Part 1: Basic Linear programming

Example1 :

A firm produces self-assembly bookshelf kits in two models, A and B. Production of the kits is limited by the availability of raw material (high quality board) and machine processing time. Each unit of A required 3 m^2 of board and each unit of B required 4 m^2 of board. The firm can obtain up to 1700 m^2 of board each week from its supplies. Each unit of A needs 12 minutes of machine time and each unit of B needs 30 minutes of machine time. Each week a total of 160 machine hours is available. If the profit on each A units is \$2, and on each B unit is \$4. How many units of each model should the firm plan to produce each week?

Solution:

Let the weekly production of A be as x_1 units while B as x_2 units.

Assume the weekly profit is P

$$\therefore \quad P = 2\,x_1 + 4\,x_2$$

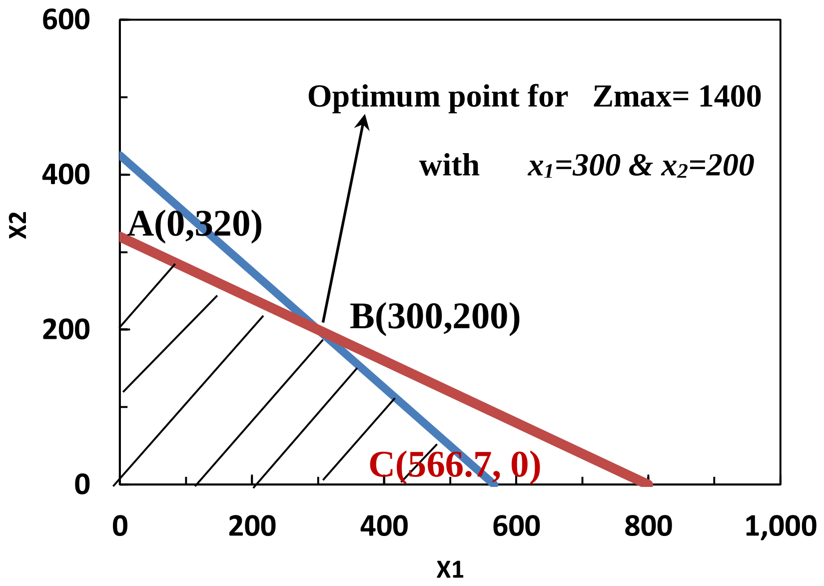
The problem is to maximize the profit P

So, $P = Z_{Max.} = 2\,x_1 + 4\,x_2$ (Objective Function)

Subject to: { Constraints }

$$3\,x_1 + 4\,x_2 \le 1700$$
$$\frac{12}{60}\,x_1 + \frac{30}{60}\,x_2 \le 160$$
With $x \ge 0$

Or
$$3\,x_1 + 4\,x_2 \le 1700$$
$$2\,x_1 + 5\,x_2 \le 1600$$
With $x \ge 0$



1- *Graphical Solution of Two-Dimensional Problem*

Example 2:

$$\text{Min. } Z = -3x_1 - 4x_2$$

Sub. to: $x_1 + x_2 \le 20$ {(20,0) & (0,20)}

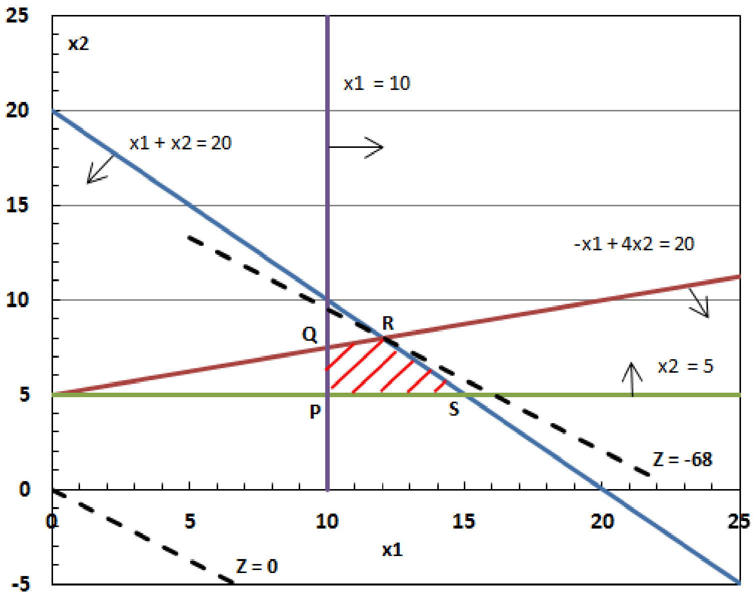
$-x_1 + 4x_2 \le 20$ {(-20,0) & (0,5)}

$x_1 \ge 10$

$x_2 \ge 5$

Solution:

Points	X_1	X_2	Z
P	10	5	-50
Q	10	7.5	-60
R	12	8	-68
S	15	5	-65



Example 3:

$$\text{Min. } Z = -6x_1 - 2x_2$$

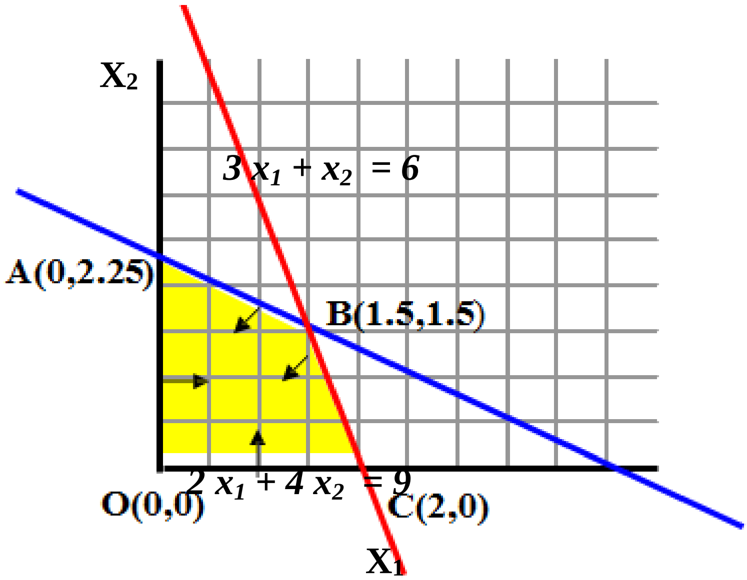
Sub. to: $2x_1 + 4x_2 \le 9$ {(0,2.25) & (4.5,0)}

$3x_1 + x_2 \le 6$ {(0,6) & (2,0)}

Solution:

Any point on BC segment represents an optimal solution

Points	X_1	X_2	Z
A	0	2.25	-4.5
B	1.5	1.5	-12
C	2	0	-12



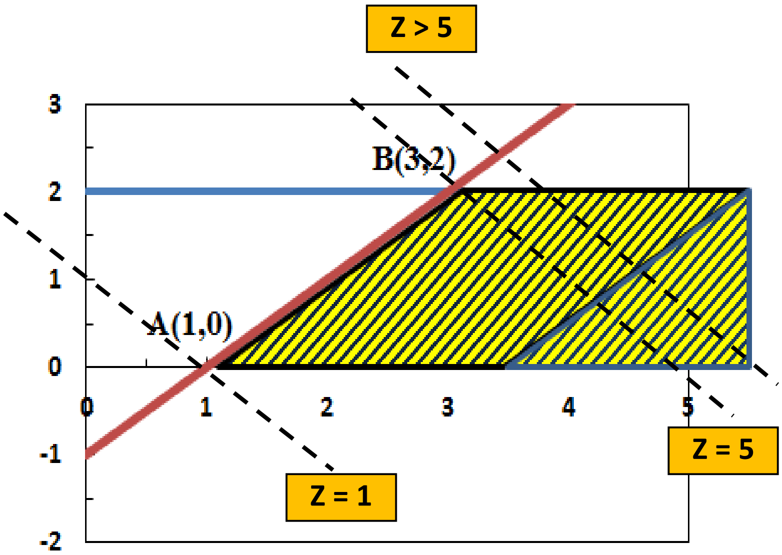
Example 4: ***Unbounded solution***

Maximize $Z = x_1 + x_2$

Subject to: $x_1 \geq 0 , x_2 \geq 0$

$x_1 - x_2 \geq 1$

$x_2 \leq 2$



Important Note

For a max problem, an unbounded ***LP*** occurs if it is possible to find points in the feasible region with arbitrarily large ***Z-values***. This corresponds to arbitrarily large profits or revenue. For a minimization problem, an ***LP*** is unbounded if there are points in the feasible region producing arbitrarily small ***Z-values***.

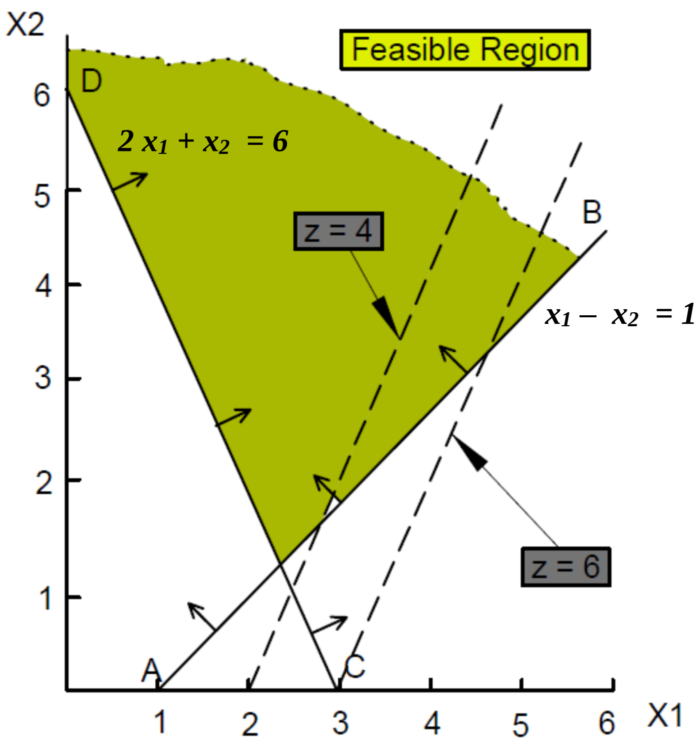
Example5:

Max $Z = 2x_1 - x_2$

s.t. $x_1 , x_2 \geq 0$

$x_1 - x_2 \leq 1$ ***(AB line)***

$2x_1 + x_2 \geq 6$ ***(CD line)***



Solution:

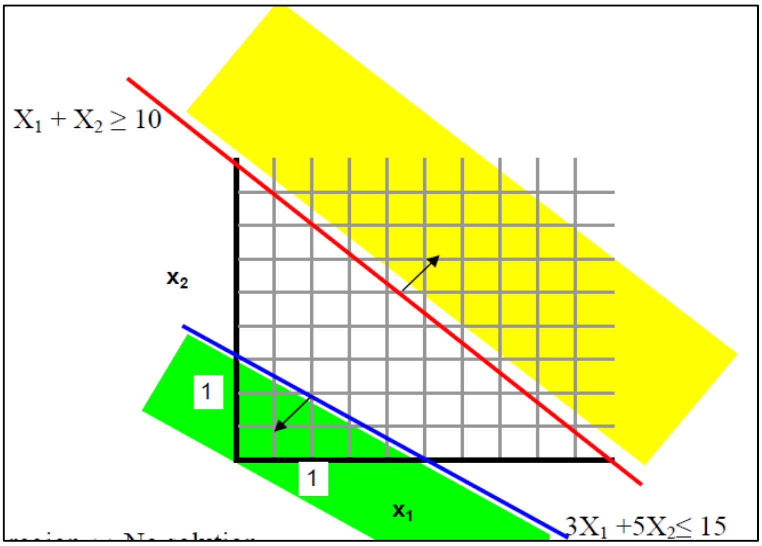
Considering to the graph shown to the right side. The constraints are satisfied by all points bounded by the x_2 axis and on or above ***AB*** and ***CD***. The isoprofit lines for $Z = 4$ and $Z = 6$ are shown. Any isoprofit line drawn will intersect the feasible region because the isoprofit line is steeper than the line

$x_1 - x_2 = 1$

Thus there are points in the feasible region which will produce **arbitrarily large z-values**. So, it is ***(unbounded LP).***

Also, sometime there is no solution at all because a feasible region does not exit

Example 6:



Minimize $Z = 2x_1 + 3x_2$

Subject to $x_1 \geq 0$, $x_2 \geq 0$

$$x_1 + x_2 \geq 10$$
$$3x_1 + 5x_2 \leq 15$$

Solution:

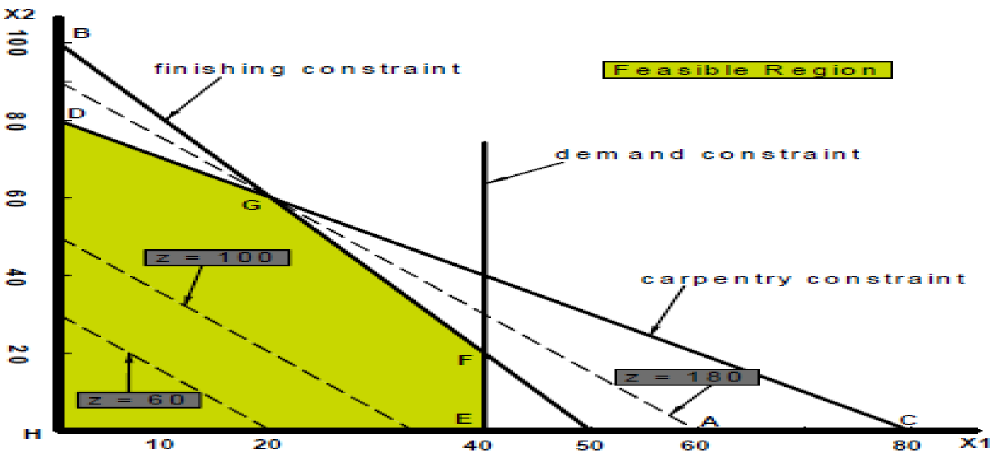
The solution is considered in the graph shown to the right side.

