Yield Line Theory

Introduction:

An elastic analysis of a reinforced concrete slab gives no indication of its ultimate load carrying capacity and further analyses have to be made for this condition. An exact solution for the ultimate flexural strength of slab can be found only rarely, but it possible to determine upper and lower bounds, to the true collapse load.

The yield line method of analysis gives an upper bounds to the ultimate load capacity of a reinforced concrete slab by a study of assumed mechanisms of collapse. This method developed by JOHANSON (1940s), is powerful tool for estimating the required bending resistance and hence the necessary reinforcement especially for slabs of non-regular geometry or loading.

Two approaches are possible in yield line theory:

1-Energy Method:

External work done by the loads during the small virtual displacement of the collapse mechanism is equated to the internal work done by the resisting moments.

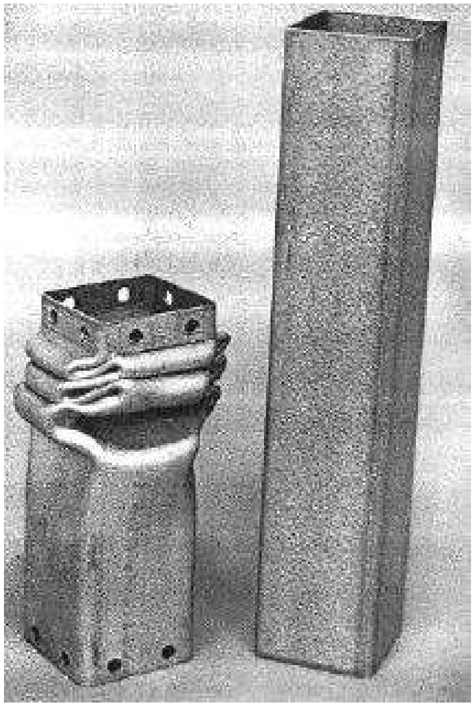
2-Equilibrium Method:

Equilibrium of the segments or parts of slab into which the slab is divided by the yield lines.

Plastic Analysis:

Plastic behavior of the material describes the deformation of the body undergoing nonreversible changes of shape in response to applied forces. The main task of the structural engineering is to design the structural members so they can carry the loads under all possible conditions including ultimate limit states. The elastic distribution of stresses can be obtained by solution of the elasticity problem. However the structural elements do not behave elastically near ultimate load and bending capacity of section is based on a plastic analysis.

Nonreversible
Plastic Deformation

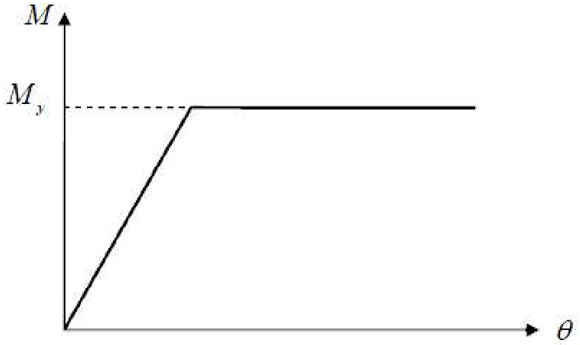


Assumptions:

a) The slab is under-reinforced, and shear failure, bond failure and over-reinforced failure are prevented.

b) The moment-curvature relationship is idealized as the elastic-perfectly plastic curve with a long horizontal portion.

It is only applicable to ductile (under-reinforced) slabs since we assume that the moment-rotation diagram shown holds.



c) The assumed collapse mechanism is defined by a pattern of yield lines along which the reinforcement has yielded.

d) The location of the yield lines depends on the support conditions, and the loading conditions.

e) The yield lines divide the slab into several regions called rigid regions which are assumed to remain plane, so that all rotations take place in the yield lines only.

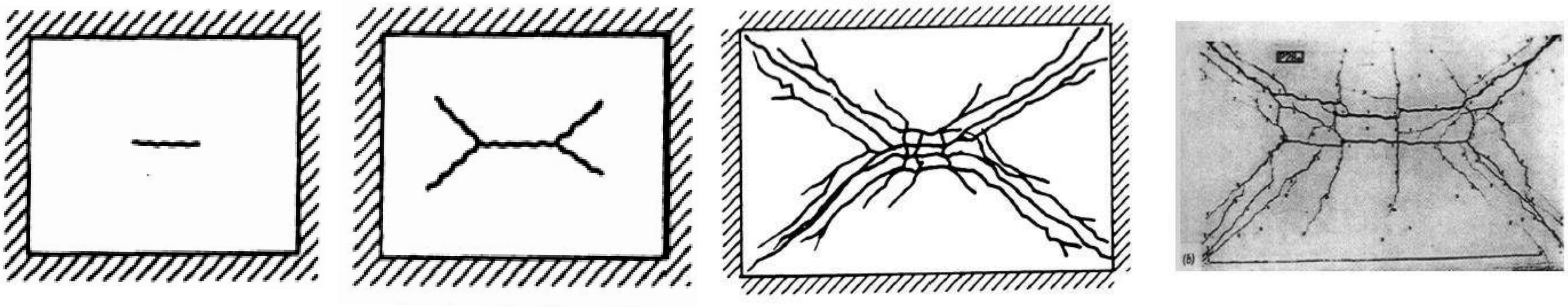
f) Yield lines are straight and they end at a slab boundary.

g)A yield line between two rigid regions must pass through the intersection of the axes of rotation of the two regions.

What is a yield line?

Consider a reinforced concrete slab that is progressively loaded to failure:

- prior to cracking, the distribution of bending moments follows the linear elastic theory.
- after cracking, the distribution of the bending moments changes due to the decrease in flexural rigidity of the cracked portions.
- with further loading, yielding of the reinforcing steel occurs, and the slab undergoes a redistribution of the bending moments.
- As the load on the slab is further increased, the lines where cracking concentrates (across which the steel has yielded) will propagate until a collapse mechanism is formed. These lines are referred to as yield lines.
- The yield line distribution at collapse is called a yield line pattern.

