx + \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1$$

$$a_0 = \frac{1}{4}$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(x)\cos\frac{n\pi}{p}x\,dx = \frac{1}{2}\int_{-3}^{1} f(x)\cos\frac{n\pi}{2}x\,dx$$

$$a_n = \frac{1}{2}\int_{-3}^{0} 0\cos\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{0}^{1} x\cos\frac{n\pi}{2}x\,dx$$

$$a_n = \frac{1}{2}\left|\frac{2x}{n\pi}\sin\frac{n\pi}{2}x + \frac{4}{(n\pi)^2}\cos\frac{n\pi}{2}x\right|_0^1$$

$$a_n = \left|\frac{x}{n\pi}\sin\frac{n\pi}{2}x + \frac{2}{(n\pi)^2}\cos\frac{n\pi}{2}x\right|_0^1$$

$$a_n = \frac{1}{n\pi}\sin\frac{n\pi}{2} + \frac{2}{(n\pi)^2}\cos\frac{n\pi}{2} - \frac{2}{(n\pi)^2}$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(x)\sin\frac{n\pi}{p}x\,dx = \frac{1}{2}\int_{-3}^{1} f(x)\sin\frac{n\pi}{2}x\,dx$$

$$b_n = \frac{1}{2}\int_{-3}^{0} 0\sin\frac{n\pi}{p}x\,dx + \frac{1}{2}\int_{0}^{1} x\sin\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{2}\int_{0}^{1} x\sin\frac{n\pi}{p}x\,dx$$

x		$\sin\frac{n\pi}{2}x$
	$+$	
1		$\frac{-2}{n\pi}\cos\frac{n\pi}{2}x$
	$-$	
0		$\frac{-4}{(n\pi)^2}\sin\frac{n\pi}{2}x$

$$b_n = \frac{1}{2}\left|-\frac{2x}{n\pi}\cos\frac{n\pi}{2}x + \frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}x\right|_0^1$$

$$b_n = \left|-\frac{x}{n\pi}\cos\frac{n\pi}{2}x + \frac{2}{(n\pi)^2}\sin\frac{n\pi}{2}x\right|_0^1$$

$$b_n = \frac{2}{(n\pi)^2}\sin\frac{n\pi}{2} - \frac{1}{n\pi}\cos\frac{n\pi}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right)$$

$$f(x) = \frac{1}{8} + \sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin\frac{n\pi}{2} + \frac{2}{(n\pi)^2}\cos\frac{n\pi}{2} - \frac{2}{(n\pi)^2}\right)\cos\frac{n\pi}{2}x + \sum_{n=1}^{\infty}\left(\frac{2}{(n\pi)^2}\sin\frac{n\pi}{2} - \frac{1}{n\pi}\cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x$$

Even and Odd Periodic Function

It is possible to take advantage of the symmetry (even or odd) that may be found in the periodic function to reduce the operation of finding the ($a_0, a_n, and\ b_n$)

For even functions

$b_n = 0$

$$a_0 = \frac{2}{p}\int_0^p f(x)dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

For odd functions

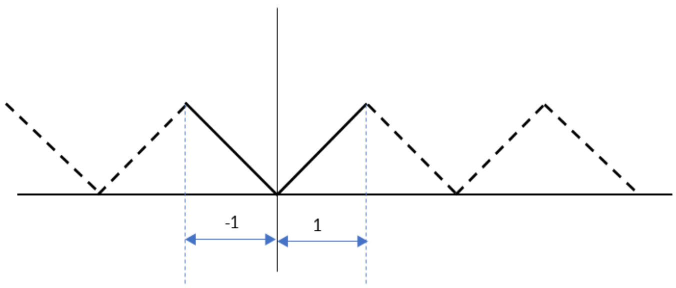
$a_0 = 0$

$a_n = 0$

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

Example.3 Find the Fourier series for the following periodic function whose defined in one period is as follows:

$f(x) = |x| \qquad -1 < x < 1$



Solution

$2p = 2 \qquad p = 1$

The function is even ($b_n = 0$)

$$a_0 = \frac{2}{p}\int_0^p f(x)dx = \frac{2}{1}\int_0^1 xdx$$

$$a_0 = 2\left|\frac{x^2}{2}\right|_0^1 = 1$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi}{1}x\,dx = 2\int_0^1 x\cos n\pi x\,dx$$

$$a_n = 2\left|\frac{x}{n\pi}\sin n\pi x + \frac{1}{(n\pi)^2}\cos n\pi x\right|_0^1$$

$$a_n = 2\left(\frac{1}{(n\pi)^2}\cos n\pi - \frac{1}{(n\pi)^2}\right) = \frac{-4}{(n\pi)^2} \qquad n = 1,3,5,7,..$$

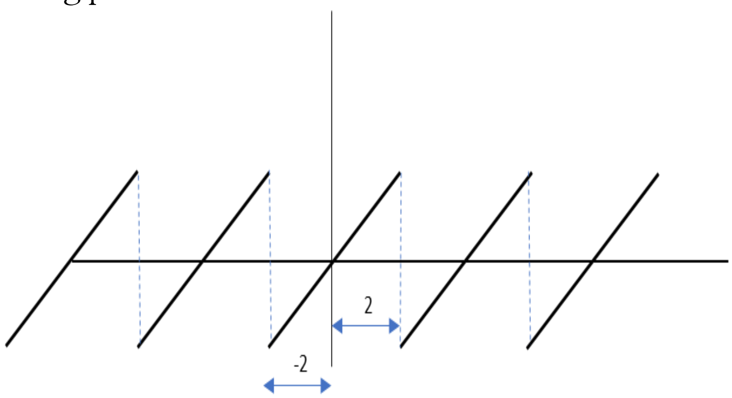
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x)$$

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,..}^{\infty}(a_n\cos\frac{n\pi}{p}x + 0\sin\frac{n\pi}{p}x)$$

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,..}^{\infty} -\frac{4}{(n\pi)^2}\cos n\pi x$$

Example.4 Find the Fourier series for the following periodic function whose defined in one period is as follows:

$f(x) = x \qquad -2 < x < 2$



Solution

$2p = 4 \qquad p = 2$

The function is odd ($a_0 = 0, a_n = 0$)

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

x $\quad$ $\sin\frac{n\pi}{2}x$

$+$

1 $\quad$ $\frac{-2}{n\pi}\cos\frac{n\pi}{2}x$

$-$

$$b_n = \frac{2}{2}\int_0^2 x \sin\frac{n\pi}{2}x\,dx = \int_0^2 x \sin\frac{n\pi}{2}x\,dx$$

$$b_n = \left| -\frac{2x}{n\pi}\cos\frac{n\pi}{2}x + \frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}x \right|_0^2$$

$$b_n = (-\frac{4}{n\pi}\cos n\pi)$$

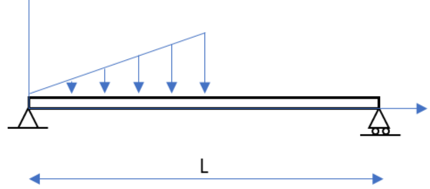
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