Example 7: ***The feasible region is the set of all points satisfying the constraints:***

The ***LP*** is bounded by the five sided polygon ***DGFEH***. Any point on or in the interior of this polygon (the shade area) is in the Subject to (s.t.)

$2x1 + x2 \leq 100$ (finishing constrain)
$x1 + x2 \leq 80$ (carpentry constraint)
$x1 \leq 40$ (constraint on demand)
$x1 \geq 0$ (sign restriction)
$x2 \geq 0$ (sign restriction)

From figure, we see that the set of points satisfying the ***L P*** is bounded by the five sided polygon ***DGFEH***. Any point on or in the interior of this polygon (the shade area) is in the feasible region.



To find the optimal solution, graph a line on which the points have the same ***Z-value***. In a ***Max- problem***, such a line is called an **isoprofit** line while in a ***Min-***

problem, this is called the **isocost** line. The figure shows the isoprofit lines for ***Z = 60, Z = 100, and Z = 180***

The last isoprofit intersecting (touching) the feasible region indicates the optimal solution for the ***LP***. For the problem, this occurs at point ***G*** $(x_1 = 20, x_2 = 60, Z = 180)$.

Some LPs have an infinite number of solutions.

Example 8: Consider the following formulation:

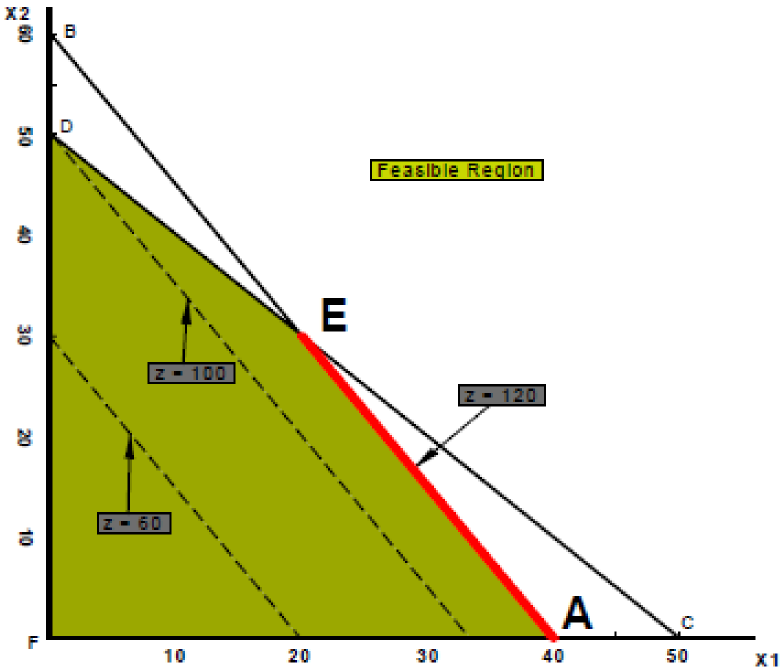
$$Max\ Z = 3x_1 + 2x_2$$

$$\frac{1}{40} \cdot x_1 + \frac{1}{60} \cdot x_2 \le 1$$

$$\frac{1}{50} \cdot x_1 + \frac{1}{50} \cdot x_2 \le 1$$

$$x_1, x_2 \ge 0$$

Any point (solution) falling on line segment **EA** will yield an optimal solution of z=120.



Example 9: ***There are two suppliers of pipes with their information as shown below in the table:***

Source	Unit Cost ($/m)	Supply Limit(m)
1	100	100
2	125	Unlimited

At least 900 meters of pipe is required. The goal is to minimize the total cost of pipe.
a. Find the optimum solution.
b. Formulate a mathematical model with the supply of pipe from source No. 2 limited to 700 meters.

Let the variables be as follows:

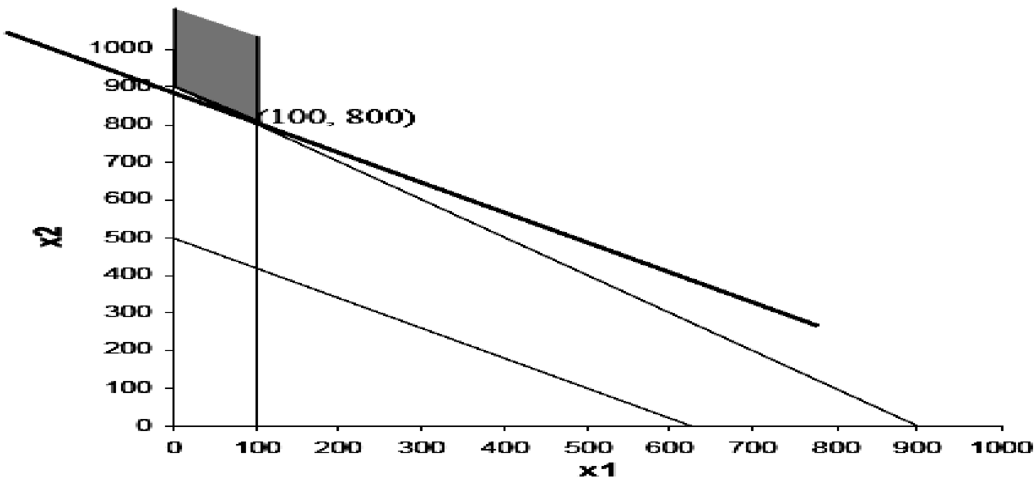
x_1 : Number of meters of pipe supplied from source 1
x_2 : Number of meters of pipe supplied from source 2

The objective is to minimize the cost, c, from the two suppliers

$$Min\ c = 100\,x_1 + 125\,x_2$$

The constraints are:

$$x_1 + x_2 \geq 900$$
$$x_1 \leq 100$$
$$x_1, x_2 \geq 0$$



b- Min c = 100 x1 + 125 x2

Sub. To: x1 + x2 ≥ 900

x1 ≤ 100

x2 ≤ 700

x1 , x2 ≥ 0

Example 10: *A contractor may purchase material from two different sand and gravel pits. The unit cost of material including delivery from pits 1 and 2 is LE50 and LE70 per cubic meter, respectively, the contractor requires at least 100cubic meter of mix. The mix must contain a minimum of 30% sand. Pit 1 contains 25% and pit 2 contains 50% sand. If the objective is to minimize the cost of material,*

a. Draw the feasible region
b. Determine the optimum solution by the graphical method
c. Label the active and inactive constraints
d. Use Solver to model and solve this problem

Solution:

Unit cost of material from pit 1	= LE50
Unit cost of material from pit 2	= LE70
Material needed (sand & gravel) from both pits	≥ 100 Cubic meters
Amount of sand in the mix	≥ 30 Cubic meters
Amount of sand in Pit 1	= 25 %
Amount of sand in Pit 2	= 50 %

Let the variables be as follows:

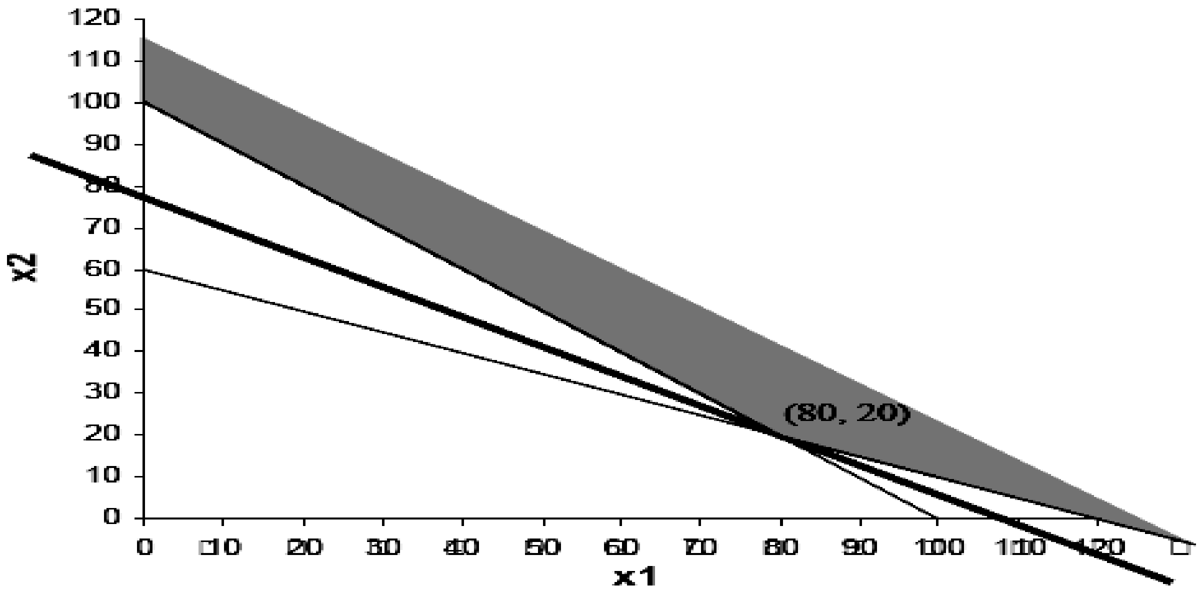
x_1 : amount of material from pit 1
x_2 : amount of material from pit 2

The objective is to minimize the cost, c, of the two types of materials

$$Min\ c = 50\,x_1 + 70\,x_2$$

The constraints are:

$x_1 + x_2$	≥ 100	*(Total material constraint)*
$0.25\,x_1 + 0.5\,x_2$	≥ 30	*(sand material constraint)*
x_1, x_2	≥ 0	



Example 11:

Someone has two farms that grow wheat and corn. Because of differing soil conditions, there are differences in the yields and costs of growing crops on the two farms. The yields and the costs are:

	FARM 1	FARM 2
Corn yield/ acre	500 bushels	650 bushels
Cost/ acre of corn	$100	$120
Wheat yield/ acre	400 bushels	350 bushels
Cost/ acre of wheat	$90	$80

(a) Each farm has 100 acres available for cultivation,
(b) at least 11,000 bushels of wheat and 7000 bushels of corn must be grown.

Formulate the LP model to determine a planting plan that will minimize the cost of meeting these demands.