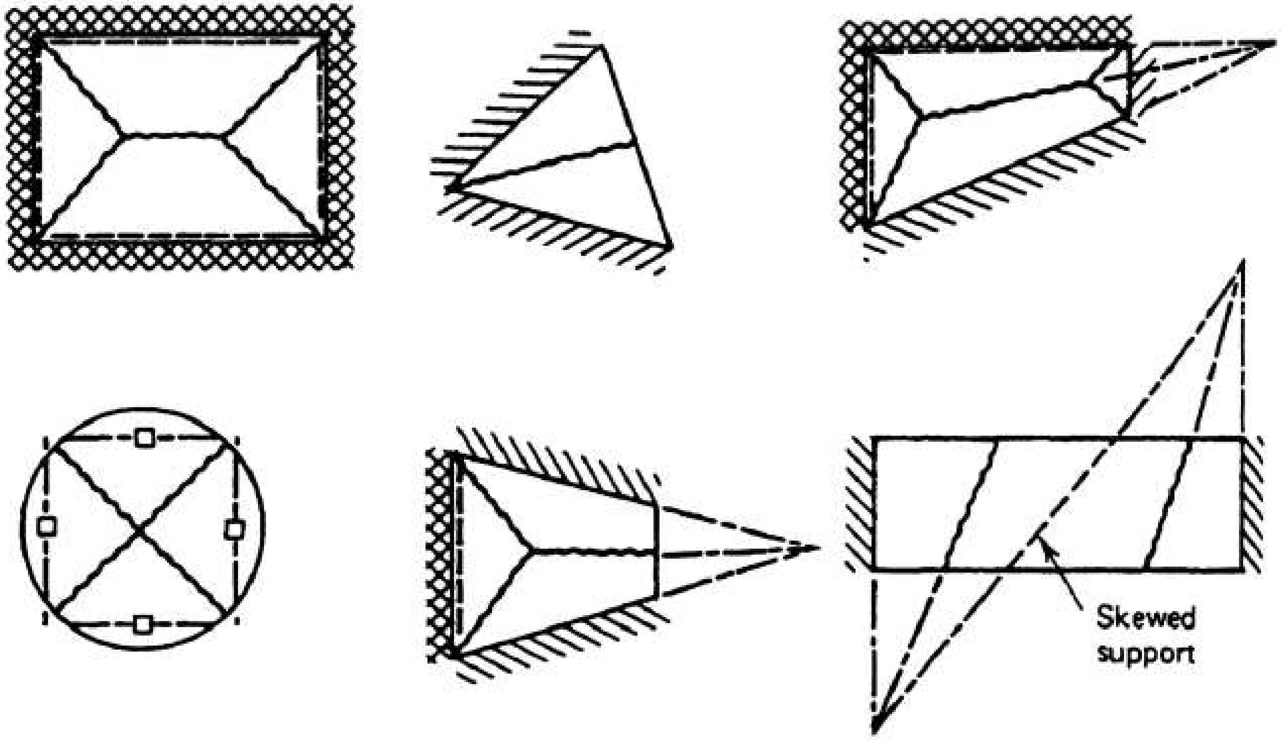


General Rules to Formulate a Yield Line Pattern

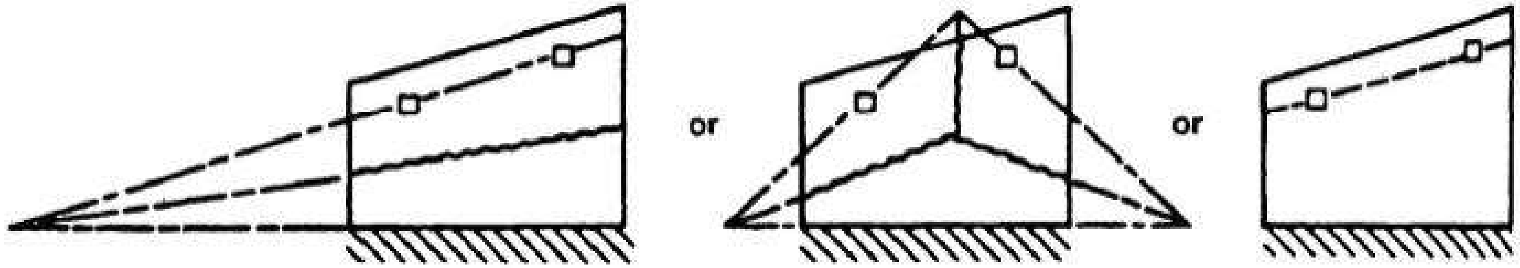
1. yield lines must be straight, acting as axes of rotation for the rigid slab segments in between
2. the supported (or clamped) edges will act as axes of rotation
3. axes of rotation pass through point supports
4. for compatibility of deformations, a yield line must pass through the intersection of the axes of rotation of adjacent segments

— — — —	Axis of Rotation
⫽⫽⫽⫽⫽⫽	Simply Supported Edge
☒☒☒☒☒☒	Fixed Edge
- - - - - -	-ve Yield Line (tension in upper face)
〰〰〰	+ve Yield Line (tension in bottom face)
■	Column
⊕	Point Load

Examples of Yield Line Patterns:



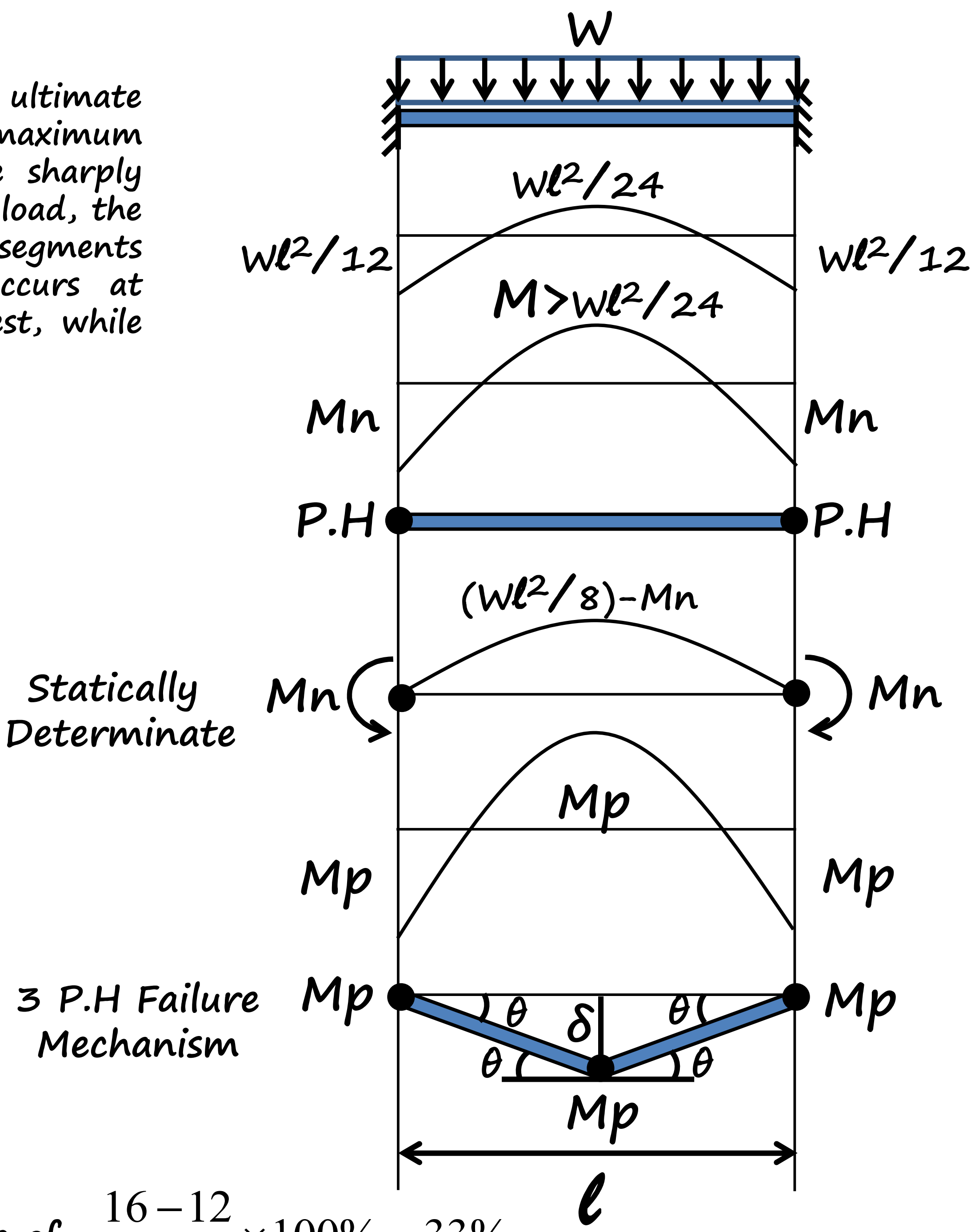
Multiple Yield Line Patterns:



If multiple yield line patterns are possible, the most likely to occur is the one corresponding to the lowest ultimate load.