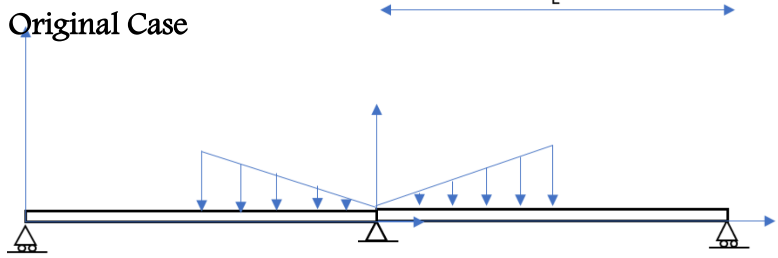
$$f(x) = \sum_{n=1}^{\infty}(-\frac{4}{n\pi}\cos n\pi)\sin\frac{n\pi}{2}x$$

Half Range Expansions

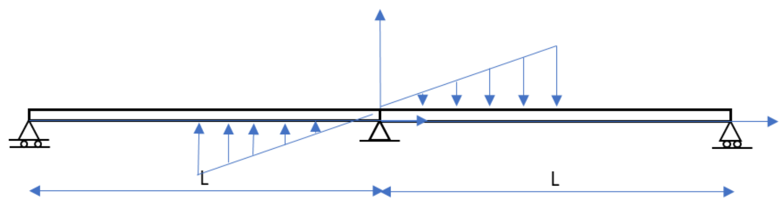
The Fourier series can be used in many applications, depending on the same idea of symmetry about the y-axis and about the origin, which was explained previously, and as shown



Original Case



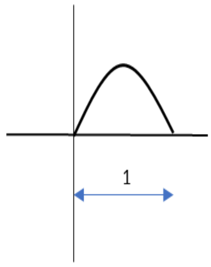
Assumed Case (symmetry about y-axis)



Assumed Case (symmetry about origin)

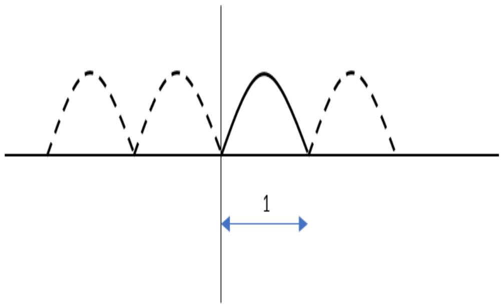
Example.1 Find the half range expansion for the following function

$f(x) = x - x^2 \qquad 0 < x < 1$



Solution

Firstly, we will solve it by assuming that the function is symmetric about the (y-axis)



$b_n = 0$

$2p = 2 \qquad p = 1$

$$a_0 = \frac{2}{p}\int_0^p f(x)dx = \frac{2}{1}\int_0^1 (x - x^2)dx$$

$$a_0 = 2\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$$a_n = \frac{2}{1}\int_0^1 (x - x^2)\cos\frac{n\pi}{1}x\,dx$$

$$a_n = 2\int_0^1 (x - x^2)\cos n\pi x\,dx$$

$$a_n = 2\left|\frac{x - x^2}{n\pi}\sin n\pi x + \frac{1 - 2x}{(n\pi)^2}\cos n\pi x + \frac{2}{(n\pi)^3}\sin n\pi x\right|_0^1$$

$$a_n = 2\left|\left(0\sin n\pi + \frac{1 - 2}{(n\pi)^2}\cos n\pi + \frac{2}{(n\pi)^3}\sin n\pi\right) - \left(0 + \frac{(1 - 0)}{(n\pi)^2}\cos 0 + 0\right)\right|$$

$$a_n = 2\left(-\frac{1}{(n\pi)^2}\cos n\pi - \frac{1}{(n\pi)^2}\right)$$

$$a_n = -\frac{2}{(n\pi)^2}(1 + \cos n\pi)$$

$$a_n = -\frac{4}{(n\pi)^2} \qquad n = 2,4,6, \ldots.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

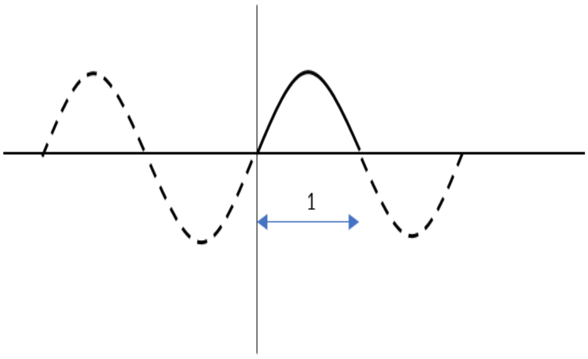
$$f(x) = \frac{1}{6} + \sum_{n=2,4,6}^{\infty} -\frac{4}{(n\pi)^2}\cos\frac{n\pi}{1}x$$

$$f(x) = \frac{1}{6} + \sum_{n=2,4,6}^{\infty} -\frac{4}{(n\pi)^2}\cos n\pi x$$

$$f(x) = \frac{1}{6} - \frac{4}{\pi^2}\left(\frac{1}{4}\cos 2\pi x + \frac{1}{16}\cos 4\pi x + \frac{1}{36}\cos 6\pi x + \cdots\right)$$

$x - x^2$	+	$\cos n\pi x$
$1 - 2x$	−	$\frac{1}{n\pi}\sin n\pi x$
-2	+	$\frac{-1}{(n\pi)^2}\cos n\pi x$
0		$\frac{-1}{(n\pi)^3}\sin n\pi x$

The second solution by assuming that the function is symmetric about the origin



$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{2}{1}\int_0^1 (x - x^2)\sin\frac{n\pi}{1}x\,dx$$

$$b_n = 2\int_0^1 (x - x^2)\sin n\pi x\,dx$$

$$b_n = 2\left|-\frac{(x - x^2)}{n\pi}\cos n\pi x + \frac{1 - 2x}{(n\pi)^2}\sin n\pi x - \frac{2}{(n\pi)^3}\cos n\pi x\right|_0^1$$

$$b_n = 2\left(\left(0 + 0 - \frac{2}{(n\pi)^3}\cos n\pi\right) - \left(0 + 0 - \frac{2}{(n\pi)^3}\right)\right)$$

$$b_n = \frac{4}{(n\pi)^3}(1 - \cos n\pi) = \frac{8}{(n\pi)^3} \qquad n = 1,3,5, \ldots$$

$x - x^2$	+	$\sin n\pi x$
$1 - 2x$	−	$\frac{-1}{n\pi}\cos n\pi x$
-2	+	$\frac{-1}{(n\pi)^2}\sin n\pi x$
0		$\frac{1}{(n\pi)^3}\cos n\pi x$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi}{p} x + b_n \sin\frac{n\pi}{p} x)$$