Solution:

Let:

x_1 = Number of acres of corn planted on farm 1
x_2 = Number of acres of wheat planted on farm 1
x_3 = Number of acres of corn planted on farm 2
x_4 = Number of acres of wheat planted on farm 2

LP model:

$$\text{Minimize } Z = 100\,x_1 + 90\,x_2 + 120\,x_3 + 80\,x_4$$

Subject to:

$$x_1 + x_2 \leq 100$$
$$x_3 + x_4 \leq 100$$
$$500\,x_1 + 650\,x_3 \geq 7{,}000$$
$$400\,x_2 + 350\,x_4 \geq 11{,}000$$
$$\text{and} \quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

2- A Standard Form for Linear Programming Problems

The L.P. problem could be put into standard form in which the objective function is to be ***minimized*** and all constraints take the form of equations in non-negative variables and as follows:

1- Objective function

Maximizing the objective function

$$Z_{max} = c_1 X_1 + c_2 X_2 + c_3 X_3 + \ldots\ldots.. + c_n X_n$$

is equivalent to minimizing the objective function

$$Z_{min} = - c_1 X_1 - c_2 X_2 - c_3 X_3 - \ldots\ldots.. - c_n X_n$$

2- Inequality constraints

The constraint $3 X_1 + 2 X_2 - X_3 \leq 6$ *can be put in equation form as*

$$3 X_1 + 2 X_2 - X_3 + X_4 = 6$$

While the constraint $3 X_1 + 2 X_2 - X_3 \geq 6$ *can be put in equation form as*

$$3 X_1 + 2 X_2 - X_3 - X_5 = 6$$

Where X_4 & X_5 are non-negative ***slack variables***.

3- Non-negative variables

If X_3 can take any value, then let $X_3 = X_4 - X_5$

Where $X_4 \geq 0$ & $X_5 \geq 0$

Example 11: **Express the following problem in standard form.**

$$Z_{max} = 2 X_1 - X_2 + 2 X_3$$

Sub. to.

$$X_1 - X_2 + X_3 \leq 1$$
$$2X_1 + X_2 \geq 6$$
$$X_1 , X_2 \geq 0 \ \& \ X_3 \text{ any value}$$

Solution: let $X_3 = X_4 - X_5$

$$Zmin = - Zmax = - \{ 2X_1 - X_2 + 2(X_4 - X_5) \} = -2X_1 + X_2 -2X_4 + 2 X_5$$

Sub. to. $X_1 - X_2 + X_4 - X_5 + X_6 = 1$

$$2 X_1 + X_2 - X_7 = 6$$

Some important definitions

- ***Feasible solution:*** Any non-negative solution of the constraints is called a feasible solution.
- ***Basic solution & basic feasible solution:*** for n-variables with m-equations, ***basic solution*** of the constraints is obtain by setting (n-m) to zero and solving the m-equations solution of the remaining m-variables provided that these equations have a ***unique*** solution. If it is also feasible (≥ 0) , it is a **basic feasible solution**.
- ***Non-basic variables & basis:*** the variables put equal to zero are called the non-basic variables. The others are the basic variables and form a basis.

Example 12: $Z_{Max.} = 2x_1 + 4x_2$

Subject to:

$$3x_1 + 4x_2 \leq 1700$$
$$2x_1 + 5x_2 \leq 1600$$
$$x \geq 0$$

Solution:

The above problem in standard form is:

$$3x_1 + 4x_2 + x_3 = 1700$$
$$2x_1 + 5x_2 + x_4 = 1600$$

Where n = 4 & m = 2

In matrix form the constraints can be written as:

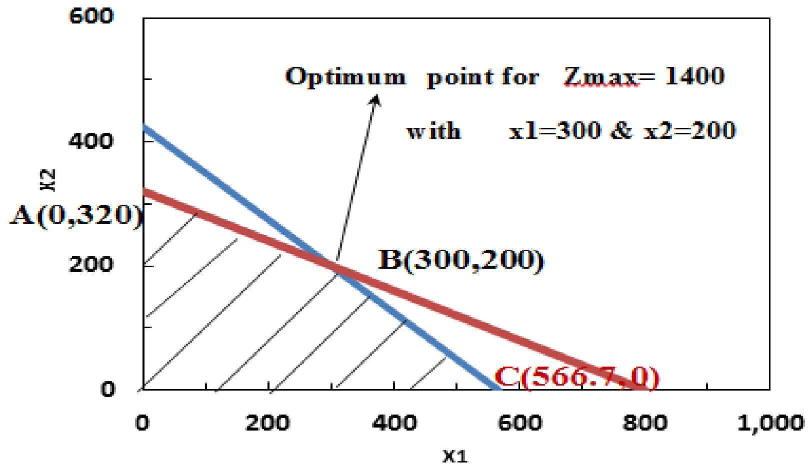
$$\begin{pmatrix} 3 & 4 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1700 \\ 1600 \end{pmatrix}$$

For this problem it could be selected the two non-basic variable in

$\frac{n!}{m!(n-m)!} = \frac{4!}{2!(2)!} = \mathbf{6}$ ways. The basic solutions are easily seen to be given by

No.	X_1	X_2	X_3	X_4	Point	Non-Basic Vs.	Basic Variables.	Remark
1	0	0	1700	1600	O	X_1 , X_2	X_3 , X_4	Basic feasible solution
2	0	425	0	-525		X_1 , X_3	X_2 , X_4	Basic solution
3	0	320	420	0	A	X_1 , X_4	X_2 , X_3	Basic feasible solution
4	566.67	0	0	466.67	C	X_2 , X_3	X_1 , X_4	Basic feasible solution
5	800	0	-700	0		X_2 , X_4	X_1 , X_3	Basic solution
6	300	200	0	0	B	X_3 , X_4	X_1 , X_2	Basic feasible solution

Of these 6 basic solutions, only 4 are also feasible, and it will be seen that these solutions correspond to the feasible vertices of the feasible region in the figure.



3- Fundamental Results for Linear Programming

- ***If the Constraints have a Feasible Region they have a Basic Feasible Solution.***
- ***The Feasible Region is a Convex Set.***
- ***Basic Feasible Solutions Correspond to Vertices of the Convex Set.***
- ***If the Objective Function has a Finite Minimum, then at Least One Optimal Solution is a Basic Feasible Solution.***

4- Simplex Method

Simplex method, developed by George Dantzing, is computational procedure gives the optimum solution from a basic feasible solution by algebraic form

Example 1:

Subject to $x_1, x_2 \geq 0$ and

$$3\,x_1 + 4\,x_2 \leq 1700$$
$$2\,x_1 + 5\,x_2 \leq 1600$$

minimize $Z_{max} = +\,2\,x_1 + 4\,x_2$

Solution: the ***appropriate canonical form*** is:

$$3\,x_1 + 4\,x_2 + x_3 = 1700$$
$$2\,x_1 + 5\,x_2 + x_4 = 1600$$
$$Z_{min} = -2\,x_1 - 4\,x_2$$

where $x_1 = x_2 = 0$

while $x_3 = 1700$ & $x_4 = 1600$

Now, if we fixed $x_1 = 0$ and increase x_2, the value of x_3 & x_4 will change such that:

For $3\,x_1 + 4\,x_2 + x_3 = 1700$

x_3 ***is made zero when*** $x_2 = \frac{1700}{4} = 425$

For $2\,x_1 + 5\,x_2 + x_4 = 1600$

x_4 ***is made zero when*** $x_2 = \frac{1600}{5} = 320$

For $x_4 = 0$ & $x_3 > 0$

Choose the minimum value for x_2

So, consider $x_2 = 320$

So the above ***canonical form*** *will take a new form by eliminating the basic variable* x_3 & x_2 *from the objective function and express the constraints as:*

$$\frac{7}{5}x_1 + 0\,x_2 + 1\,x_3 - \frac{4}{5}x_4 = 420$$

$$\frac{2}{5}x_1 + 1\,x_2 + 0\,x_3 + \frac{1}{5}x_4 = 320$$

$$-\frac{2}{5}x_1 + 0\,x_2 + 0\,x_3 + \frac{4}{5}x_4 = z + 1280$$

New canonical form with a non-basic variables of $X_1 = 0$, $X_4 = 0$ ***and basic or basis of*** $X_3 = 420$ & $X_2 = 320$ ***which represents a basic feasible solution. Also,*** $Z = -1280$

Now, if we fixed $x_4 = 0$ and increase x_1, the value of x_3 & x_2 will change such that:

For $\frac{7}{5}x_1 + 0x_2 + 1\,x_3 - \frac{4}{5}x_4 = 420$

x_3 *is made zero when* $x_1 = \frac{420}{7/5} = 300$

For $\frac{2}{5}x_1 + 1\,x_2 + 0x_3 + \frac{1}{5}x_4 = 320$

x_2 *is made zero when* $x_1 = \frac{320}{2/5} = 800$

For x3 = 0 & x2 > 0
Choose the minimum value for x_1
So, consider $x_1 = 300$

By the same way the new canonical form will take a new form by eliminating the basic variable x_1 ***&*** x_2 ***from the objective function and express the constraints as:***

$$x1 + 0x2 + \frac{5}{7}x3 - \frac{4}{7}x4 = 300$$

$$0\,x1 + x2 - \frac{2}{7}x3 + \frac{3}{7}x4 = 200$$

$$0\,x1 + 0x2 + \frac{2}{7}x3 + \frac{4}{7}x4 = z + 1400$$

Thus it is not the possible to reduce Z further since there is no (–ve signs) and we have reached the minimum of Z which is -1400 at the basic feasible solution of $x_1 = 300$ ***&*** $x_2 = 200$ ***with*** $x_3 = x_4 = 0$

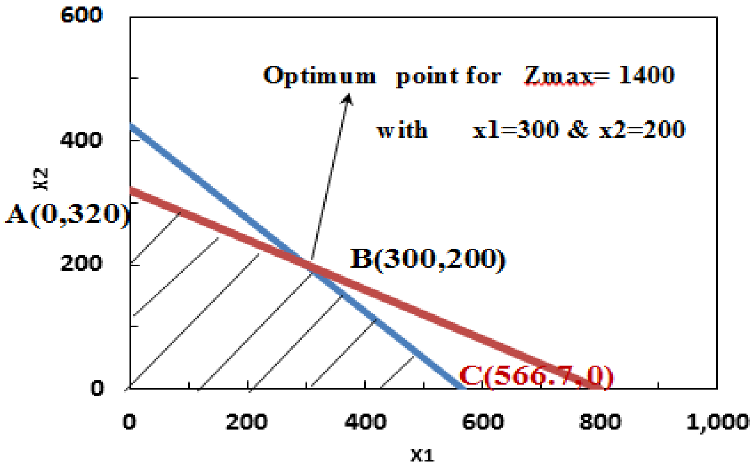
The calculation of this ***iterative process*** can be set out conveniently in the so called ***Simplex method.***

Example 2: *Re-solve example 1 using Simplex method.*

Solution:

The appropriate canonical form is:

$$3x_1 + 4x_2 + x_3 = 1700$$
$$2x_1 + 5x_2 + x_4 = 1600$$
$$-Z - 2x_1 - 4x_2 = 0$$



It.	Basis	Value	X_1	X_2	X_3	X_4	Row	Value/a	Remark
0	X_3	1700	3	4	1	.	1	1700/4=425	
	X_4	1600	2	5*	.	1	2	1600/5=320	
	-Z	0	-2	-4	.	.	3		
1	X_3	420	7/5*	.	1	-4/5	4	420/(7/5)=300	-4Pv + R1
	X_2	320	2/5	1	.	1/5	5	320/(2/5)=800	Pv = Pivot = R2/5
	-Z	1280	-2/5	.	.	4/5	6		-(-4)Pv + R3
2	X_1	300	1	.	5/7	-4/7	7		Pivot = R4/(7/5)
	X_2	200	.	1	-2/7	3/7	8		-(2/5)Pv + R5
	-Z	1400	.	.	2/7	4/7	9		-(-2/5)Pv + R6

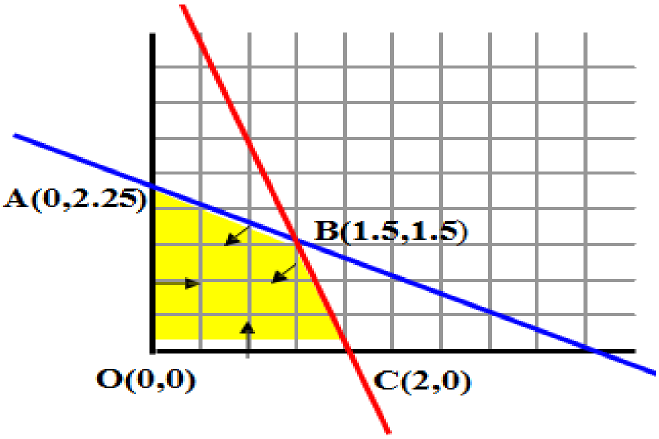
Example 3:

Min. $Z = -6X_1 - 2X_2$

Subject to:

$2X_1 + 4X_2 \le 9$

$3X_1 + X_2 \le 6$



Solution: *the problem in standard form is:*

$-Z - 6X_1 - 2X_2 = 0$

$2X_1 + 4X_2 + X_3 = 9$

$3X_1 + X_2 + X_4 = 6$

It.	Basis	Value	X_1	X_2	X_3	X_4	Row	Value/a	Remark
0	X_3	9	2	4	1	.	1	4.5	
	X_4	6	3*	1	.	1	2	2	
	-Z	0	-6	-2	.	.	3		
1	X_3	5	.	10/3*	1	-2/3	4	5/(10/3)=15/10	-2Pv + R1
	X_1	2	1	1/3	.	1/3	5	2/(1/3)=6	Pv = Pivot = R2/3
	-Z	12	.	0	.	2	6		-(-6)Pv + R3
2	X_2	3/2	.	1	3/10	-2/10	7		Pivot = R4/(10/3)
	X_1	3/2	1	.	-1/10	4/10	8		(-1/3)Pv + R5
	-Z	12	.	.	0	2			(Zero)Pv + R6

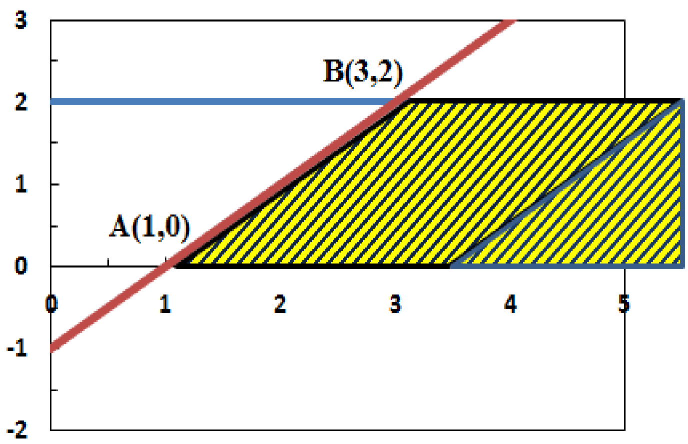
Example 4: ***Unbounded solution***

Maximize $Z = X_1 + X_2$

Subject to $X_1 \geq 0, X_2 \geq 0$

$X_1 - X_2 \geq 1$

$X_2 \leq 2$



Solution:

The problem in standard form is:

$X_1 - X_2 - X_3 = 1$

$X_2 + X_4 = 2$

Minimize $-Z - X_1 - X_2 = 0$

it is clear by inspection that $x_1 = 1, x_4 = 2, x_2 = x_3 = 0$ ***is the basic feasible solution and that the constraints are in the correct form to apply the method. The objective function contains*** x_1***, one of the basic variables. Thus, we must use the first constraint to eliminate*** x_1 ***from Z to obtain as follows:***

$$-2X_2 - X_3 - Z = 1$$

It.	Basis	Value	X_1	X_2	X_3	X_4	Row	Value/a	Remark
0	X_1	1	1	-1	-1	.	1	-1	
	X_4	2	.	1*	0	1	2	2	
	-Z	1	.	-2	-1	.	3		
1	X_1	3	1	.	-1	1	4	-3	-(-1) pv + R1
	X_2	2	.	1	0	1	5	∞	Pv = Pivot = R2/1
	-Z	5	.	.	-1	2	6		-(-2) pv + R3

There are no strictly positive coefficients in the x_3 column in the constraints. Thus, however much we increase x_3 we will never drive a basic variable to zero. Indeed x_1 will be increased and x_2 will remain unchanged. We have a case of an unbounded solution which is quite clear from the figure above. It manifested itself in the Simplex method through the fact that all ***ais'*** ≤ 0

4- Generating an Initial Basic Feasible Solution

The previous examples chosen to illustrate the simplex method were such that an initial basic feasible solution and the corresponding canonical form are obvious or else easy to obtain as in example 4. The difficulty arises from the "≥" and "=" constraints.

Example 4: *Min.* $Z = -3X_1 - 4X_2$

Sub. to:
$X_1 \geq 10$
$X_2 \geq 5$
$X_1 + X_2 \leq 20$
$-X_1 + 4X_2 \leq 20$

Solution: *the problem in standard form is:*

$X_1 - X_3 = 10$
$X_2 - X_4 = 5$
$X_1 + X_2 + X_5 = 20$
$-X_1 + 4X_2 + X_6 = 20$
$-Z - 3X_1 - 4X_2 = 0$

The problem is that we have no obvious basic feasible solution. The basic solution obtained by equating the slack variables to R.H.S. values is not feasible. This solution is x1=x2=0, x3=-10, x4=-5, x5=20, x6=20 where x3 & x4 are negative.

To generate the basic feasible solution, we modify the first two constraints (for ≥) by introducing ***artificial variables such as*** x_7 **&** x_8 **(which are non-negative)** into ***L.H.S.*** So, the **modified** constraints are:

$X_1 - X_3 + X_7 = 10$
$X_2 - X_4 + X_8 = 5$
$X_1 + X_2 + X_5 = 20$
$-X_1 + 4X_2 + X_6 = 20$
$-Z - 3X_1 - 4X_2 = 0$

For this form the basic feasible solution is clear. it is x1=x2=x3=x4=0 (non-basic variables) while x7=10 ,x8=5, x5=20 and x6=20 (which are the basis)

Now, we use the Simplex method to minimize $w = x_7 + x_8$

Where ***w: is called the artificial objective function.***

In this case the problem will be solved by two phases as follows:

Phase I: Minimize w and will end when w with both the artificial variables reduced to zero.

Phase II: Minimize Z starting with the end of phase I and using the results as the initial basic feasible solution to minimize Z and ignore w with all artificial variables.

Express $\boldsymbol{w}$ in term of non-basic variables by eliminating x_7 & x_8 (basic variable) from $\boldsymbol{w}$. So: $-x_1 - x_2 + x_3 + x_4 - w = -15$

1- The tableaux for phase I for minimum *w* of the problem are as follows:

It.	Basis	Value	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
0	X_7	10	1*	0	-1	0	.	.	1	.
	X_8	5	0	1	0	-1	.	.	.	1
	X_5	20	1	1	0	0	1	.	.	.
	X_6	20	-1	4	0	0	.	1	.	.
	-Z	0	-3	-4	0	0	.	.	.	.
	-*w*	-15	-1	-1	1	1	.	.	.	.
1	X_1	10	1	0	-1	0	.	.	1	.
	X_8	5	.	1*	0	-1	.	.	0	1
	X_5	10	.	1	1	0	1	.	-1	.
	X_6	30	.	4	-1	0	.	1	1	.
	-Z	30	.	-4	-3	0	.	.	3	.
	-*w*	-5	.	-1	0	1	.	.	1	.
2	X_1	10	1	.	-1	0	.	.	1	0
	X_2	5	.	1	0	-1	.	.	0	1
	X_5	5	.	.	1	1	1	.	-1	-1
	X_6	10	.	.	-1	4	.	1	1	-4
	-Z	50	.	.	-3	-4	.	.	3	4
	-*w*	0	.	.	0	0	.	.	1	1

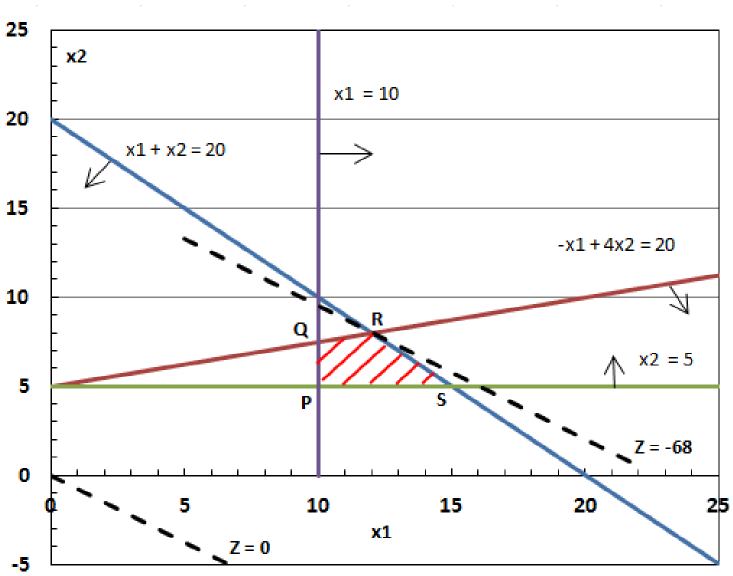
2- The tableaux for phase II for minimum *Z* of the problem are as follows:

It.	Basis	Value	X_1	X_2	X_3	X_4	X_5	X_6
2	X_1	10	1	.	-1	0	.	.
	X_2	5	.	1	0	-1	.	.
	X_5	5	.	.	1	1	1	.
	X_6	10	.	.	-1	4*	.	1
	-Z	50	.	.	-3	-4	.	.
3	X_1	10	1	.	-1	.	.	0
	X_2	15/2	.	1	-1/4	.	.	1/4
	X_5	5/2	.	.	5/4*	.	1	-1/4
	X_4	5/2	.	.	-1/4	1	.	1/4
	-Z	60	.	.	-4	.	.	1
4	X_1	12	1	.	.	.	4/5	-1/5
	X_2	8	.	1	.	.	1/5	1/5
	X_3	2	.	.	1	.	4/5	-1/5
	X_4	3	.	.	.	1	1/5	1/5
	-Z	68	.	.	.	.	16/5	1/5

∴ *The optimal so that the minimum of Z is -68 for x1 = 12, x2=8, x3=2, x4=3.*

The successive tableaux 2,3,4 refer to the points P, Q, R in the figure.

Points	X_1	X_2	Z
P	10	5	-50
Q	10	7.5	-60
R	12	8	-68



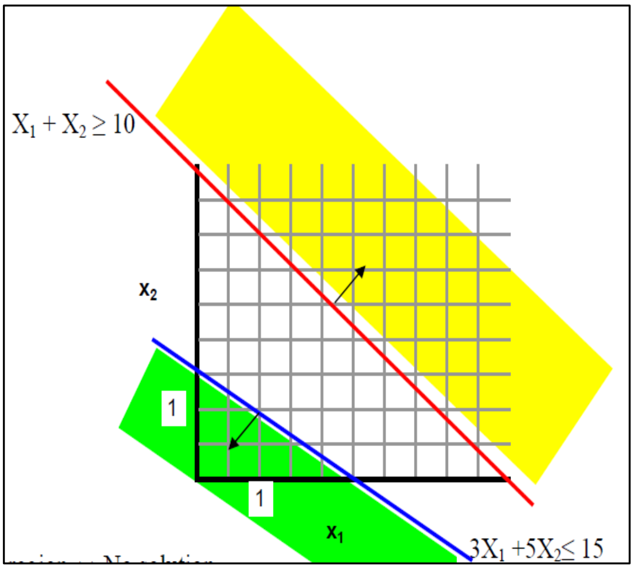
Example 5: No feasible solution

Maximize $Z = 2X_1 + 3X_2$

Subject to: $X_1 + X_2 \geq 10$

$3X_1 + 5X_2 \leq 15$

$X_1 \geq 0 \ , \ X_2 \geq 0$



Solution:

The problem in standard form is:

$X_1 + X_2 - X_3 + X_5 = 10$

$3X_1 + 5X_2 + X_4 = 15$

$-Z - 2X_1 - 3X_2 = 0$

$W = X_5 = 10 - X_1 - X_2 + X_3$

Or $-W - X_1 - X_2 + X_3 = -10$

It.	Basis	Value	X_1	X_2	X_3	X_4	X_5
0	X_5	10	1	1	-1	.	1
	X_4	15	3*	5	0	1	.
	-Z	0	2	3	0	.	.
	-w	-10	-1	-1	1	.	.
1	X_5	5	.	-2/3	-1	-1/3	1
	X_1	5	1	5/3	0	1/3	.
	-Z	-10	.	-1/3	0	-2/3	.
	-w	-5	.	2/3	1	1/3	.

At this stage ***W*** is minimized since all the coefficients in the ***W*** row are positive. But ***W*** has not been reduced to zero and **x5 is 5**. ***Phase I*** is complete but we cannot start on ***phase II*** because the original constraints do not have a basic feasible solution.

Example 6: ***A firm requires coal with phosphorus content no more than 0.03% and no more than 3.25% ash impurity. Three grades of coal A, B, C are available at the price shown. What is the quantity from each grade to minimize the cost***

Grade	% phosphorus	% Ash	Cost ($/ton)
A	0.06	2	30
B	0.04	4	30
C	0.02	3	45

Solution:

Let 1 tone of the blend contain x_1, x_2, x_3 tones of A, B, and C respectively.