Ultimate Behavior of Fixed Beam:

At load increases, maximum moment reaches to ultimate moment and tension steel yields along line of maximum moment. The curvature and deflection increase sharply causing plastic deformation. With the increase of load, the moment will increase to form plastic hinge and segments of slab move (Mp=Mn). The plastic hinges occurs at supports first because the moment is the greatest, while the plastic hinge forms at mid span later.



Elastic Behavior :

$$Mn = \frac{wl^2}{12} \rightarrow w_{elastic} = \frac{12Mn}{l^2}$$

Elastic-Plastic Behavior :

$$Mn = \frac{wl^2}{8} - Mn$$

$$2Mn = \frac{wl^2}{8} \rightarrow w_{plastic} = \frac{16Mn}{l^2}$$

Plastic Behavior :

$$ExternalWork = InternalWork$$

$$2(w \times \frac{l}{2} \times \frac{\delta}{2}) = 4Mp\theta \rightarrow \frac{wl\delta}{2} = 4Mp\frac{\delta}{l/2}$$

$$\frac{wl}{2} = 4Mp\frac{2}{l} \rightarrow w_{plastic} = \frac{16Mp}{l^2}$$

Increase of load $= \frac{16-12}{12} \times 100\% = 33\%$

Example 1:

Determine the load capacity of one way uniformly loaded continuous slab shown if (Mn)A =25 kN.m/m , (Mn)B =25 kN.m/m and (Mn)C =40 kN.m/m

Sol:

A virtual unit deflection is assumed at B

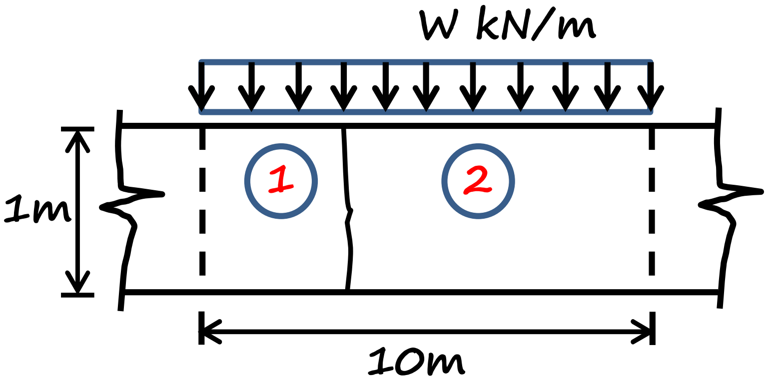
External Work= load X area X deflection at center of rigid segment

$$E.W=E.W_1+E.W_2=w(1\times x)\times\frac{1}{2}+w\left[1\times(10-x)\right]\times\frac{1}{2}$$

Internal Work=moment X rotation X projection of Y.L on axis of rotation

$$I.W=m\times\theta\times l=M_A\times\frac{1}{x}\times 1+M_B\times\frac{1}{x}\times 1+M_B\times\frac{1}{10-x}\times 1+M_C\times\frac{1}{10-x}\times 1$$

External Work = Internal Work

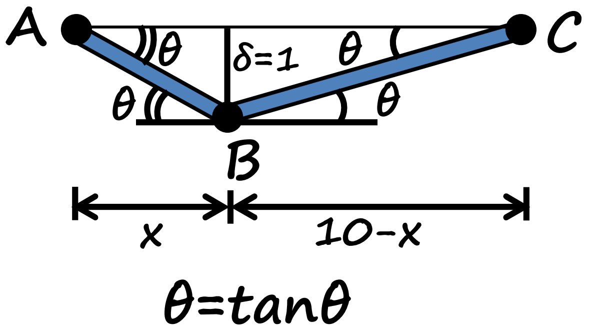


$$w(1\times x)\times\frac{1}{2}+w\left[1\times(10-x)\right]\times\frac{1}{2}=\frac{25}{x}+\frac{25}{x}+\frac{25}{10-x}+\frac{40}{10-x}$$

$$w=\frac{10}{x}+\frac{13}{10-x}$$ Minimum load occurs at : $\frac{dw}{dx}=0$

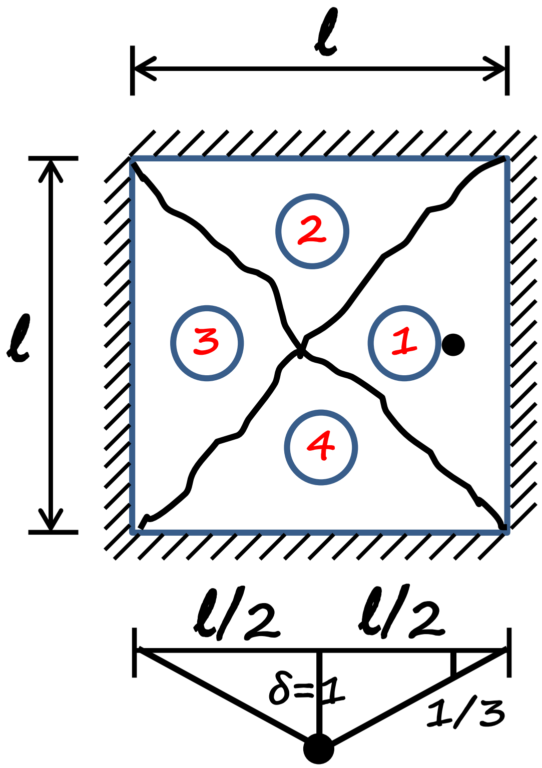
$$\therefore\frac{dw}{dx}=\frac{-10}{x^2}+\frac{13}{(10-x)^2}=0\rightarrow x=4.67m$$

$$\therefore w=\frac{10}{4.67}+\frac{13}{10-4.67}=4.58kN/m^2$$



Example 2:

Find collapse load (w) for the square slab of length ℓ shown if the resisting moment in every direction= m kN.m/m (Isotropic slab).



Sol:

$$E.W = I.W$$

$$w\left(\frac{1}{2}\times l\times\frac{l}{2}\right)\times\frac{1}{3}\times 4 = m\times\theta\times l\times 4$$

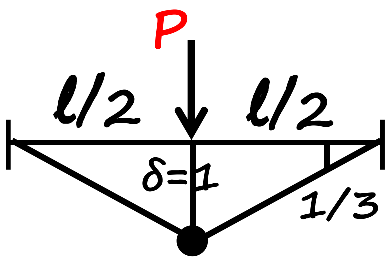
$$\frac{wl^2}{3} = m\times\frac{1}{l/2}\times l\times 4$$

$$\frac{wl^2}{3} = 8m \rightarrow w = \frac{24m}{l^2}$$

Example 3:

Find collapse concentrated load (P) kN at the center of the square slab shown in the previous example.