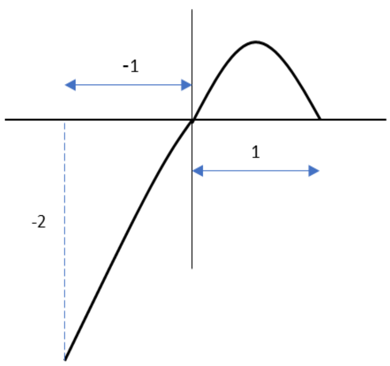
$$f(x) = \sum_{n=1,3,5,..}^{\infty} b_n \sin\frac{n\pi}{1} x$$

$$f(x) = \sum_{n=1,3,5,..}^{\infty} \frac{8}{(n\pi)^3} \sin n\pi x$$

$$f(x) = \frac{8}{\pi^3}\left(\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{27} + \frac{\sin 5\pi x}{125} + \cdots\right)$$

اما اذا اردنا إيجاد متسلسلة فوريير عندما تكون غير متناظرة لا حول المحور الصادي ولا حو نقطة الأصل ففي هذه الحالة سيتم إيجاد جميع المعاملات للمتسلسلة من غير ان تكون هناك قيمة صفرية ولعمل ذلك توجد طريقتان

1- توسيع الدالة لمجال غبر المجال المعطى وكما موضح



$2p = 2 \qquad p = 1$

$$a_0 = \frac{1}{p}\int_{-1}^{1} f(x)dx = \frac{1}{1}\int_{-1}^{1} (x - x^2)dx$$

$$a_n = \frac{1}{p}\int_{-1}^{1} f(x) \cos\frac{n\pi}{p} x\, dx$$

$$a_n = \frac{1}{1}\int_{-1}^{1} (x - x^2) \cos\frac{n\pi}{1} x\, dx$$

$$a_n = \int_{-1}^{1} (x - x^2) \cos n\pi x\, dx$$

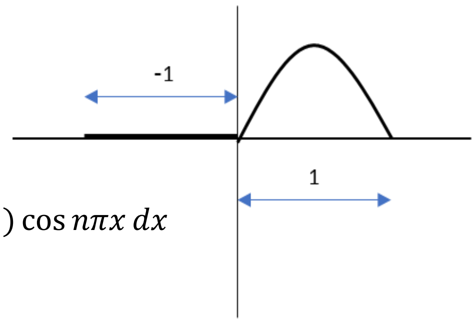
$$b_n = \frac{1}{p}\int_{-1}^{1} f(x) \sin\frac{n\pi}{p} x\, dx$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x) \sin\frac{n\pi}{1} x\, dx$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin n\pi x\, dx$$

ثم نكمل الحل من خلال إيجاد التكاملات

2- هو ان نضيف على جهة اليمين دالة صفرية وكما موضح بالشكل



$2p = 2 \qquad p = 1$

$$a_0 = \frac{1}{p}\int_{-1}^{1} f(x)dx = \int_{-1}^{0} 0dx + \int_{-1}^{0}(x - x^2)dx$$

$$a_n = \frac{1}{p}\int_{-1}^{1} f(x)\cos\frac{n\pi}{p}x\,dx = \frac{1}{1}\int_{-1}^{0} 0\cos\frac{n\pi}{1}x\,dx + \frac{1}{1}\int_{01}^{1}(x - x^2)\cos n\pi x\,dx$$

$$b_n = \frac{1}{p}\int_{-1}^{1} f(x)\sin\frac{n\pi}{p}x\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi}{1}x\,dx$$

$$b_n = \int_{-1}^{0} 0\sin n\pi x\,dx + \int_{0}^{1}(x - x^2)\sin n\pi x\,dx +$$

ومن ثم نكمل الحل

Applications of Fourier Series

1-Deflection of Simply supported Beam:

The differential equations which used here are:

$$EI\frac{d^4y}{dx^4} = q(x) \qquad Load$$

$$EI\frac{d^3y}{dx^3} = V(x) \qquad Shear\ Force$$

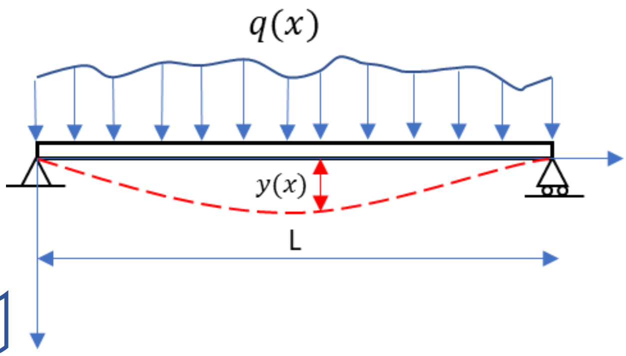
$$EI\frac{d^2y}{dx^2} = M(x) \qquad Bending\ Moment$$

The differential equations above would be the beginning to calculate the deflection in the beam by choosing one after which several sequential integrals would be made to reach the deflection

. And to find the deflection equation of the simply supported beam, as shown in fig.1 and according to the following boundaries conditions

$M(0) = 0 \qquad M(L) = 0$

$y(0) = 0 \qquad y(L) = 0$



$$EIy'''' = b_n \sin\frac{n\pi}{L}x$$

$$EIy''' = \frac{-L}{n\pi}\mathrm{b_n}\cos\frac{n\pi}{L}x + c_1$$

$$EIy'' = -\left(\frac{L}{n\pi}\right)^2 b_n \sin\frac{n\pi}{L}x + c_1 x + c_2$$

$at\ x = 0 \quad EIy'' = M = 0$

$at\ x = L \quad EIy'' = M = 0$

$$0 = -\left(\frac{L}{n\pi}\right)^2 b_n \sin\frac{n\pi}{L}(0) + c_1\,(0) + c_2 \qquad c_2 = 0$$

$$0 = -\left(\frac{L}{n\pi}\right)^2 b_n \sin\frac{n\pi}{L}L + c_1\,L \qquad c_1 = 0$$

Hence

$$EIy'' = -\left(\frac{L}{n\pi}\right)^2 b_n \sin\frac{n\pi}{L}x$$

$$EIy' = \left(\frac{L}{n\pi}\right)^3 \mathrm{b_n}\cos\frac{n\pi}{L}x + c_3$$

$$EIy = \left(\frac{L}{n\pi}\right)^4 b_n \sin\frac{n\pi}{L}x + c_3 x + c_4$$

$at\ x = 0 \quad y = 0$

$at\ x = L \quad y = 0$

$$0 = \left(\frac{L}{n\pi}\right)^4 \mathrm{b_n}\sin\frac{n\pi}{L}0 + c_3 0 + c_4 \qquad c_4 = 0$$

$$0 = \left(\frac{L}{n\pi}\right)^4 b_n \sin\frac{n\pi}{L}L + c_3 L \qquad c_3 = 0$$