Then $x_1, x_2, x_3 \geq 0$. ***Further we require that:***

$x_1 + x_2 + x_3 = 1$

$0.06\, x_1 + 0.04\, x_2 + 0.02\, x_3 \leq 0.03$ or $(6x1 + 4x2 + 2x3 \leq 3)$

$2\, x_1 + 4\, x_2 + 3\, x_3 \leq 3.25$

Thus, subject to these constraints we require that:

$$Cost = Z\ min = 30\, x_1 + 30\, x_2 + 45\, x_3$$

No, for an obvious basic feasible solution put the system above such that:

$x_1 + x_2 + x_3 + x_6 = 1$

$6\, x_1 + 4\, x_2 + 2\, x_3 + x_4 = 3$

$2\, x_1 + 4\, x_2 + 3\, x_3 + x_5 = 3.25$

$-Z + 30\, x_1 + 30\, x_2 + 45\, x_3 = 0$

$W = x_6$

or $-x_1 - x_2 - x_3 - w = -1$

It.	*Basis*	Value	X_1	X_2	X_3	X_4	X_5	X_6
0	X_6	1	1	1	1*	.	.	1
	X_4	3	6	4	2	1	.	.
	X_5	3.25	2	4	3	.	1	.
	-Z	0	30	30	45	.	.	.
	-w	-1	-1	-1	-1	.	.	.
1	X_3	1	1	1	1	.	.	1
	X_4	1	4	2	.	1	.	-2
	X_5	0.25	-1	1*	.	.	1	-3
	-Z	- 45	-15	-15	.	.	.	-45
	-w	0	0	0	.	.	.	1
2	X_3	3/4	2	.	1	.	-1	
	X_4	1/2	6*	.	.	1	-2	
	X_2	1/4	-1	1	.	.	.	
	-Z	- 41.25	-30	.	.	.	15	
3	X_3	7/12	.	.	1	-1/3	-1/3	
	X_1	1/12	1	.	.	1/6	-1/3	
	X_2	1/3	.	1	.	1/6	2/3	
	-Z	- 38.75	.	.	.	5	5	

∴ *The minimum value of Z is 38.75 for x_1 = 1/12, x_2 =1/3, x_3 = 7/12.*

5-Special Cases

5.1 linearization of some non-linear problems:

In some cases, we could solve some non-linear problems using the simplex method after linearize them as follows:

Example

Find $Z_{max} = x_1 \cdot x_2^2$

Sub. to: $x_1 x_2 \leq 5$

$x_1 \geq 2 x_2$

$1 \leq x_1 \leq 10$

$1 \leq x_2 \leq 10$

Solution:

1 - $\ln Z_{max} = \ln x_1 + 2 \ln x_2$

Sub. to: $\ln x_1 + \ln x_2 \leq 1.61$

$\ln x_1 - \ln x_2 \geq 0.693$

$0 \leq \ln x_1 \leq 2.303$

$0 \leq \ln x_2 \leq 2.303$

2- Let $P = \ln Z_{max}$

$y_1 = \ln x_1$; $y_2 = \ln x_2$

So the problem will take the form:

$P_{max} = y_1 + 2 y_2$

Sub. to: $y_1 + y_2 \leq 1.61$

$y_1 - y_2 \geq 0.693$

$0 \leq y_1 \leq 2.303$

$0 \leq y_2 \leq 2.303$

3- In standard form the problem is:

$P_{min} = - P_{max} = - y_1 - 2 y_2$

Sub. to: $y_1 + y_2 + y_3 = 1.61$

$y_1 - y_2 - y_4 + y_7 = 0.693$

$y_1 + y_5 = 2.303$

$y_2 + y_6 = 2.303$

$w = y_7 = - y_1 + y_2 + y_4 + 0.693$

or $-w - y_1 + y_2 + y_4 = -0.693$

5.2 Absolute Problems:

Example $Z_{min} = Abs\,(2 x_1 - x_2) = |2 x_1 - x_2|$

Sub. to: $x_1 + x_2 \leq 5$

Solution: let $Abs\,(2x_1 - x_2) = |\varepsilon|$

Where the value of ε may be positive or negative (any value).

So let: $\varepsilon = x_3 - x_4$ with $(x_3 \& x_4) \geq 0$

$\therefore \quad |\varepsilon| = |x_3 - x_4|$

So, $Z_{min} = |2x_1 - x_2| = |\varepsilon| = |x_3 - x_4| = x_3 + x_4$

Sub. to: $x_1 + x_2 \leq 5$

$2x_1 - x_2 = x_3 - x_4$

Therefore, the problem in standard form is:

$Z_{min} = x_3 + x_4$

Sub. to: $x_1 + x_2 + x_5 = 5$

$2x_1 - x_2 - x_3 + x_4 + x_6 = 0$

$w = x_6 = -2x_1 + x_2 + x_3 - x_4$

Or $-w - 2x_1 + x_2 + x_3 - x_4 = 0$

5.3 Multiobjective Planning Model

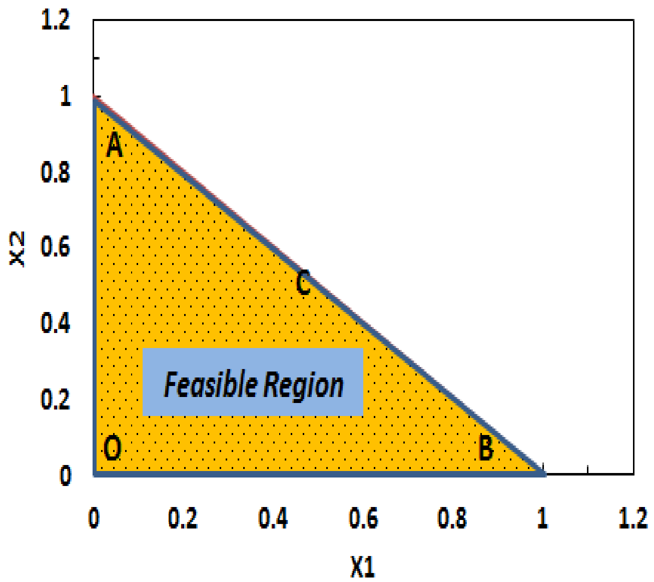
Example: Objective Function Min, Max{ $(2 - x_1 - 2x_2)$, $(1 - x_1)$ }
Subject to $x_1 + x_2 \leq 1$

Solution: let $Z_1 = 2 - x_1 - 2x_2$ & $Z_2 = 1 - x_1$

So, the problem is:
Min of Max (Z_1, Z_2)
Sub. to : $x_1 + x_2 \leq 1$

The solution by graphical method is:

Point	X_1	X_2	Z_1	Z_2	Z_{max}
O	0	0	2	1	2
A	0	1	0	1	1
B	1	0	1	0	1
C	0.5	0.5	0.5	0.5	0.5



By the Simplex Method the problem will take the following form:

Objective Function is Min, Max{ $(2 - x_1 - 2x_2)$, $(1 - x_1)$}

Subject to $x_1 + x_2 \le 1$; $X \ge 0$

let $x_3 \ge 2 - x_1 - 2x_2$ ***or*** $x_3 + x_1 + 2x_2 \ge 2$ --------------- 1
and $x_3 \ge 1 - x_1$ ***or*** $x_3 + x_1 \ge 1$ --------------- 2
$x_1 + x_2 \le 1$ --------------- 3
With $\text{Min } Z = x_3$

In standard form the problem is:

$$x_3 + x_1 + 2x_2 - x_4 + x_7 = 2$$

$$x_3 + x_1 - x_5 + x_8 = 1$$

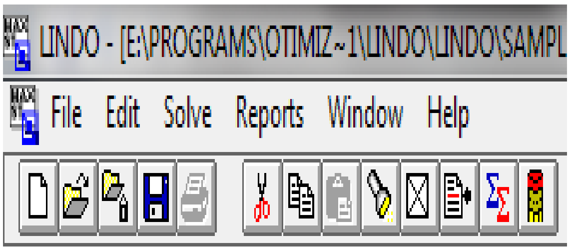
$$x_1 + x_2 + x_6 = 1$$

$$-Z + x_3 = 0$$

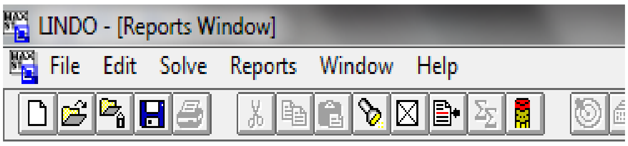
$$w = x_7 + x_8 = 3 - 2x_3 - 2x_1 - 2x_2 + x_4 + x_5$$

Or $-w \;\; -2x_3 - 2x_1 - 2x_2 + x_4 + x_5 = -3$

Using Lindo software, the results are as follows:




```
Min     x3
SUBJECT TO
       2)   x3 + x1 + 2 x2 >= 2
       3)   x3 + x1  >= 1
       4)   x1 + x2 <= 1

END
```




```
 LP OPTIMUM FOUND AT STEP      2

        OBJECTIVE FUNCTION VALUE

        1)      0.5000000

  VARIABLE        VALUE          REDUCED COST
        X3         0.500000          0.000000
        X1         0.500000          0.000000
        X2         0.500000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         -0.500000
        3)         0.000000         -0.500000
        4)         0.000000          1.000000

 NO. ITERATIONS=       2
```

6- Applications

6.1 Water Resource Management

Example:

Streams 1 and 2 each possessed of a reservoir, joint to form a common stream 3, as shown below. The total benefits derived from annual releases x_1 and x_2 from each reservoir are:

$$Z = 5\,x_1 + 3\,x_2$$

The maximum capacity of the reservoir 1 is 11 M.m³ and the maximum capacity of the reservoir 2 is 10 M.m³. Initial annual inflow of streams 1 & 2 are 5 & 4 M.m³ respectively. The maximum capacity of channels 1, 2 and 3 are (6, 5 and 9) M.m³ respectively. Formulate the optimization problem for maximizing annual benefit.

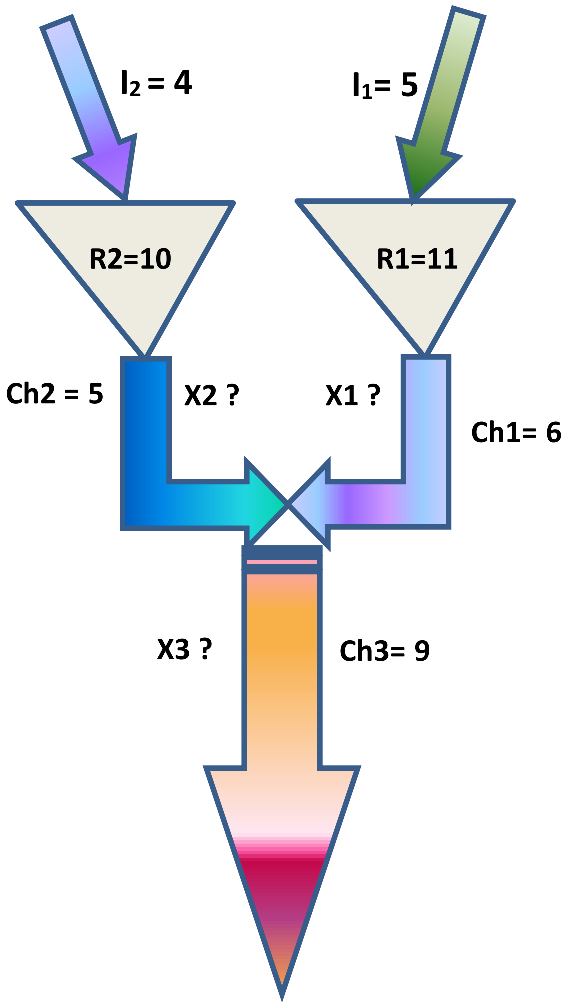
Given :

Inflow: $I_1 = 5$ & $I_2 = 4$

Initial Storage: $S_{1,o} = 7$ & $S_{2,o} = 7$

	Capacity:
Res.1	R1 = 11
Res.2	R2 = 10
River 1	Ch1 = 6
River 2	Ch1 = 5
River 3	Ch1 = 9

The required is the out flow (X1 & X2) for maximum benefit.