Sol:



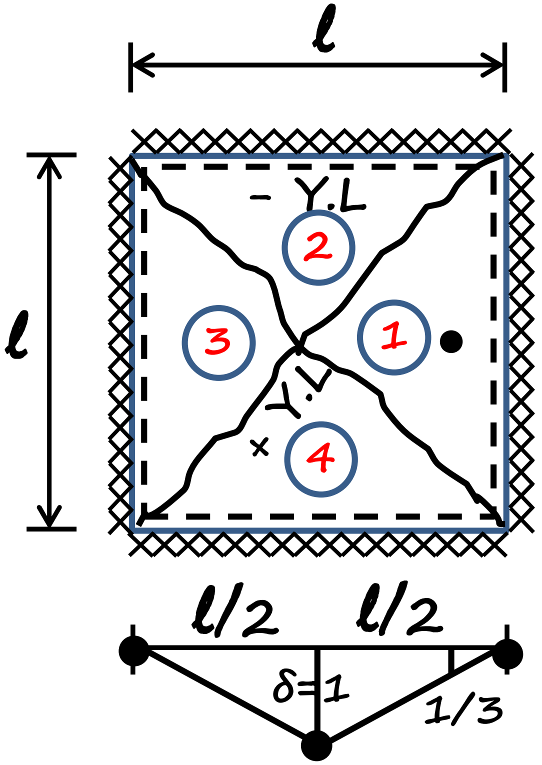
$$E.W = I.W$$

$$P\times 1 = m\times\theta\times l\times 4$$

$$P = 8m$$

Example 4:

Find collapse load (w) for the square slab of length ℓ shown, if the resisting moment in every direction= m kN.m/m (Isotropic slab).



Sol:

$$E.W = I.W$$

$$w(\frac{1}{2}\times l\times\frac{l}{2})\times\frac{1}{3}\times 4 = m^{+}\times\frac{1}{l/2}\times l\times 4 + m^{-}\times\frac{1}{l/2}\times l\times 4$$

$$\frac{wl^2}{3} = 16m$$

$$w = \frac{48m}{l^2}$$

Example 5:

Find collapse concentrated load (P) kN at the center of the square slab shown In the previous example.

Sol:

$$E.W = I.W$$

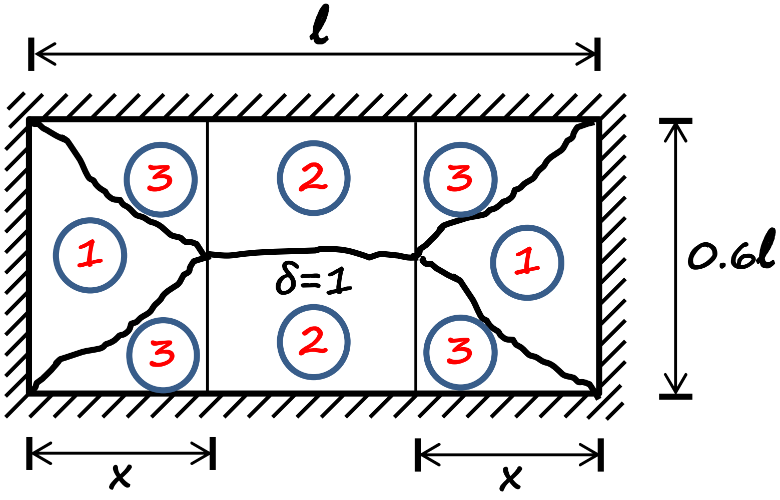
$$P\times 1 = 16m$$

$$P = 16m$$

Example 6:

Find collapse load (w) for the simply supported rectangular slab shown, assume the ultimate moment of resistance in each direction= Mp kN.m/m (Isotropic slab).

Sol: $E.W=2E.W_1+2E.W_2+4E.W_3$



$$E.W_1=w\times(\frac{1}{2}\times0.6l\times x)\times\frac{1}{3}=0.1wlx$$

$$E.W_2=w\times(l-2x)\times0.3l\times\frac{1}{2}=0.15wl(l-2x)$$

$$E.W_3=w\times(\frac{1}{2}\times0.3l\times x)\times\frac{1}{3}=0.05wlx$$

$$E.W=2(0.1wlx)+2[0.15wl(l-2x)]+4(0.05wlx)=\frac{wl}{10}(3l-2x)$$

$$I.W=2(Mp\times0.6l\times\frac{1}{x}+Mp\times l\times\frac{1}{0.3l})=(\frac{3.6l+20x}{3x})\times Mp$$

$$E.W=I.W\rightarrow\frac{wl}{10}(3l-2x)=(\frac{3.6l+20x}{3x})\times Mp\rightarrow w=\frac{10}{l}\left[\frac{3.6l+20x}{3x(3l-2x)}\right]Mp$$

$$For:w_{\min}\rightarrow\frac{dw}{dx}=0\rightarrow\frac{60x(3l-2)-(3.6l+20x)(9l-12x)}{[3x(3l-2x)]^2}=0\rightarrow120x^2+43.2lx-32.4l^2=0$$

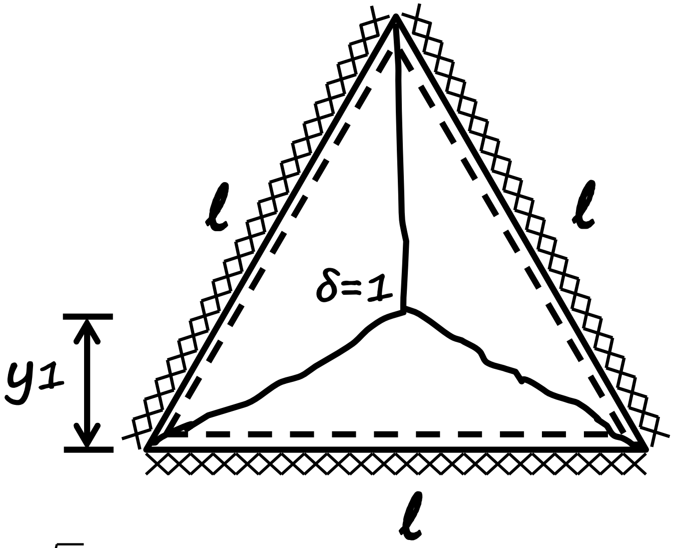
$$For:l=10m\rightarrow x=3.7m\rightarrow w=0.438Mp$$

H.W: Find collapse load (w) if the rectangular slab is fixed supported.

Example 7:

Find collapse load (w) for the fixed supported equilateral triangular slab shown, if +veMp=20 kN.m/m, -veMp=25 kN.m/m and l =6m.