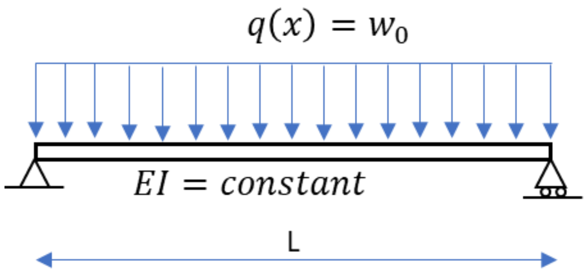
$$EIy = \left(\frac{L}{n\pi}\right)^4 b_n \sin\frac{n\pi}{L}x$$

$$\boldsymbol{y = \frac{1}{EI}\left(\frac{L}{n\pi}\right)^4 b_n \sin\frac{n\pi}{L}x}$$

So that

$$b_n = \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi}{L}x\ dx$$

Example.1 Find the deflection equation at any point by using the Fourier series



$$b_n = \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi}{L}x\ dx$$

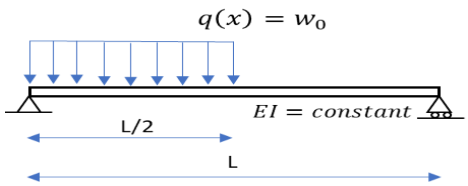
$$b_n = \frac{2}{L}\int_0^L w_0\sin\frac{n\pi}{L}x\ dx = \left| -\frac{2w_0}{L}\left(\frac{L}{n\pi}\right)\cos\frac{n\pi}{L}x\right|_0^L$$

$$b_n = -\frac{2w_0}{n\pi}(\cos n\pi - 1) = \frac{4w_0}{n\pi} \qquad n = 1,3,5,\dots$$

$$y = \frac{1}{EI}\left(\frac{L}{n\pi}\right)^4 b_n\sin\frac{n\pi}{L}x$$

$$y = \frac{1}{EI}\left(\frac{L}{n\pi}\right)^4\frac{4w_0}{n\pi}\sin\frac{n\pi}{L}x = \frac{4w_0}{EI}\frac{L^4}{(n\pi)^5}\sin\frac{n\pi}{L}x \qquad n = 1,3,5,\dots.$$

Example.2 Find the deflection equation at any point by using the Fourier series



$$b_n = \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi}{L}x\ dx$$

$$b_n = \frac{2}{L}\int_0^{L/2} w_0\sin\frac{n\pi}{L}x\ dx + \frac{2}{L}\int_{L/2}^L 0\sin\frac{n\pi}{L}x\ dx = \left| -\frac{2w_0}{L}\left(\frac{L}{n\pi}\right)\cos\frac{n\pi}{L}x\right|_0^{L/2}$$

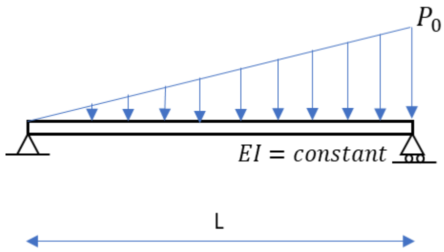
$$b_n = -\frac{2w_0}{n\pi}(\cos\frac{n\pi}{2} - 1) \qquad n = 1,2,3,\dots$$

$$y = \frac{1}{EI}\left(\frac{L}{n\pi}\right)^4 b_n\sin\frac{n\pi}{L}x$$

$$y = \frac{1}{EI}\left(\frac{L}{n\pi}\right)^4\left(-\frac{2w_0}{n\pi}(\cos\frac{n\pi}{2} - 1)\right)\sin\frac{n\pi}{L}x$$

$$y = \frac{-2w_0}{EI}\frac{L^4}{(n\pi)^5}\left((\cos\frac{n\pi}{2} - 1)\right)\sin\frac{n\pi}{L}x$$

Example.3 Find the deflection equation at any point by using the Fourier series



$$b_n = \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi}{L}x\ dx$$

$$b_n = \frac{2}{L}\int_0^L \left(\frac{P_0}{L}x\right)\sin\frac{n\pi}{L}x\ dx$$

$$b_n = \frac{2P_o}{L^2}\int_0^L x\sin\frac{n\pi}{L}x\ dx$$

x	+	$\sin\frac{n\pi}{L}x$
1	−	$-\frac{L}{n\pi}\cos\frac{n\pi}{L}x$
0		$-\left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi}{L}x$

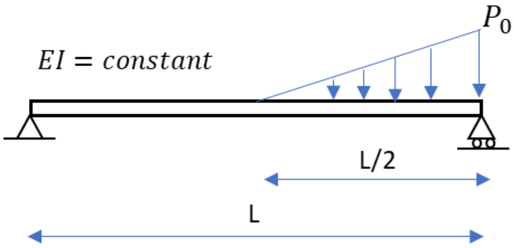
$$b_n = \frac{2P_o}{L^2}\left|-\frac{xL}{n\pi}\cos\frac{n\pi}{L}x + \left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi}{L}x\right|_0^L$$

$$b_n = \frac{-2P_o}{L^2}\frac{L^2}{n\pi}\cos n\pi = -\frac{2P_0}{n\pi}\cos n\pi$$

$$\boldsymbol{y = \frac{1}{EI}\left(\frac{L}{n\pi}\right)^4 b_n\sin\frac{n\pi}{L}x}$$

$$y = \frac{1}{EI}\left(\frac{L}{n\pi}\right)^4\left(-\frac{2P_0}{n\pi}\cos n\pi\right)\sin\frac{n\pi}{L}x = -\frac{2P_0}{EI}\frac{L^4}{(n\pi)^5}\cos n\pi\sin\frac{n\pi}{L}x$$

Example.4 Find the deflection equation at any point by using the Fourier series



$$b_n = \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi}{L}x\ dx$$

$$q(x) = \frac{P_0}{\frac{L}{2}}x + B$$

$$q(x) = \frac{2P_0}{L}x + B$$

$$at\ x = \frac{L}{2} \qquad q = 0$$

$$0 = \frac{2P_0}{L}\left(\frac{L}{2}\right) + B \qquad B = -P_0$$

$x-\frac{L}{2}$ $\sin\frac{n\pi}{L}x$

$$q(x) = \frac{2P_0}{L}x - P_0 = \frac{2P_0}{L}(x - \frac{L}{2})$$

$$b_n = \frac{2}{L}\int_0^{L/2} 0 \sin\frac{n\pi}{L}x \ dx + \frac{2}{L}\int_{L/2}^{L} \left(\frac{2P_0}{L}(x - \frac{L}{2})\right) \sin\frac{n\pi}{L}x \ dx$$

$$b_n = \frac{4P_0}{L^2}\int_{L/2}^{L} \left(x - \frac{L}{2}\right) \sin\frac{n\pi}{L}x \ dx$$

$$b_n = \frac{4P_0}{L^2}\left|\left(x - \frac{L}{2}\right)\left(\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\right) + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi}{L}x\right|_{L/2}^{L}$$

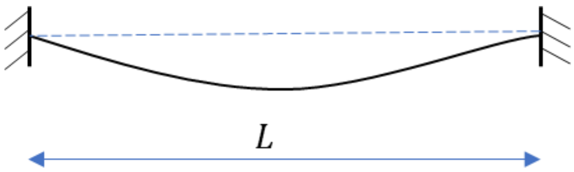
Then complete the solution

Partial Differential Equations

1- The Wave equation

We will consider a vertical string of length L that has been tightly stretched between two points at $x = 0$ and $x = L$