Solution:

1- *For reservoir 1*

Storage 2 = storage 1 + inflow - outflow

$S_{1,2} = S_{1,1} + I_1 - O_1$ for $0 \le S_{1,2} \le Rc1$

$0 \le 7 + 5 - x1 \le 11$

So, for $7 + 5 - X1 \le 11$ yield $X1 \ge 1$

and for $0 \le 7 + 5 - X1$ yield $X1 \le 12$

2- For Reservoir 2

$S_{2,2} = S_{2,1} + I_2 - O_2$ for $0 \le S_{2,2} \le Rc2$

$0 \le 7 + 4 - X2 \le 10$

So, for $7 + 4 - X2 \le 10$ yield $X2 \ge 1$

and for $0 \le 7 + 4 - X2$ yield $X2 \le 11$

3- For River 1

$X1 \le ch1$ or $X1 \le 6$

4- For River 2

$X2 \le ch2$ or $X2 \le 5$

5- For River 3

$X1 + X2 \le Ch3$ or $X1 + X2 \le 9$

6- Now , the problem is:

$Z\,max = 5\,X1 + 3\,X2$

Subject to:

$X1 \ge 1$

$X1 \le 12$

$X2 \ge 1$

$X2 \le 11$

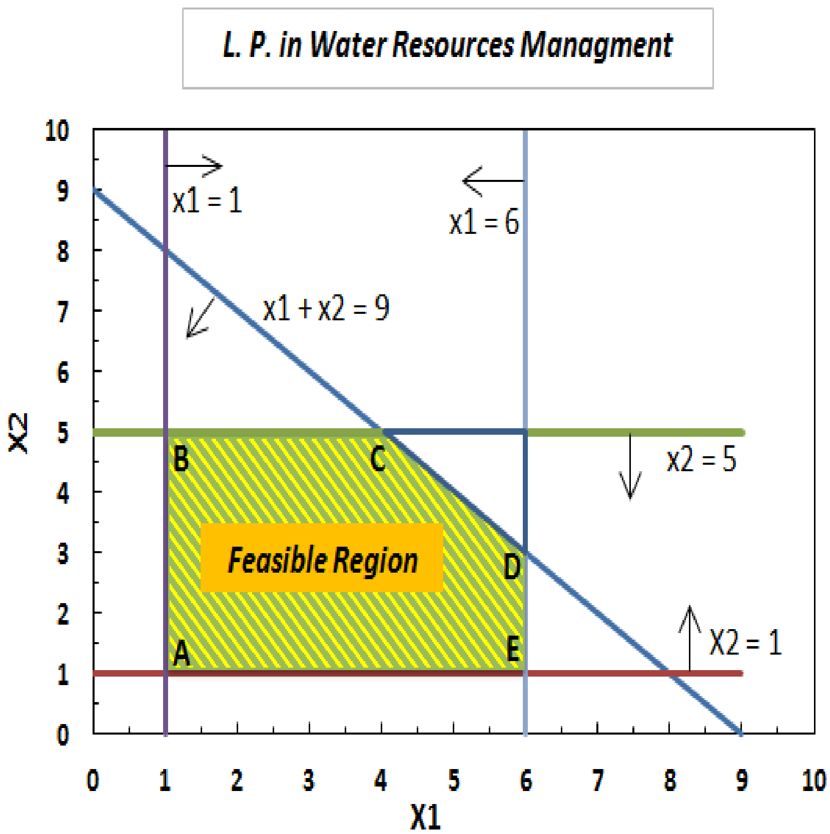
$X1 \le 6$

$X2 \le 5$

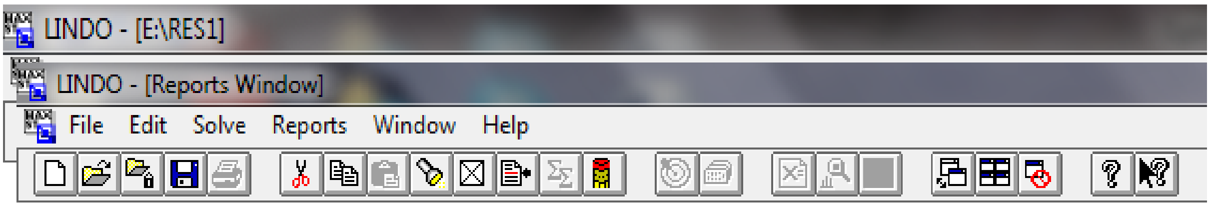
$X1 + X2 \le 9$

The following figuer and table show the solution by graphical method.

Points	X1	X2	Z_{max}
A	1	1	8
B	1	5	20
C	4	5	35
D	6	3	39
E	6	1	33



Using Lindo software, the results are as follows:

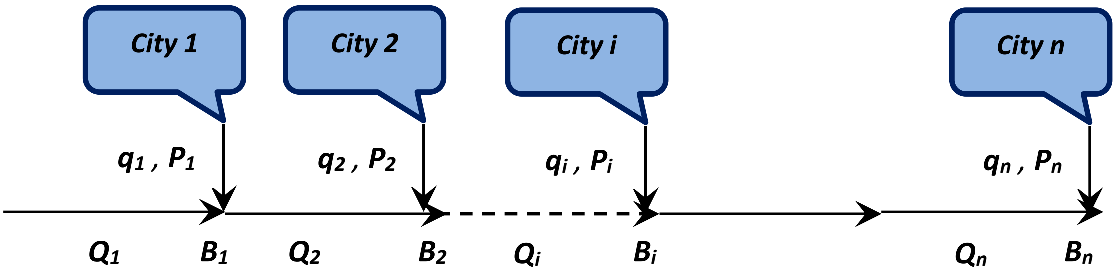



```
LP OPTIMUM FOUND AT STEP      2

       OBJECTIVE FUNCTION VALUE

       1)      39.00000

 VARIABLE        VALUE          REDUCED COST
       X1         6.000000         0.000000
       X2         3.000000         0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         5.000000         0.000000
       3)         0.000000         2.000000
       4)         2.000000         0.000000
       5)         2.000000         0.000000
       6)         0.000000         3.000000

NO. ITERATIONS=       2
```

H.W: try to provide the solution by the Simplex method.

6.2 Water Quality Management

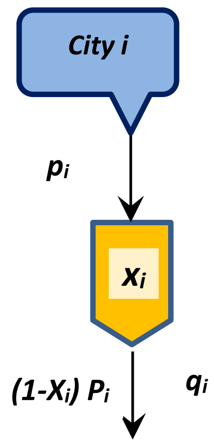
Assume that we have the following system



Where:

Q_i : Stream flow rate at i-point, (volume / time).
q_i : waste water rate of i-city, (volume / time).
P_i : amount of a disposal pollution from i-city, (mass / time)
B_i : Maximum allowable pollution at i-point, (mass / volume).

If the system above has no waste water treatment plant the pollution of each city P_i will reach the river or the stream at the same value or load. With a waste water treatment plant the pollution will be equals to $(1-x_i)P_i$, where x_i is removal ratio of the treatment plant of the ***i-city.***



Conservation and non-conservation pollution

Any pollution may be conservation or non-conservation. Consequently, the formulation of the problem will be considered. Therefore:

A- *For conservation pollution.*

For point 1: $\frac{(1-X_1)\,P_1}{Q_1+q_1} \le B_1$ Or $\frac{X_1\,P_1}{Q_1+q_1} \ge \frac{P_1}{Q_1+q_1} - B_1$

For point 2: $\frac{(1-X_1)\,P_1}{Q_1+q_1+q_2} + \frac{(1-X_2)\,P_2}{Q_1+q_1+q_2} \le B_2$

$$Or\quad \frac{X_1\,P_1}{Q_1+q_1+q_2} + \frac{X_2\,P_2}{Q_1+q_1+q_2} \ge \frac{P_1}{Q_1+q_1+q_2} + \frac{P_2}{Q_1+q_1+q_2} - B_2$$

For point i:

$$\frac{(1-X_1)\,P_1}{Q_1+q_1+\cdots+q_i}+\frac{(1-X_2)\,P_2}{Q_1+q_1+\cdots+q_i}+\cdots\cdots+\frac{(1-X_i)\,P_i}{Q_1+q_1+\cdots+q_i}\leq B_i$$

$$Or\quad \frac{X_1\,P_1}{Q_1+q_1+\cdots+q_i=Q_T}+\frac{X_2\,P_2}{Q_T}+\cdots\cdots+\frac{X_i\,P_i}{Q_T}\geq\frac{P_1}{Q_T}+\frac{P_2}{Q_T}+\cdots\cdots+\frac{P_i}{Q_T}-B_i$$

And so on continue until the n-city

For point n:

$$\frac{(1-X_1)\,P_1}{Q_1+q_1+\cdots+q_n}+\frac{(1-X_2)\,P_2}{Q_1+q_1+\cdots+q_n}+\cdots\cdots\cdots+\frac{(1-X_n)\,P_n}{Q_1+q_1+\cdots+q_n}\leq B_n$$

$$Or\quad \frac{X_1\,P_1}{Q_1+q_1+\cdots+q_n=Q_T}+\frac{X_2\,P_2}{Q_T}+\cdots\cdots+\frac{X_n\,P_n}{Q_T}\geq\frac{P_1}{Q_T}+\frac{P_2}{Q_T}+\cdots\cdots+\frac{P_n}{Q_T}-B_n$$

The objective function is:

$$Z_{min}=C_1x_1P_1+C_2x_2P_2+C_ix_iP_i+\cdots\cdots C_nx_nP_n$$

B- ***For non-conservation pollution.***

For point 1: $\frac{(1-X_1)\,\rho_{11}P_1}{Q_1+q_1}\leq B_1$ **Or** $\frac{X_1\rho_{11}\,P_1}{Q_1+q_1}\geq\frac{\rho_{11}\,P_1}{Q_1+q_1}-B_1$

For point 2: $\frac{(1-X_1)\rho_{12}\,P_1}{Q_1+q_1+q_2}+\frac{(1-X_2)\rho_{22}\,P_2}{Q_1+q_1+q_2}\leq B_2$

$$Or\quad \frac{X_1\rho_{12}\,P_1}{Q_1+q_1+q_2}+\frac{X_2\rho_{22}\,P_2}{Q_1+q_1+q_2}\geq\frac{\rho_{12}\,P_1}{Q_1+q_1+q_2}+\frac{\rho_{22}\,P_2}{Q_1+q_1+q_2}-B_2$$

For point 3: $\frac{(1-X_1)\rho_{13}\,P_1}{Q_1+q_1+q_2+q_3}+\frac{(1-X_2)\rho_{23}\,P_2}{Q_1+q_1+q_2+q_3}+\frac{(1-X_3)\rho_{33}\,P_3}{Q_1+q_1+q_2+q_3}\leq B_3$

$$Or\quad \frac{X_1\rho_{13}\,P_1}{Q_1+q_1+q_2+q_3}+\frac{X_2\rho_{23}\,P_2}{Q_1+q_1+q_2+q_3}+\frac{X_3\rho_{33}\,P_3}{Q_1+q_1+q_2+q_3}\geq\frac{\rho_{13}\,P_1}{Q_1+q_1+q_2+q_3}+\frac{\rho_{23}\,P_2}{Q_1+q_1+q_2+q_3}+\frac{\rho_{33}\,P_3}{Q_1+q_1+q_2+q_3}-B_3$$

For point i:

$$\frac{(1-X_1)\rho_{1i}P_1}{Q_1+q_1+\cdots+q_i}+\frac{(1-X_2)\rho_{2i}P_2}{Q_1+q_1+\cdots+q_i}+\cdots\cdots+\frac{(1-X_i)\rho_{ii}P_i}{Q_1+q_1+\cdots+q_i}\leq B_i$$

Or
$$\frac{X_1\rho_{1i}P_1}{Q_1+q_1+\cdots+q_i=Q_T}+\frac{X_2\rho_{2i}P_2}{Q_T}+\cdots\cdots+\frac{X_i\rho_{ii}P_i}{Q_T}\geq\frac{\rho_{1i}P_1}{Q_T}+\frac{\rho_{2i}P_2}{Q_T}+\cdots\cdots+\frac{\rho_{ii}P_i}{Q_T}-B_i$$

And so on continue until the n-city

For point n:
$$\frac{(1-X_1)\rho_{1n}P_1}{Q_1+q_1+\cdots+q_n}+\frac{(1-X_2)\rho_{2n}P_2}{Q_1+q_1+\cdots+q_n}+\cdots\cdots\cdots\cdots+\frac{(1-X_n)\rho_{nn}P_n}{Q_1+q_1+\cdots+q_n}\leq B_n$$