Sol:



$$\tan 30 = \frac{y_1}{l/2} \rightarrow \frac{1}{\sqrt{3}} = \frac{y_1}{l/2}$$

$$y_1 = \frac{l}{2\sqrt{3}}$$

$$E.W = w \times (\frac{1}{2} \times l \times \frac{l}{2\sqrt{3}}) \times \frac{1}{3} \times 3 = \frac{wl^2}{4\sqrt{3}}$$

$$I.W = \left[Mp^+ \times l \times (\frac{1}{l/2\sqrt{3}}) + Mp^- \times l \times (\frac{1}{l/2\sqrt{3}}) \right] \times 3 = 270\sqrt{3}$$

$$E.W = I.W$$

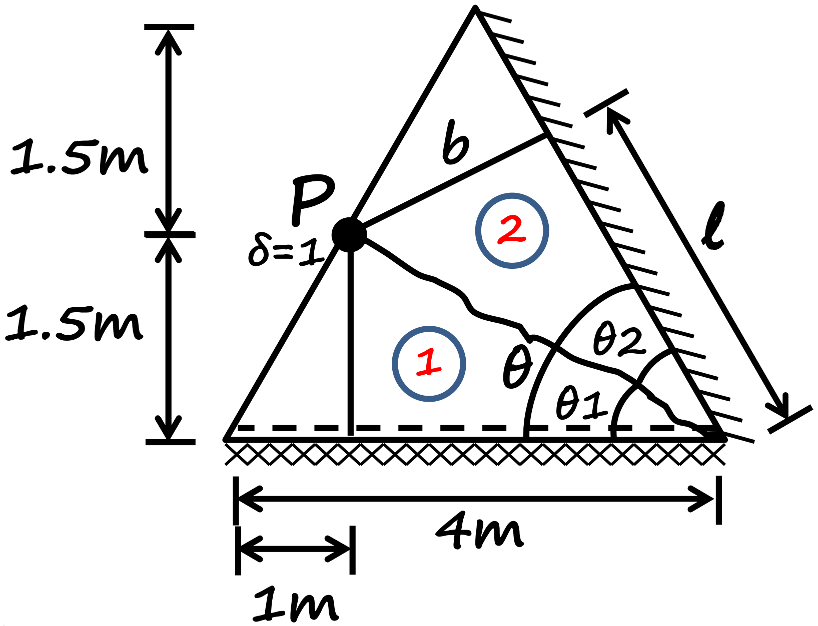
$$\frac{wl^2}{4\sqrt{3}} = 270\sqrt{3}$$

$$w = 90kN/m^2$$

Example 8:

Find collapse load (P) for the isotropic triangular slab shown, if:
+veMp=-veMp=Mp.

Sol:



$$E.W = P \times 1 = P$$

$$I.W = I.W_1 + I.W_2$$

$$I.W_1 = (Mp^+ \times 3 + Mp^- \times 4) \times \frac{1}{1.5} = \frac{14}{3} Mp$$

$$I.W_2 = Mp^+ \times l \times \frac{1}{b}$$

$$\theta_1 = \tan^{-1}(\frac{1.5}{3}) = 26.57^\circ \text{，} \theta = \tan^{-1}(\frac{3}{2}) = 56.31^\circ \text{，} \theta_2 = \theta - \theta_1 = 29.74^\circ$$

$$Length : (+Y.L) = \sqrt{1.5^2 + 3^2} = 3.354m$$

$$\therefore b = 3.354 \sin\theta_2 = 1.664m$$

$$l = 3.354 \cos\theta_2 = 2.912m$$

$$\therefore I.W_2 = Mp^+ \times l \times \frac{1}{b} = Mp^+ \times 2.912 \times \frac{1}{1.664} = 6.417 Mp$$

$$E.W = I.W_1 + I.W_2$$

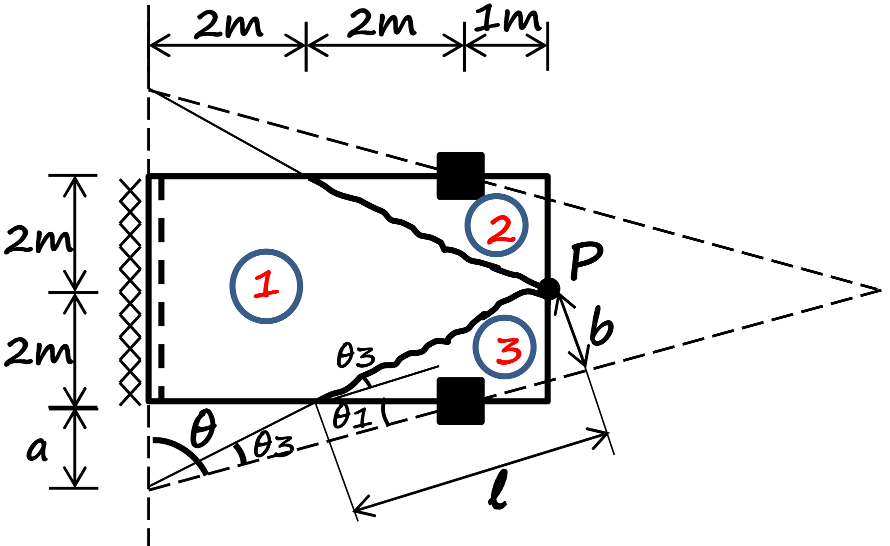
$$\therefore P = \frac{14}{3} Mp + 6.417 Mp = 11.08 Mp$$

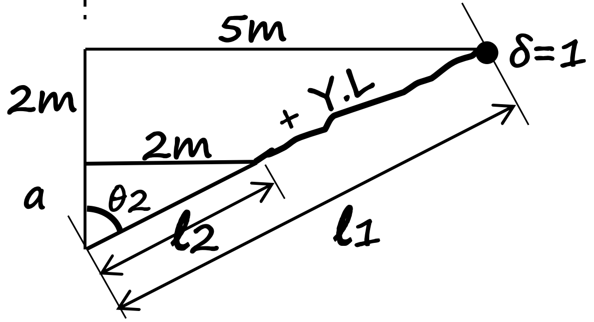
Example 9:

Find concentrated collapse load (P) for the isotropic slab with yield lines shown.

Sol:

Isotropic slab: +veMp=-veMp=Mp





$$\frac{a}{a+2}=\frac{2}{5}\rightarrow 5a=2a+4\rightarrow a=1.33$$

$$E.W=P\times 1=P$$

$$I.W=I.W_1+2I.W_2$$

$$I.W=(Mp^{+}\times 4+Mp^{-}\times 4)\times\frac{1}{5}+2(Mp^{+}\times l\times\frac{1}{b})$$

$$\theta_1=\tan^{-1}(\frac{1.33}{4})=18.4^{\circ}$$

$$\theta=180-90-18.4=71.56^{\circ}$$

$$\theta_2=\tan^{-1}(\frac{2}{1.33})=56.31^{\circ}$$

$$\therefore\theta_3=\theta-\theta_2=71.56-56.31=15.25^{\circ}$$

$$l_1=\sqrt{5^2+3.33^2}=6m$$

$$\therefore b=l_1\sin\theta_3=1.58m$$

$$l_2=\sqrt{2^2+1.33^2}=2.4m$$

$$Length:(+Y.L)=6-2.4=3.6m$$

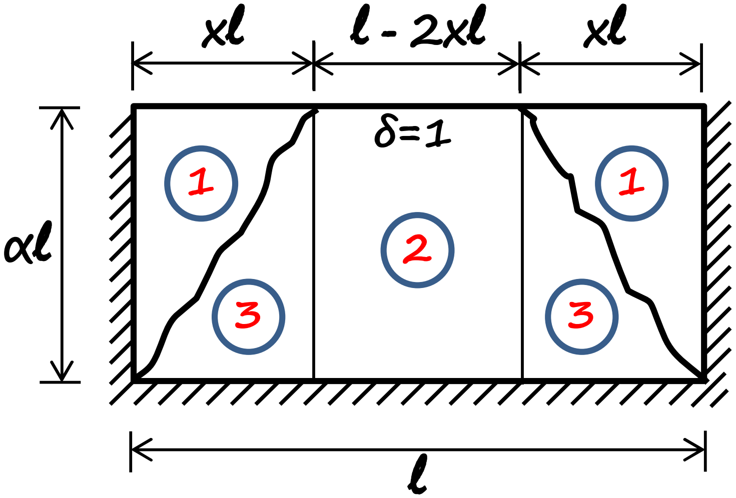
$$l=3.6\cos\theta_3=3.46m$$ Projection of Y.L on axis of rotation

$$\therefore I.W=\frac{8}{5}Mp+2Mp\times 3.46\times\frac{1}{1.58}=6Mp$$

$$E.W=I.W\rightarrow P=6Mp$$

Example 10:

Find max. collapse load (w) that the slab can sustain if it reinforced in each direction by (As1) and (As2).