Because the string has been tightly stretched, we can assume that the slope of the displaced string at any point is small. Further, in most cases the only external force that will act upon the string is gravity and if the string light enough the effects of gravity on the vertical displacement will be small. This leads to

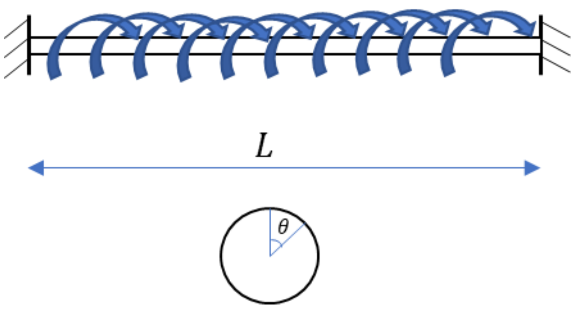


$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Where $a^2 = \frac{T.g}{w} = constant$

$a = velocity\ of\ the\ wave$

2-Torsional Vibration of Circular Shaft



$$\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2}$$

Where $a^2 = \frac{G.g}{\gamma} = constant$

$\theta = twisting\ angle$

3- Heat Flow

$$a^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad One - Dimensional\ Heat\ Flow$$

$$a^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad Two - Dimensional\ Heat\ Flow$$

$$a^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad Three - Dimensional\ Heat\ Flow$$

4- Consolidation Differential Equation

$$\frac{\partial u}{\partial t} = C_{vx} \frac{\partial^2 u}{\partial x^2} \qquad One - Dimensional\ Consolidation$$

$$\frac{\partial u}{\partial t} = C_{vx} \frac{\partial^2 u}{\partial x^2} + C_{vy} \frac{\partial^2 u}{\partial y^2} \qquad Two - Dimensional\ Consolidation$$

$$\frac{\partial u}{\partial t} = C_{vx} \frac{\partial^2 u}{\partial x^2} + C_{vy} \frac{\partial^2 u}{\partial y^2} + C_{vz} \frac{\partial^2 u}{\partial z^2} \qquad Three - Dimensional\ Consolidation$$

Method of Separation of Variables

In mathematics, separation of variables (also known as the Fourier method) is any of several methods for solving ordinary and partial differential equations, in which algebra allows one to rewrite an equation so that each of two variables occurs on a different side of the equation.

Example.1 Solve the following differential equation

$$\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2} \qquad For\ all \quad 0 < x < L \quad and\ t > 0$$

The boundary conditions are

$\theta(0,t)=0 \qquad For\ all\ t>0$

$\theta(L,t)=0 \qquad For\ all\ \ t>0$

Initial Conditions

$\theta(x,0)=f(x) \qquad 0<x<L$

$\frac{\partial\theta}{\partial t}(x,0)=g(x) \qquad 0<x<L$

Solution

Let $\theta(x,t)=X(x)T(t)$

$\frac{\partial\theta}{\partial x}=X'(x)T(t)$

$\frac{\partial^2\theta}{\partial x^2}=X''(x)T(t)$

$\frac{\partial\theta}{\partial t}=X(x)T'(t)$

$\frac{\partial^2\theta}{\partial t^2}=X(x)T''(t)$

$$\frac{\partial^2\theta}{\partial t^2}=a^2\frac{\partial^2\theta}{\partial x^2}$$

$$X(x)T''(t)=a^2X''(x)T(t)$$

$$\frac{T''(t)}{T(t)}=a^2\frac{X''(x)}{X(x)}$$

$$\frac{T''(t)}{T(x)}=\mu$$

$$T''(t)=\mu T(t)$$

$$T''(t)-\mu T(t)=0$$

$$m^2-\mu=0$$

$$m_{1,2}=\pm\sqrt{\mu}$$

$$T(x)=c_1e^{\sqrt{\mu}\,t}+c_2e^{-\sqrt{\mu}\,t}$$

$$a^2\frac{X''(x)}{X(x)}=\mu$$

$$a^2X''(x) = \mu X(x)$$

$$a^2X''(x) - \mu X(x) = 0$$

$$a^2m^2 - \mu = 0$$

$$m_{1,2} = \pm\frac{\sqrt{\mu}}{a}$$

$$X(x) = c_3e^{\frac{\sqrt{\mu}}{a}x} + c_4e^{-\frac{\sqrt{\mu}}{a}x}$$

$$\theta(x,t) = X(x)T(t)$$

$$\theta(x,t) = \left(c_3e^{\frac{\sqrt{\mu}}{a}x} + c_4e^{-\frac{\sqrt{\mu}}{a}x}\right)\left(c_1e^{\sqrt{\mu}\,t} + c_2e^{-\sqrt{\mu}\,t}\right)$$

Case 1: $\boldsymbol{\mu > 0}$

$$\theta(x,t) = \left(c_3e^{\frac{\sqrt{\mu}}{a}x} + c_4e^{-\frac{\sqrt{\mu}}{a}x}\right)\left(c_1e^{\sqrt{\mu}\,t} + c_2e^{-\sqrt{\mu}\,t}\right)$$

This solution is rejected

Case 2: $\boldsymbol{\mu = 0}$

$$T''(t) = \mu T(t)$$

$$T''(t) - 0T(t) = 0$$

$$m^2 - 0 = 0$$

$$m_{1,2} = 0$$

$$T(t) = c_1 + c_2t$$

$$a^2\frac{X''(x)}{X(x)} = \mu$$

$$a^2X''(x) = 0X(x)$$

$$a^2X''(x) - 0X(x) = 0$$

$$a^2m^20 = 0$$

$$m_{1,2} = 0$$

$$X(x) = c_3 + c_4x$$

$$\theta(x,t) = X(x)T(t)$$

$$\theta(x,t) = (c_3 + c_4 x)(c_1 + c_2 t)$$

This solution is rejected

<u>Case 3:</u> $\boldsymbol{\mu < 0}$

$$\frac{T''(t)}{T(x)} = -\lambda^2$$

$$T''(t) = -\lambda^2 T(t)$$

$$T''(t) + \lambda^2 T(t) = 0$$

$$m^2 + \lambda^2 = 0$$

$$m_{1,2} = \pm \lambda\, i$$

$$T(x) = c_1 \cos \lambda\, t + c_2 \sin \lambda\, t$$

$$a^2 \frac{X''(x)}{X(x)} = -\lambda^2$$

$$a^2 X''(x) = -\lambda^2 X(x)$$

$$a^2 X''(x) + \lambda^2 X(x) = 0$$

$$a^2 m^2 + \lambda^2 = 0$$

$$m_{1,2} = \pm \frac{\lambda}{a} i$$

$$X(x) = c_3 \cos \frac{\lambda}{a} x + c_4 \sin \frac{\lambda}{a} x$$

$$\theta(x,t) = X(x)T(t)$$

$$\theta(x,t) = (c_1 \cos \lambda\, t + c_2 \sin \lambda\, t)\left(c_3 \cos \frac{\lambda}{a} x + c_4 \sin \frac{\lambda}{a} x\right)$$