Or
$$\frac{X_1\rho_{1n}P_1}{Q_1+q_1+\cdots+q_n=Q_T}+\frac{X_2\rho_{2n}P_2}{Q_T}+\cdots\cdots\cdots+\frac{X_n\rho_{nn}P_n}{Q_T}\geq\frac{\rho_{1n}P_1}{Q_T}+\frac{\rho_{2n}P_2}{Q_T}+\cdots\cdots\cdots+\frac{\rho_{nn}P_n}{Q_T}-B_n$$

The objective function is:

$$Zmin = C_1x_1P_1+C_2x_2P_2+C_ix_iP_i+\cdots\cdots C_nx_nP_n$$

In general the formulation is:

$$\sum_{i=1}^{k}\frac{x_i\,\rho_{ik}P_i}{[Q_k+\sum_{i=1}^{k}q_i]}\geq\sum_{i=1}^{k}\frac{\rho_{ik}P_i}{[Q_k+\sum_{i=1}^{k}q_i]}-B_k$$

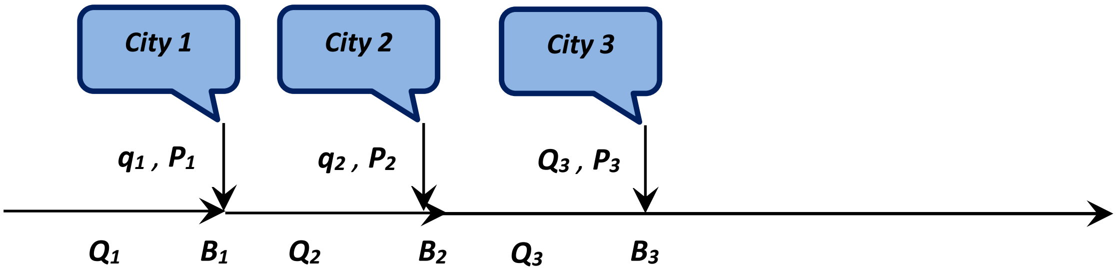
The objective function is:

$$Zmin = \sum_{i=1}^{n}C_ix_iP_i = C_1x_1P_1+\cdots+C_ix_iP_i+\cdots C_nx_nP_n$$

Not that:

- ρ_{ik} *is the decomposition ratio, fraction ratio of i*[th] *waste still present at location k.*
- *For conservation pollution* $\rho_{ik}=1$
- *For non-conservation pollution* $\rho_{11}=\rho_{22}=\rho_{ii}=1$
- $0\leq\rho_{ik}\leq1$
- $\rho_{ii}=1>\rho_{i,i+1}>\rho_{i,i+2}>\rho_{i,i+3}>\cdots\cdots>\rho_{i,k}$

Example: ***For the following waste water treatments system, formulate the problem as a linear programing model.***



Solution

1- For conservation pollution.

For point 1: $\frac{(1-X_1)\,P_1}{Q_1+q_1} \leq B_1$ **Or** $\frac{X_1\,P_1}{Q_1+q_1} \geq \frac{P_1}{Q_1+q_1} - B_1$

For point 2: $\frac{(1-X_1)\,P_1}{Q_1+q_1+q_2} + \frac{(1-X_2)\,P_2}{Q_1+q_1+q_2} \leq B_2$

$$O\,r \quad \frac{X_1\,P_1}{Q_1+q_1+q_2} + \frac{X_2\,P_2}{Q_1+q_1+q_2} \geq \frac{P_1}{Q_1+q_1+q_2} + \frac{P_2}{Q_1+q_1+q_2} - B_2$$

For point 3:

$$\frac{(1-X_1)\,P_1}{Q_1+q_1+q_2+q_3} + \frac{(1-X_2)\,P_2}{Q_1+q_1+q_2+q_3} + \frac{(1-X_3)\,P_3}{Q_1+q_1+q_2+q_3} \leq B_3$$

$$O\,r \quad \frac{X_1\,P_1}{Q_1+q_1+q_2+q_3=Q_T} + \frac{X_2\,P_2}{Q_T} + \frac{X_3\,P_3}{Q_T} \geq \frac{P_1}{Q_T} + \frac{P_2}{Q_T} + \frac{P_3}{Q_T} - B_3$$

$$Zmin \;=\; C_1x_1P_1 + C_2x_2P_2 + C_3x_3P_3$$

2- For non-conservation pollution.

For point 1: $\frac{(1-X_1)\,\rho_{11}P_1}{Q_1+q_1} \leq B_1$ **Or** $\frac{X_1(1)P_1}{Q_1+q_1} \geq \frac{1\,P_1}{Q_1+q_1} - B_1$

For point 2: $\frac{(1-X_1)\rho_{12}\,P_1}{Q_1+q_1+q_2} + \frac{(1-X_2)\rho_{22}\,P_2}{Q_1+q_1+q_2} \leq B_2$

$$O\,r \quad \frac{X_1\rho_{12}\,P_1}{Q_1+q_1+q_2} + \frac{X_2(1)\,P_2}{Q_1+q_1+q_2} \geq \frac{\rho_{12}\,P_1}{Q_1+q_1+q_2} + \frac{1\,P_2}{Q_1+q_1+q_2} - B_2$$

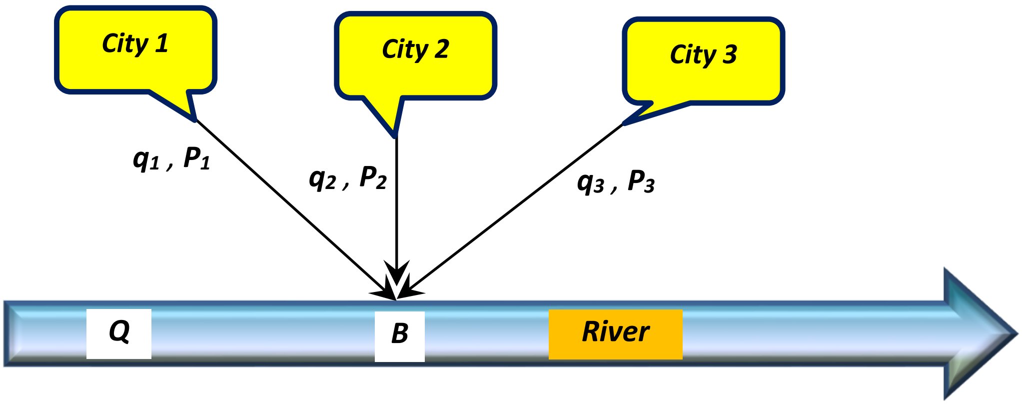
For point 3:

$$\frac{(1-X_1)\,\rho_{13}\,P_1}{Q_1+q_1+q_2+q_3=Q_T}+\frac{(1-X_2)\,\rho_{23}\,P_2}{Q_T}+\frac{(1-X_3)\,\rho_{33}\,P_3}{Q_T}\leq B_3$$

$$Or\quad \frac{X_1\rho_{13}\,P_1}{Q_1+q_1+q_2+q_3=Q_T}+\frac{X_2\rho_{23}\,P_2}{Q_T}+\frac{X_3\,(1)\,P_3}{Q_T}\geq\frac{\rho_{13}\,P_1}{Q_T}+\frac{\rho_{23}\,P_2}{Q_T}+\frac{1\,P_3}{Q_T}-B_3$$

$$Zmin\ =\ C_1x_1P_1+C_2x_2P_2+C_3x_3P_3$$

Home Work. ***For the following waste water treatments system, formulate the problem as a linear programing model.***



Solution

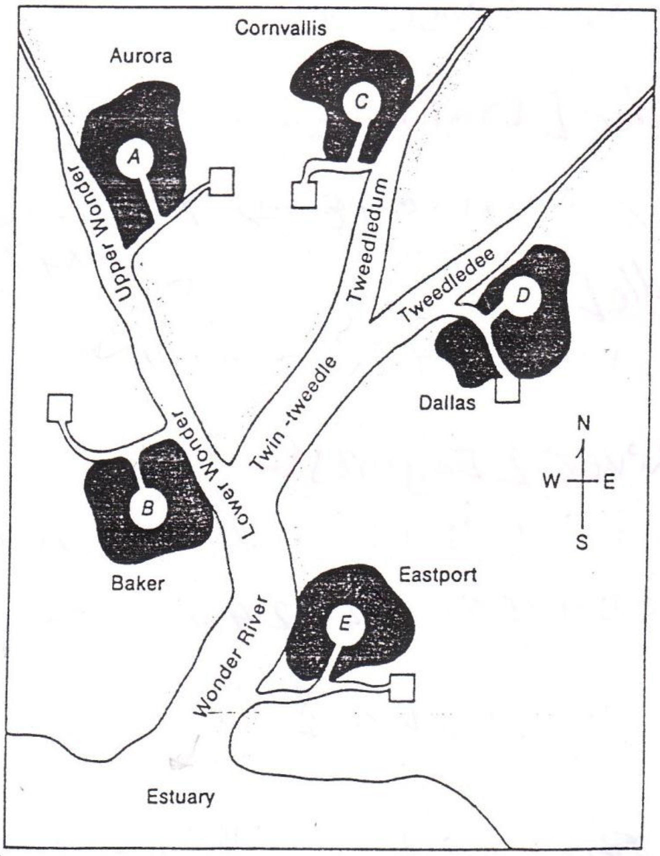
$$\frac{(1-X_1)\,P_1+(1-X_2)\,P_2+(1-X_3)\,P_3}{Q_1+q_1+q_2+q_3=QT}\leq B_1$$

$$Or\quad \frac{X_1\,P_1}{QT}+\frac{X_2\,P_2}{QT}+\frac{X_3\,P_3}{QT}\geq\frac{P_1}{QT}+\frac{P_2}{QT}+\frac{P_3}{QT}-B_1$$

$$Zmin\ =\ C_1x_1P_1+C_2x_2P_2+C_3x_3P_3$$

Example (Application Problem):

- . *Wonder River sewage-treatment program* Wonder Valley is criss-crossed by Wonder River and its two tributaries, Tweedledum and Tweedledee. Five cities are located on this tributary system; Aurora, Baker, Cornvallis, Dallas, and Eastport. These communities discharge wastes into the river system but are now being asked by the Wonderland government to submit a consolidated plan for building sewage-treatment plants. Because of the distance involved, it is necessary to build a plant in each community and/or discharge the waste into the river. Individual cities must cooperate to satisfy specific biochemical oxygen demand (BOD) levels at minimum cost. Neglecting waste decomposition in the river, the following data were obtained. See Figure . Formulation only.



City	BOD load waste discharged by the community, lb/day	Sewage plant treatment cost, $/lb of BOD	Streamflow at the city, ft³/sec	Maximum BOD in stream, lb/ft³	Maximum BOD pounds allowed in the river, lb/day
Aurora	2,200	.18	3.2	0.0012	332
Baker	800	.22	3.6	0.0012	373
Cornvallis	2,000	.16	4.5	0.0010	389
Dallas	500	.25	5.0	0.0015	648
Eastport	1,200	.20	13.1	0.0013	1,452

Maximum allowable Twin-tweedle river BOD is 829 lb/day.