Sol:

Let Mp1 and Mp2 are the moments of resistance corresponding to As1 and As2 respectively.

$$\rho = \frac{As}{bd} \rightarrow Mp = Mu = \phi \rho b d^2 fy(1 - 0.59\rho \frac{fy}{f'c})$$

$$E.W = 2E.W_1 + E.W_2 + 2E.W_3$$

$$E.W = 2(w \times \frac{1}{2} \times \alpha l \times xl \times \frac{1}{3}) + w \times \alpha l \times (l - 2xl) \times \frac{1}{2} + 2(w \times \frac{1}{2} \times \alpha l \times xl \times \frac{1}{3})$$

$$E.W = \frac{1}{6} w \alpha l^2 (3 - 2x)$$

$$I.W = 2I.W_1 + I.W_{2+3}$$

$$I.W = 2(Mp_1 \times \alpha l \times \frac{1}{xl}) + Mp_2 \times 2xl \times \frac{1}{\alpha l}$$

$$I.W = \frac{2}{x\alpha}(\alpha^2 Mp_1 + x^2 Mp_2)$$

$$E.W = I.W$$

$$\frac{1}{6} w \alpha l^2 (3 - 2x) = \frac{2}{x\alpha}(\alpha^2 Mp_1 + x^2 Mp_2)$$

$$w = \frac{12(\alpha^2 Mp_1 + x^2 Mp_2)}{\alpha^2 l^2 (3x - 2x^2)}$$

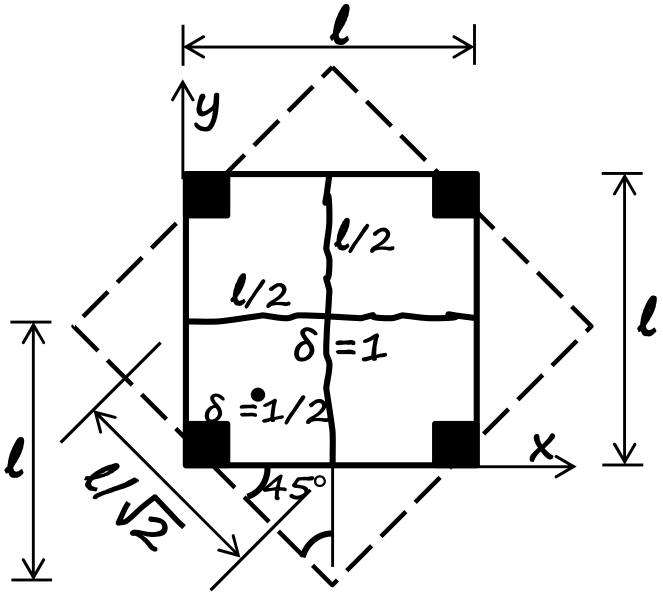
$$w_{\max} \rightarrow \frac{dw}{dx} = 0$$

$$x = ? \rightarrow w_{\max} = ?$$

Example 11:

Find collapse load (w) for the slab shown.

Sol:



First Method:

$$E.W = 4(w \times \frac{l}{2} \times \frac{l}{2} \times \frac{1}{2}) = \frac{wl^2}{2}$$

$$I.W = 4(Mp \times \frac{l}{\sqrt{2}} \times \theta) \rightarrow \theta = \frac{1}{\sqrt{(l/2)^2 + (l/2)^2}} = \frac{\sqrt{2}}{l}$$

$$I.W = 4(Mp \times \frac{l}{\sqrt{2}} \times \frac{\sqrt{2}}{l}) = 4Mp$$

$$E.W = I.W \rightarrow \frac{wl^2}{2} = 4Mp \rightarrow \therefore w = \frac{8Mp}{l^2}$$

Second Method:

$$I.W = Mp \times l_x \times \theta_x + Mp \times l_y \times \theta_y$$

$$\theta_x = \frac{\text{Deflection of any point in the segment}}{\text{Vertical distance from the point to the axis of rotation of the segment}}$$

$$\theta_y = \frac{\text{Deflection of any point in the segment}}{\text{Horizontal distance from the point to the axis of rotation of the segment}}$$

$$I.W = 4(Mp \times \frac{l}{2} \times \frac{1}{l} + Mp \times \frac{l}{2} \times \frac{1}{l}) = 4Mp$$

Example 12:

Find the plastic moment for the slab shown if Wu= 15 kN/m².

Sol:

Each segment considers as triangle and θ is very small

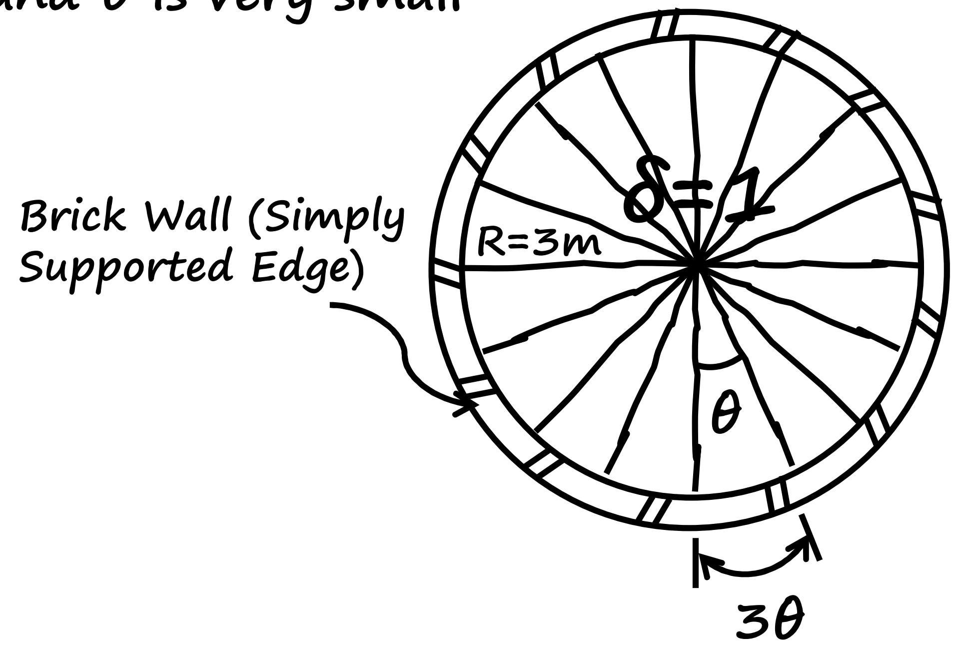
$$E.W = w\times\frac{3\times 3\theta}{2}\times\frac{1}{3}\times\frac{2\pi}{\theta} = 3\pi w$$

$$I.W = Mp\times 3\theta\times\frac{1}{3}\times\frac{2\pi}{\theta} = 2Mp\pi$$

$$E.W = I.W$$

$$3\pi w = 2Mp\pi$$

$$Mp = 1.5w \rightarrow Mp = 22.5kN.m/m$$



Example 13:

Find concentrated collapse load (P) for the isotropic slab shown.