Now we will use the boundary conditions

$\theta(0,t) = 0$ $\qquad$ *For all* $t > 0$

$\theta(L,t) = 0$ $\qquad$ *For all* $t > 0$

$$\theta(x,t) = (c_1 \cos \lambda\, t + c_2 \sin \lambda\, t)\left(c_3 \cos \frac{\lambda}{a} x + c_4 \sin \frac{\lambda}{a} x\right)$$

$$0 = (c_1 \cos \lambda\, t + c_2 \sin \lambda\, t)\left(c_3 \cos \frac{\lambda}{a} 0 + c_4 \sin \frac{\lambda}{a} 0\right)$$

$$0 = (c_1 \cos \lambda t + c_2 \sin \lambda t)(c_3) \ \rightarrow c_3 = 0$$

$$\theta(x,t) = (c_1 \cos \lambda t + c_2 \sin \lambda t)\left(c_4 \sin \frac{\lambda}{a} x\right)$$

$$0 = (c_1 \cos \lambda t + c_2 \sin \lambda t)\left(c_4 \sin \frac{\lambda}{a} L\right)$$

$$c_4 \sin \frac{\lambda}{a} L = 0$$

$$\sin \frac{\lambda}{a} L = 0 \ \ \rightarrow \frac{\lambda}{a} L = n\pi \ \ \rightarrow \lambda = \frac{n\pi a}{L}$$

$$\theta(x,t) = (c_1 \cos \lambda t + c_2 \sin \lambda t)\left(c_4 \sin \frac{\frac{n\pi a}{L}}{a} x\right)$$

$$\theta(x,t) = \left(c_1 \cos \frac{n\pi a}{L} t + c_2 \sin \frac{n\pi a}{L} t\right)\left(c_4 \sin \frac{n\pi}{L} x\right)$$

$$\theta(x,t) = \left(c_1 \cos \frac{n\pi a}{L} t + c_2 \sin \frac{n\pi a}{L} t\right)\left(\sin \frac{n\pi}{L} x\right)$$

$$\theta(x,t) = \left(\sin \frac{n\pi}{L} x\right)\left(c_1 \cos \frac{n\pi a}{L} t + c_2 \sin \frac{n\pi a}{L} t\right)$$

$$\theta(x,t) = \left(\sin \frac{n\pi}{L} x\right)\left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t\right)$$

Now we will use the initial conditions

$\theta(x,0) = f(x) \qquad\qquad 0 < x < L$

$\frac{\partial \theta}{\partial t}(x,0) = g(x) \qquad\qquad 0 < x < L$

$$\theta(x,t) = \left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t\right)\left(\sin \frac{n\pi}{L} x\right)$$

$$f(x) = \left(A_n \cos \frac{n\pi a}{L} 0 + B_n \sin \frac{n\pi a}{L} 0\right)\left(\sin \frac{n\pi}{L} x\right)$$

$$f(x) = A_n \left(\sin \frac{n\pi}{L} x\right)$$

According to Fourier series

$$\boldsymbol{A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx}$$

Use the second initial boundary

$$\theta(x,t) = \left(A_n\cos\frac{n\pi a}{L}t + B_n\sin\frac{n\pi a}{L}t\right)\left(\sin\frac{n\pi}{L}x\right)$$

$$\frac{\partial\theta}{\partial t} = \left(-A_n\frac{n\pi a}{L}\sin\frac{n\pi a}{L}t + B_n\frac{n\pi a}{L}\cos\frac{n\pi a}{L}t\right)\left(\sin\frac{n\pi}{L}x\right)$$

$$g(x) = \left(-A_n\frac{n\pi a}{L}\sin\frac{n\pi a}{L}0 + B_n\frac{n\pi a}{L}\cos\frac{n\pi a}{L}0\right)\left(\sin\frac{n\pi}{L}x\right)$$

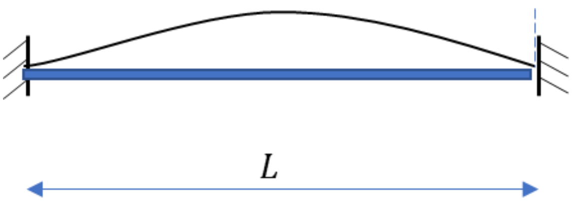
$$g(x) = B_n\frac{n\pi a}{L}\left(\sin\frac{n\pi}{L}x\right)$$

According to Fourier series

$$B_n\frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}xdx$$

$$\boldsymbol{B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}xdx}$$

Example.2 Find the solution of this problem (torsion)



The boundary conditions are

$\theta(0,t) = 0$ $For\ all\ t > 0$

$\theta(L,t) = 0$ $For\ all\ \ t > 0$

Initial Conditions

$\theta(x,0) = x(L-x)$ $0 < x < L$

$\frac{\partial\theta}{\partial t}(x,0) = 0 \quad (initial\ velocity)$ $0 < x < L$

$$\theta(x,t) = \left(A_n\cos\frac{n\pi a}{L}t + B_n\sin\frac{n\pi a}{L}t\right)\left(\sin\frac{n\pi}{L}x\right)$$

$xL - x^2$		$\sin\frac{n\pi}{L}x$
	$+$	
$L - 2x$		$\frac{-L}{n\pi}\cos\frac{n\pi}{L}x$
	$-$	
-2		$\frac{-L^2}{(n\pi)^2}\sin\frac{n\pi}{L}x$
	$+$	
0		$\frac{L^3}{(n\pi)^3}\cos\frac{n\pi}{L}x$

$$\boldsymbol{A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx}$$

$$A_n = \frac{2}{L}\int_0^L x(L-x)\sin\frac{n\pi}{L}x\,dx$$

$$A_n = \frac{2}{L}\left|-\frac{L}{n\pi}(xL-x^2)\cos\frac{n\pi}{L}x + \frac{L^2}{(n\pi)^2}(L-2x)\sin\frac{n\pi}{L}x - 2*\frac{L^3}{(n\pi)^3}\cos\frac{n\pi}{L}x\right|_0^L$$

$$A_n = \frac{2}{L}\left(2\frac{L^3}{(n\pi)^3} - 2*\frac{L^3}{(n\pi)^3}\cos n\pi\right)$$

$$A_n = \frac{4L^2}{(n\pi)^3}(1-\cos n\pi) = \frac{8L^2}{(n\pi)^3} \qquad n = 1,3,5, \ldots.$$

$$\boldsymbol{B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}xdx}$$

$$B_n = \frac{2}{n\pi a}\int_0^L 0\sin\frac{n\pi}{L}xdx$$

$$B_n = 0$$

$$\theta(x,t) = \left(A_n\cos\frac{n\pi a}{L}t + B_n\sin\frac{n\pi a}{L}t\right)\left(\sin\frac{n\pi}{L}x\right)$$

$$\theta(x,t) = \left(A_n\cos\frac{n\pi a}{L}t\right)\left(\sin\frac{n\pi}{L}x\right)$$

$$\theta(x,t) = \frac{8L^2}{(n\pi)^3}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x$$

$$\theta(x,t) = \frac{8L^2}{\pi^3}\left(\frac{1}{\mathrm{n}^3}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x\right)$$

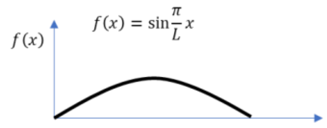
$$\theta(x,t) = \frac{8L^2}{\pi^3}\left(\frac{1}{1^3}\cos\frac{1\pi a}{L}t\sin\frac{1\pi}{L}x + \frac{1}{3^3}\cos\frac{3\pi a}{L}t\sin\frac{3\pi}{L}x + \frac{1}{5^3}\cos\frac{5\pi a}{L}t\sin\frac{5\pi}{L}x + \cdots\right)$$

Example.3 Find the solution of this problem

The boundary conditions are

$\theta(0,t)=0 \qquad For\ all\ t>0$

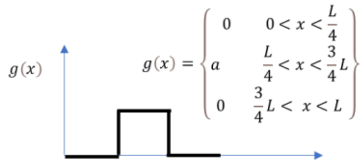
$\theta(L,t)=0 \qquad For\ all\ \ t>0$



Initial Conditions

$\theta(x,0)=\sin\frac{\pi}{L}x \qquad 0<x<L$

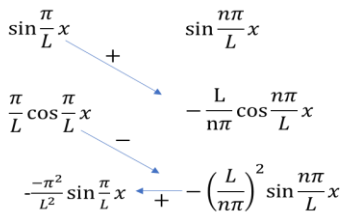
$$\frac{\partial\theta}{\partial t}(x,0)=g(x)=\begin{cases}0 & 0<x<\frac{L}{4}\\ a & \frac{L}{4}<x<\frac{3}{4}L\\ 0 & \frac{3}{4}L<x<L\end{cases}$$



$$\theta(x,t)=\left(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t\right)\left(\sin\frac{n\pi}{L}x\right)$$

$$\boldsymbol{A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx}$$

$$A_n=\frac{2}{L}\int_0^L \sin\frac{\pi}{L}x\sin\frac{n\pi}{L}xdx$$



$$\int_0^L\sin\frac{\pi}{L}x\sin\frac{n\pi}{L}xdx=\left|\left(-\frac{L}{n\pi}\right)\sin\frac{\pi}{L}x\cos\frac{n\pi}{L}x+\left(\frac{\pi}{L}\right)\left(\frac{L}{n\pi}\right)^2\cos\frac{\pi}{L}x\sin\frac{n\pi}{L}x\right|_0^L+\int_0^L\left(\frac{\pi^2}{L^2}\right)\left(\frac{L}{n\pi}\right)^2\sin\frac{\pi}{L}x\sin\frac{n\pi}{L}xdx$$

$$\int_0^L\sin\frac{\pi}{L}x\sin\frac{n\pi}{L}xdx=\left|\left(-\frac{L}{n\pi}\right)\sin\frac{\pi}{L}x\cos\frac{n\pi}{L}x+\left(\frac{\pi}{L}\right)\left(\frac{L}{n\pi}\right)^2\cos\frac{\pi}{L}x\sin\frac{n\pi}{L}x\right|_0^L+\int_0^L\frac{1}{n^2}\sin\frac{\pi}{L}x\sin\frac{n\pi}{L}xdx$$

$$\int_0^L\sin\frac{\pi}{L}x\sin\frac{n\pi}{L}xdx-\int_0^L\frac{1}{n^2}\sin\frac{\pi}{L}x\sin\frac{n\pi}{L}xdx=\left|\left(-\frac{L}{n\pi}\right)\sin\frac{\pi}{L}x\cos\frac{n\pi}{L}x+\left(\frac{\pi}{L}\right)\left(\frac{L}{n\pi}\right)^2\cos\frac{\pi}{L}x\sin\frac{n\pi}{L}x\right|_0^L$$

ثم نكمل الحل

One-Dimensional Heat Equation or Consolidation

$$a^2\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

$$u(x,t)=X(x)T(t)$$

$$\frac{\partial^2 u}{\partial x^2}=X''(x)T(t)$$

$$\frac{\partial u}{\partial t}=X(x)T'(t)$$

$$a^2X(x)T'(t)=X''(x)T(t)$$

$$a^2\frac{T'(t)}{T(t)}=\frac{X''(x)}{X(x)}=\mu$$

<u>Case 1:</u> $\boldsymbol{\mu>0}$

$$a^2\frac{T'(t)}{T(t)}=\mu$$

$$a^2T'(t)=\mu T(t)$$

$$a^2T'(t)-\mu T(t)=0$$

$$m-\mu=0 \quad\rightarrow\quad m=\frac{\mu}{a^2}$$

$$T(t)=c_1e^{\frac{\mu}{a^2}t}$$

$$\frac{X''(x)}{X(x)}=\mu$$

$$X''(x)-\mu X(x)=0$$

$$m^2-\mu=0 \quad\rightarrow\quad m=\pm\sqrt{\mu}$$

$$X(x)=c_2e^{\sqrt{\mu}x}+c_3e^{-\sqrt{\mu}x}$$

$$u(x,t)=X(x)T(t)$$

$$u(x,t)=\left(c_2e^{\sqrt{\mu}x}+c_3e^{-\sqrt{\mu}x}\right)c_1e^{\frac{\mu}{a^2}t}$$

This solution is rejected

<u>Case 2:</u> $\boldsymbol{\mu=0}$

$$a^2\frac{T'(t)}{T(t)}=0$$

$$T'(t)=0 \qquad\qquad T(t)=c_1$$

$$\frac{X''(x)}{X(x)} = 0$$

$$X''(x) = 0 \qquad X(x) = c_2 x + c_3$$

$$u(x,t) = X(x)T(t)$$

$$u(x,t) = (c_2 x + c_3)c_1$$

This solution is rejected because there is no effect of time

<u>Case 3</u>: $\boldsymbol{\mu < 0}$

$$a^2 \frac{T'(t)}{T(t)} = -\lambda^2$$

$$a^2 T'(t) + \lambda^2 T(t) = 0$$

$$a^2 m + \lambda^2 = 0 \qquad m = \frac{-\lambda^2}{a^2}$$

$$T(t) = c_1 e^{\frac{-\lambda^2}{a^2}t}$$

$$\frac{X''(x)}{X(x)} = -\lambda^2$$

$$X''(x) + \lambda^2 X(x) = 0$$

$$m^2 + \lambda^2 = 0 \qquad \rightarrow \quad m = \pm\lambda i$$

$$X(x) = c_2 \cos\lambda x + c_3 \sin\lambda x$$

$$u(x,t) = X(x)T(t)$$

$$u(x,t) = (c_2 \cos\lambda x + c_3 \sin\lambda x)c_1 e^{\frac{-\lambda^2}{a^2}t}$$