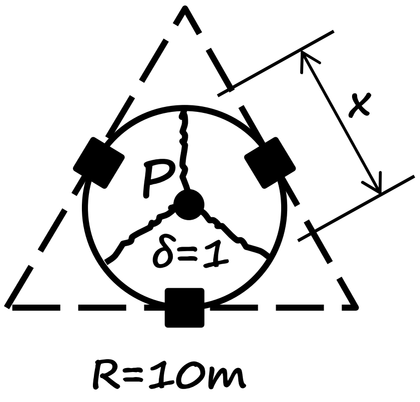
Sol:

$$x = 2R\cos 30 = 17.32$$

$$E.W = I.W$$

$$P\times 1 = 3(Mp\times 17.32\times\frac{1}{10})$$

$$P = 5.196Mp$$



Example 14:

Find concentrated collapse load (P) for the isotropic slab shown.

Sol:

Isotropic slab: +veMp=-veMp=Mp

$$E.W = P\times 1 = P$$

$$I.W = I.W_1 + I.W_2 + I.W_3$$

$$I.W_1 = (Mp\times 9 + Mp\times 9)\times \frac{1}{3} = 6Mp$$

$$I.W = Mp\times l_x\times \theta_x + Mp\times l_y\times \theta_y$$

$$I.W_2 = Mp\times 3\times \frac{0.5}{3.5} + Mp\times 3.5\times \frac{0.714}{3}$$

$$I.W_2 = 1.262Mp$$

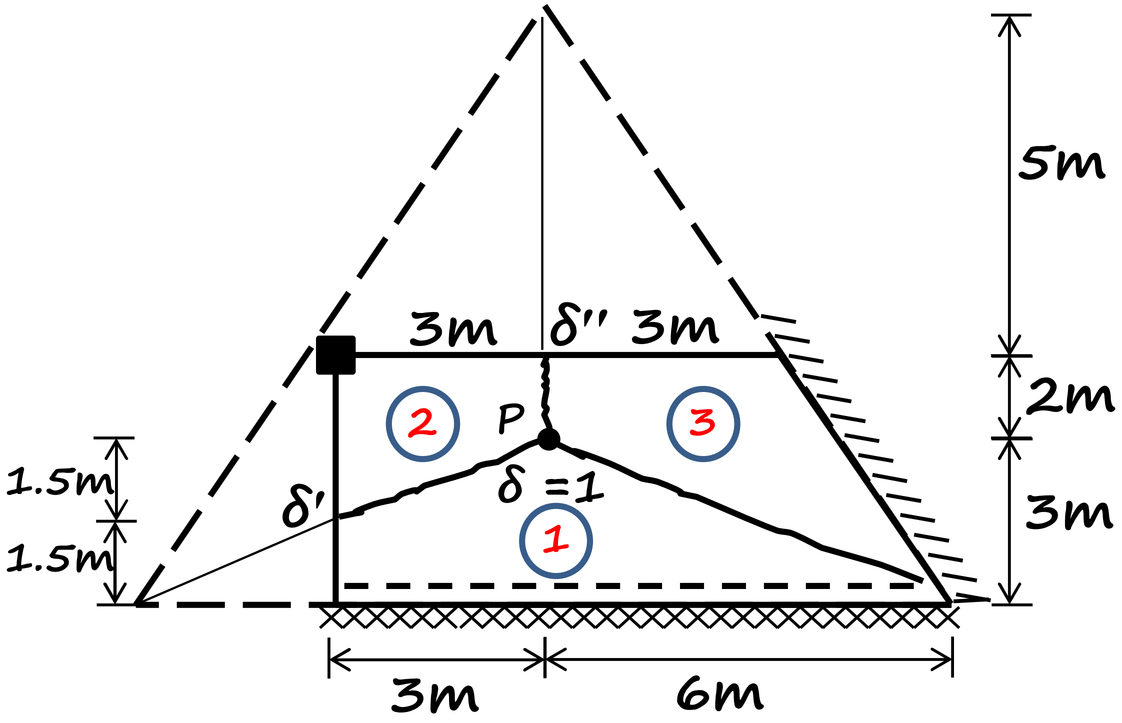
$$I.W_3 = Mp\times 6\times \frac{0.714}{5} + Mp\times 5\times \frac{0.714}{3}$$

$$I.W_3 = 2.05Mp$$

$$I.W = 6Mp + 1.262Mp + 2.05Mp = 9.312Mp$$

$$E.W = I.W$$

$$\therefore P = 9.312Mp$$



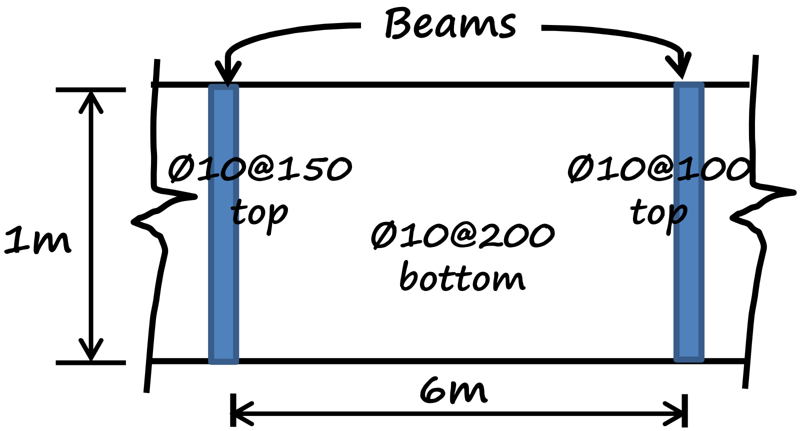
$$\frac{\delta}{3} = \frac{\delta'}{1.5} \rightarrow \delta' = 0.5$$

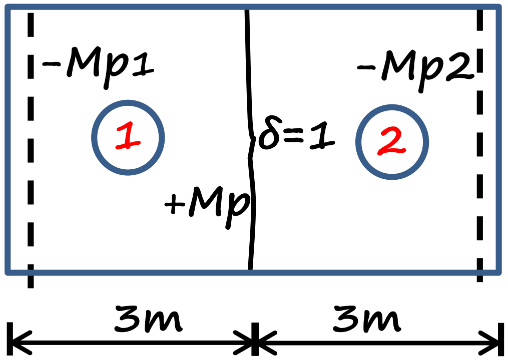
$$\frac{\delta}{7} = \frac{\delta''}{5} \rightarrow \delta'' = 0.714$$

Example 15:

Find collapse load (w) for the continuous slab shown, if fy=420 MPa, f'c=30 MPa, t=160 mm.