$$u(x,t) = (k_1 \cos\lambda x + k_2 \sin\lambda x)e^{\frac{-\lambda^2}{a^2}t}$$

The boundary conditions $(u(0,t) = 0 \quad \textit{permeable top})$

The boundary conditions $(u(L,t) = 0 \quad \textit{permeable bottom})$

$$0 = (k_1 \cos\lambda 0 + k_2 \sin\lambda 0)e^{\frac{-\lambda^2}{a^2}t}$$

$$k_1 = 0$$

$$u(x,t) = k_2 \sin \lambda x \, e^{\frac{-\lambda^2}{a^2}t}$$

$$0 = k_2 \sin \lambda L \, e^{\frac{-\lambda^2}{a^2}t}$$

$$\sin \lambda L = 0 \qquad \lambda L = n\pi \qquad \lambda = \frac{n\pi}{L}$$

$$u(x,t) = B_n \sin \frac{n\pi}{L} x \, e^{\frac{-\lambda^2}{a^2}t}$$

The initial temperature distribution is ($t = 0 \quad u(x,0) = f(x)$)

$$f(x) = B_n \sin \frac{n\pi}{L} x \, e^{\frac{-\lambda^2}{a^2}0}$$

$$f(x) = B_n \sin \frac{n\pi}{L} x$$

Depending on Fourier series

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Example.1 Find the solution of consolidation of soil

$$a^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

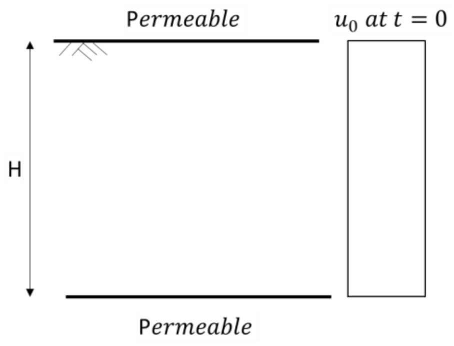
The boundary conditions are

$u(0,t) = 0 \qquad$ *For all* $t > 0$

$u(H,t) = 0 \qquad$ *For all* $t > 0$

Initial Conditions

$u(y,0) = u_0 \qquad 0 < y < H$



Solution

$$u(y,t) = B_n \sin \frac{n\pi}{L} y \, e^{\frac{-\lambda^2}{a^2}t}$$

$$\lambda = \frac{n\pi}{L}$$

$$B_n = \frac{2}{L} \int_0^L f(y) \sin \frac{n\pi}{L} y \, dy$$

$$B_n = \frac{2}{H}\int_0^H u_0 \sin\frac{n\pi}{H} y\, dy$$

$$B_n = \frac{2}{H}\left| \frac{-u_0 H}{n\pi} \cos\frac{n\pi}{H} y \right|_0^H$$

$$B_n = \frac{2u_0}{n\pi}(1 - \cos n\pi) = \frac{4u_0}{n\pi} \qquad n = 1,3,5 \ldots$$

$$u(x,t) = B_n \sin\frac{n\pi}{L} x\, e^{\frac{-\lambda^2}{a^2}t}$$

$$u(y,t) = \frac{4u_0}{n\pi} \sin\frac{n\pi}{H} y\, e^{\frac{-\lambda^2}{a^2}t}$$

$$u(y,t) = \frac{4u_0}{n\pi} \sin\frac{n\pi}{H} y\, e^{\frac{-(n\pi)^2}{a^2H^2}t}$$

Example.2 A rod of length L bars its lateral surface insulated and is so that heat flow in the rod can be regarded as one-dimensional. Initially the rod is at the temperature $100^0\ C$ through t, at time $t = 0$, the temperature at the left end of the rod is suddenly reduced to $50^0\ C$ and maintained thereafter at the right-hand end it maintained at $100^0\ C$. Find the temperature at any point of the rod at any subsequent time

Solution

$$u(x,t) = B_n \sin \lambda x\, e^{\frac{-\lambda^2}{a^2}t}$$

هنا يجب ان نضيف معادلة المستقيم لان درجات الحرارة سوف لا تصبح صفرا فيما بعد بل انها ستستقر عند القيم الموضحة في الرسم

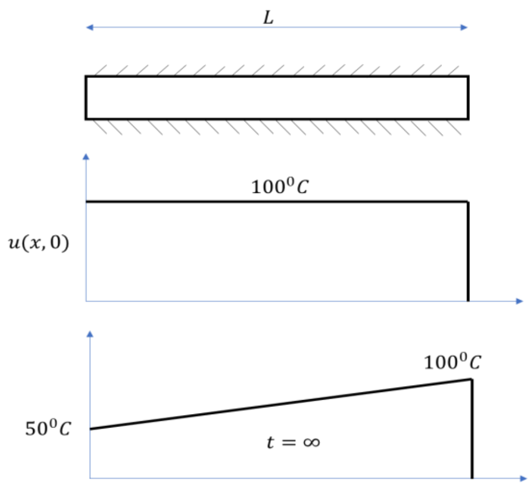
$$u(x,\infty) = Ax + B$$

$$u(x,\infty) = \frac{100-50}{L}x + 50 = \frac{50}{L}x + 50$$

$$u(x,t) = \frac{50}{L}x + 50 + B_n \sin \lambda x\, e^{\frac{-\lambda^2}{a^2}t}$$

Now use the boundary conditions

$at\ t = 0 \qquad u = 100$



$$100 = \frac{50}{L}x + 50 + B_n \sin \lambda x \, e^{\frac{-\lambda^2}{a^2}0}$$

$$50 - \frac{50}{L}x = B_n \sin \lambda x$$

$$at\ t = 0 \quad x = L \quad u = 50$$

$$50 - \frac{50}{L}L = B_n \sin \lambda L$$

$$0 = B_n \sin \lambda L \qquad \lambda L = n\pi \qquad \lambda = \frac{n\pi}{L}$$

$$50 - \frac{50}{L}x = B_n \sin \frac{n\pi}{L}x$$

وهنا نلاحظ ان شكل المعادلة هي لفوريير سيرسز أي بمعنى اخر ان المعامل سيكون

$$B_n = \frac{2}{L}\int_0^L \left(50 - \frac{50}{L}x\right) \sin \frac{n\pi}{L}x \, dx$$

$50 - \frac{50}{L}x$		$\sin \frac{n\pi}{L}x$
$-\frac{50}{L}$	$+$	$-\frac{L}{n\pi}\cos \frac{n\pi}{L}x$
0	$-$	$-\left(\frac{L}{n\pi}\right)^2 \sin \frac{n\pi}{L}x$

$$B_n = \frac{100}{n\pi}$$

$$u(x,t) = \frac{50}{L}x + 50 + B_n \sin \lambda x \, e^{\frac{-\lambda^2}{a^2}t}$$

$$u(x,t) = \frac{50}{L}x + 50 + \frac{100}{n\pi} \sin \frac{n\pi}{L}x \, e^{\frac{-n^2\pi^2}{a^2L^2}t}$$

$$u(x,t) = \frac{50}{L}x + 50 + \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{L}x \, e^{\frac{-n^2\pi^2}{a^2L^2}t}$$