Sol:





$$E.W = 2\times(w\times3\times1\times\frac{1}{2}) = 3w$$

$$I.W = I.W_1 + I.W_2$$

$$I.W_1 = Mp_1^- \times1\times\frac{1}{3} + Mp^+ \times1\times\frac{1}{3} = \frac{Mp_1^-}{3} + \frac{Mp^+}{3}$$

$$I.W_2 = Mp_2^- \times1\times\frac{1}{3} + Mp^+ \times1\times\frac{1}{3} = \frac{Mp_2^-}{3} + \frac{Mp^+}{3}$$

$$I.W = \frac{Mp_1^-}{3} + \frac{Mp_2^-}{3} + \frac{2Mp^+}{3}$$

$$d = 160 - 20 - \frac{10}{2} = 135mm$$

$$As = \frac{1000\times A_b}{S}, \rho = \frac{As}{bd}, b = 1000, Mp = \rho bd^2 fy(1-0.59\rho\frac{fy}{f'c})$$

$$As_1^- = \frac{1000\times78}{150} = 520mm^2/m, \rho_1^- = 0.00385, Mp_1^- = 28.5kN.m/m$$

$$As^+ = \frac{1000\times78}{200} = 390mm^2/m, \rho^+ = 0.0029, Mp^+ = 21.66kN.m/m$$

$$As_2^- = \frac{1000\times78}{100} = 780mm^2/m, \rho_2^- = 0.0058, Mp_2^- = 42.27kN.m/m$$

$$I.W = \frac{28.5}{3} + \frac{42.27}{3} + \frac{2\times21.66}{3} = 38.03$$

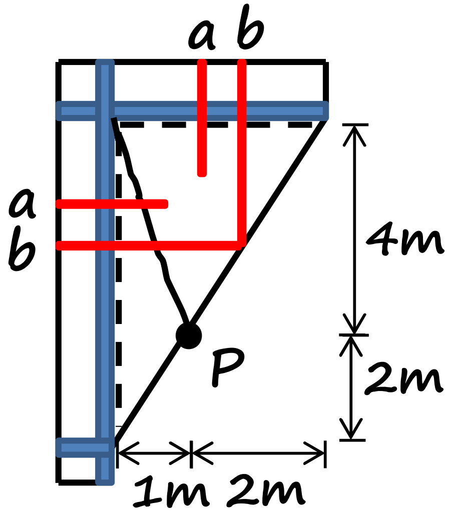
$$E.W = I.W$$

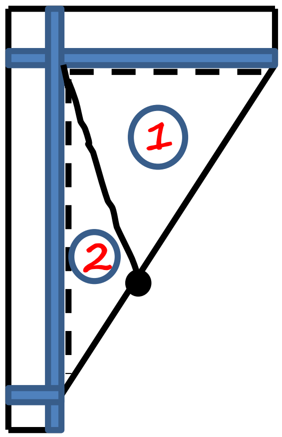
$$\therefore 3w = 38.03 \rightarrow w = 12.67kN/m^2$$

Example 16:

Find concentrated collapse load (P) that the triangular slab can sustain if it reinforced as shown, d=140mm, fy=420MPa, f'c where:
a = Ø12@200mm top. , b = Ø12@200mm bottom. (neglect slab weight)

Sol:





$$E.W = P \times 1 = P$$

$$I.W = I.W_1 + I.W_2$$

$$I.W_1 = Mp^- \times 3 \times \frac{1}{4} + Mp^+ \times 1 \times \frac{1}{4} = \frac{3Mp^-}{4} + \frac{Mp^+}{4}$$

$$I.W_2 = Mp^- \times 6 \times \frac{1}{1} + Mp^+ \times 4 \times \frac{1}{1} = 6Mp^- + 4Mp^+$$

$Mp^+ = Mp^-$ **Same reinforcement**

$$\therefore I.W = \frac{4Mp}{4} + 10Mp = 11Mp$$

$$As = \frac{1000 \times A_b}{S} = \frac{1000 \times 113}{200} = 556mm^2 / m$$

$$\rho = \frac{As}{bd} = \frac{565}{1000 \times 140} = 0.00403$$

$$Mp = \rho b d^2 fy (1 - 0.59 \rho \frac{fy}{f'c}) = 32.1 kN.m / m$$

$$I.W = 11 \times 32.1 = 353.1$$

$$E.W = I.W \rightarrow P = 353.1kN$$

Example 17:

Find the plastic moment for the slab with opening shown if Wu= 15 kN/m²

Sol:

$$E.W = 2E.W_1 + 2E.W_2$$

$$E.W_1 = w\times(\frac{9\times6}{2}\times\frac{1}{3}-\frac{3\times2}{2}\times0.77)=6.69w$$

$$E.W_2 = w\times(\frac{12\times4.5}{2}\times\frac{1}{3}-\frac{4\times1.5}{2}\times0.77)=6.69w$$

$$E.W = 2(6.69w)+2(6.69w)$$

$$E.W = 26.76w = 26.76\times15 = 401.4$$

$$I.W = 2I.W_1 + 2I.W_2$$

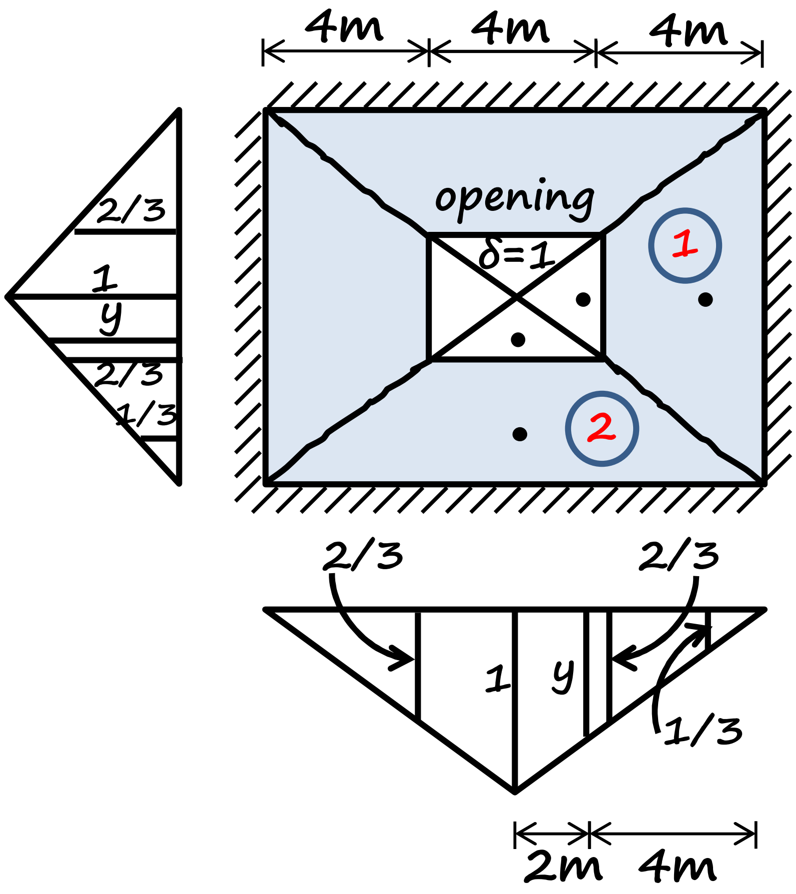
$$I.W_1 = (Mp\times6\times\frac{1}{6}) = Mp$$

$$I.W_2 = (Mp\times8\times\frac{1}{4.5}) = 1.78Mp$$

$$I.W = 2Mp + 2\times1.78Mp = 5.55Mp$$

$$E.W = I.W$$

$$\therefore 401.4 = 5.55Mp \rightarrow Mp = 72.32kN.m/m$$



$$\frac{1}{6}=\frac{y}{4.66}\rightarrow y=0.77$$

$$\frac{1}{4.5}=\frac{y}{3.5}\rightarrow y=0.77$$