Solution:

1- For A-City (Aurora):

$$\frac{(1-X_1)\,P_1}{q_1} \leq B_1 \quad Or \quad \frac{X_1\,P_1}{q_1} \geq \frac{P_1}{q_1} - B_1$$

$$\frac{2200\,X_1}{3.2*24*60^2} \geq \frac{2200}{3.2*24*60^2} - 0.0012$$

$$\therefore 0.00796\,x1 \geq 0.00676 \qquad \text{----------------- } 1$$

2- For Upper wonder:

$(1 - x_1)\,P_1 \leq 332$

Or $(1 - x_1)\,2200 \leq 332$

So, $2200\,x_1 \geq 1868$ ----------------- 2

3- For B-City (Baker):

$$\frac{(1-X_2)\,P_2}{q_2} \leq B_2 \quad Or \quad \frac{X_2\,P_2}{q_2} \geq \frac{P_2}{q_2} - B_2$$

$$\frac{800\,X_2}{3.6*24*60^2} \geq \frac{800}{3.2*24*60^2} - 0.0012$$

$$\therefore 0.002572\,x2 \geq 0.001372 \qquad \text{----------------- } 3$$

4- For Lower wonder

$(1 - x_1)\,P_1 + (1 - x_2)\,P_2 \leq 373$

$(1 - x_1)\,2200 + (1 - x_2)\,800 \leq 373$

So, $2200\,x_1 + 800\,x_2 \geq 2627$ ----------------- 4

5- For C-City (Cornvallis):

$$\frac{(1-X_3)\,P_3}{q_3} \leq B_3 \quad Or \quad \frac{X_3\,P_3}{q_3} \geq \frac{P_3}{q_3} - B_3$$

$$\frac{2000\,X_3}{4.5*24*60^2} \geq \frac{2000}{4.5*24*60^2} - 0.001$$

$$\therefore 0.005144\,x3 \geq 0.004144 \qquad \text{----------------- } 5$$

6- For Tweedledum

$(1 - x_3)\, P_3 \leq 389$

$(1 - x_3)\, 2000 \leq 389$

So, $2000\, x_3 \geq 1611$ ----------------- 6

7- For D-City (Dallas):

$$\frac{(1-X_4)\, P_4}{q_4} \leq B_4 \quad Or \quad \frac{X_4\, P_4}{q_4} \geq \frac{P_4}{q_4} - B_4$$

$$\frac{500\, X_4}{5 * 24 * 60^2} \geq \frac{500}{5 * 24 * 60^2} - 0.0015$$

$$\therefore\ 0.001157\, x4 \ \geq\ -0.000343$$

$$\therefore -0.001157\, x4 \ \leq\ 0.000343$$ ----------------- 7

8- For Tweedledee

$(1 - x_4)\, P_4 \leq 648$

$(1 - x_4)\, 500 \leq 648$

So, $-500\, x4 \leq 148$ ----------------- 8

9- For Tiwn-tweedle

$(1 - x_3)\, P_3 + (1 - x_4)\, P_4 \leq 829$

$(1 - x_3)\, 2000 + (1 - x_4)\, 500 \leq 829$

So, $2000\, x_3 + 500\, x_4 \geq 1671$ ------------------ 9

10- For E-City (Eastport):

$$\frac{(1-X_5)\, P_5}{q_5} \leq B_4 \quad Or \quad \frac{X_5\, P_5}{q_5} \geq \frac{P_5}{q_5} - B_5$$

$$\frac{1200\, x_5}{13.1 * 24 * 60^2} \geq \frac{1200}{13.1 * 24 * 60^2} - 0.0013$$

$$\therefore\ 0.00106\, x_5 \ \geq\ -0.00024$$

$$\therefore -0.00106\, x_5 \ \leq\ 0.00024$$ ----------------- 10

11- For Wonder River

$(1 - x_1)\,P_1 + (1 - x_2)\,P_2 + (1 - x_3)\,P_3 + (1 - x_4)\,P_4 + (1 - x_5)\,P_5 \le 1452$

$(1 - x_1)\,2200 + (1 - x_2)\,800 + (1 - x_3)\,2000 + (1 - x_4)\,500 + (1 - x_5)\,1200 \le 1425$

So, $2200\,x_1 + 800\,x_2 + 2000\,x_3 + 500\,x_4 + 1200\,x_5 \ge 5275$ ----------------- 11

Also, Remember that (x_1, x_2, x_3, x_4 & x_5) should be less than one

The objective function is:

$$Cost = Z_{min} = 0.18 * 2200\,x_1 + 0.22 * 800\,x_2 + 0.16 * 2000\,x_3 + 0.25 * 500\,x_4 + 0.2 * 1200\,x_5$$

$$\therefore \quad Z_{min} = 396\,x_1 + 176\,x_2 + 320\,x_3 + 125\,x_4 + 240\,x_5$$

Briefly the problem is:

Minimize $Z = 396\,x_1 + 176\,x_2 + 320\,x_3 + 125\,x_4 + 240\,x_5$

Subject to:

$0.00796\,x1 \ge 0.00676$ ----------------- 1

$2200\,x_1 \ge 1868$ ----------------- 2

$0.002572\,x2 \ge 0.001372$ ----------------- 3

$2200\,x_1 + 800\,x_2 \ge 2627$ ----------------- 4

$0.005144\,x3 \ge 0.004144$ ---------------- 5

$2000\,x_3 \ge 1611$ ---------------- 6

$-0.001157\,x4 \le 0.000343$ ----------------- 7

$-500\,x4 \le 148$ ----------------- 8

$2000\,x_3 + 500\,x_4 \ge 1671$ ----------------- 9

$-0.00106\,x5 \le 0.00024$ ---------------- 10

$2200\,x_1 + 800\,x_2 + 2000\,x_3 + 500\,x_4 + 1200\,x_5 \ge 5275$ ----------- 11

With $x_1 \le 1$, $x_2 \le 1$, $x_3 \le 1$, $x_4 \le 1$ & $x_5 \le 1$

Using Lindo software, the results are as follows:

LINDO - [E:\WQMAN.LTX]

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```
Min     396 x1 + 176 x2 + 320 x3 + 125 x4 + 240 x5
SUBJECT TO
       2)   0.00796 x1  >= 0.00676
       3)   2200 x1  >=  1868
       4)   0.002572 x2  >= 0.001372
       5)   2200 x1 +  800 x2  >= 2627
       6)   0.005144 x3  >= 0.004144
       7)  2000 x3 >= 1611
       8)  - 0.001157 x4  <= 0.000343
       9)  -500 x4 <= 148
       10) 2000 x3 +  500 x4  >= 1671
       11) - 0.00106 x5  <= 0.00024
       12) 2200 x1 + 800 x2 + 2000 x3 +  500 x4 + 1200 x5  >= 5275
       13) x1 <= 1
       14) x2 <= 1
       15) x3 <= 1
       16) x4 <= 1
       17) x5 <= 1
END
```


```
LP OPTIMUM FOUND AT STEP       8

       OBJECTIVE FUNCTION VALUE

       1)       939.5400

 VARIABLE        VALUE          REDUCED COST
       X1         1.000000          0.000000
       X2         0.533750          0.000000
       X3         1.000000          0.000000
       X4         0.000000         25.000000
       X5         0.540000          0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.001200          0.000000
       3)       332.000000          0.000000
       4)         0.000001          0.000000
       5)         0.000000         -0.020000
       6)         0.001000          0.000000
       7)       389.000000          0.000000
       8)         0.000343          0.000000
       9)       148.000000          0.000000
      10)       329.000000          0.000000
      11)         0.000812          0.000000
      12)         0.000000         -0.200000
      13)         0.000000         88.000000
      14)         0.466250          0.000000
      15)         0.000000         80.000000
      16)         1.000000          0.000000
      17)         0.460000          0.000000

 NO. ITERATIONS=       8
```

6.3 Least Square Method

***Example**: find the best fit line for the following table using the Simplex method and compare the results with the standard method (least square method).*

Solution:

any line could be express as: y^ = a + b x

from the least square method the value of ***a & b*** estimated as:

x	y	$x \cdot y$	x^2
1	4	4	1
3	6	18	9
5	10	50	25
7	12	84	49
$\sum x = 16$	$\sum y = 32$	$\sum yx = 156$	$\sum x^2 = 84$

Where: $\sum y = Na + b\sum x$

$\sum yx = a\sum x + b\sum x^2$

So, 32 = 4 a + 16 b

156 = 16 a + 84 b

Which gives a = 2.4 & b = 1.4

$\therefore\ y^{\wedge} = 2.4 + 1.4\,x$

By the L. P. method the problem is formulated as follows:

$$Z_{min} = \sum (y_i - y_i^{\wedge})^2 = \sum (y_i - a - bx_i)^2$$

***Assume that**:*

$$Z_{min} = \sum Abs(y_i - y_i^{\wedge}) = \sum \left| y_i - y_i^{\wedge} \right|$$

Let $\varepsilon_i = y_i - y_i^{\wedge} = y_i - a - bx_i = X_{3i} - X_{4i}$

$$\therefore\ Z_{min} = \sum |\varepsilon_i| = \sum | X_{3i} - X_{4i} | = \sum (X_{3i} + X_{4i})$$

So for ; $\varepsilon_1 = y_1 - y_1^{\wedge} = y_1 - a - bx_1 = X_{3i} - X_{4i}$

The result is *4 - a - 1 b = x3 - x4*

Let the unknown variables (a , b) as: a = x1 & b = x2

$\therefore\ 4 - x1 - 1\,x2 = x3 - x4$ --------------- 1

By the same way:

For ε_2: $6 - x1 - 3\,x2 = x5 - x6$ --------------- 2

For ε_3: $10 - x1 - 5\,x2 = x7 - x8$ --------------- 3

For ε_4: $12 - x1 - 7\,x2 = x9 - x10$ --------------- 4

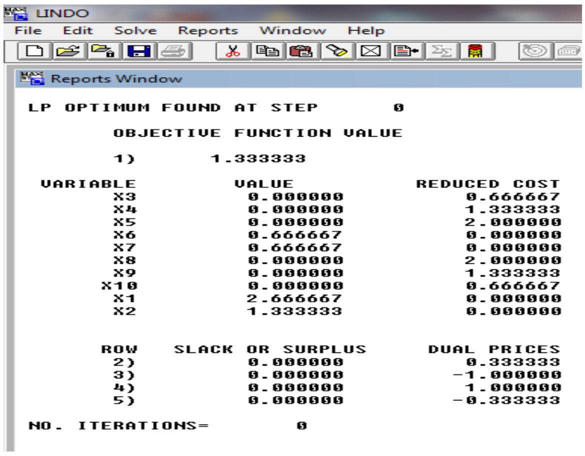
The objective function is:

$$Z_{min} = \sum|\varepsilon_i| = \sum|X_{3i} - X_{4i}| = \sum(X_{3i} + X_{4i})$$

$$\therefore\ Z_{min} = (x3 + x4) + (x5 + x6) + (x7 + x8) + (x9 + x10)$$

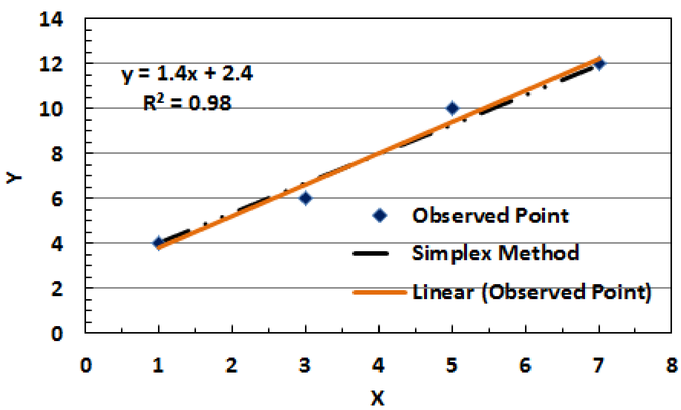
By using Lindo software, the results are as follows:

$a = x1 = 2.667$ & $b = x2 = 1.33$ with $Z_{min} = \sum|\varepsilon_i| = 1.333$



Not that: *the valuse of (a and b) of the best fit line are not compatable from the both methods (the last squer and the Simplex), Why ?. For more explanation see the following Table and figuer.*

x	y	Y^ = 2.4 + 1.4 X (Least S. M.)	Y^= 2.667 + 1.333X (Simplex M.)
1	4	3.8	3.997
3	6	6.6	6.66
5	10	9.4	9.32
7	12	12.2	11.977



7- Sensitivity Anaiysis

7.1 The Inverse of Basis and the Simplex Multipliers

For the general ***L. P.*** problem with ***m-equations*** constraints in ***n non-nagative*** variables of the form:

$$A\,x = b \qquad \text{------------- 7.1}$$

If ***B*** is *m-columns* of ***A*** which correspond to the basic variable, so that ***A*** can be written as:

$$A = (BR) \qquad \text{------------- 7.2}$$

Where: B: is the m x ***m matrix of the basis.***

R: is the m x ***(n-m) matrix of the non-basic variables.***

So, the canonical form for the basis is obtasined by multiplying:

$$(BR)\,x = b \qquad \text{------------- 7.3}$$

By B^{-1} to obtain:

$$(I_m\ B^{-1}R)\,x = B^{-1}\,b = b' \qquad \text{------------- 7.4}$$

Which will represent the canonical form for the constraints.

If a_j represents the column of coefficioents of the variable x_j in the first equation form of the constraints, then:

$$a_j' = B^{-1}\,a_j \qquad \text{------------- 7.5}$$

will represent the column of coeffiocients of x_j in the canonical form. If x_j is a slack variable that arose from a $" \leq "$ constraint then:

$$a_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ +1 \\ 0 \\ 0 \end{pmatrix} \leftarrow pth\ row\ say \quad ; \quad \textbf{So that } a_j' \textbf{ will be the pth column of } B^{-1}.$$

Also, if x_k is the a slack variable that arose from a $" \geq "$ constraint then:

$$a_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \\ 0 \\ 0 \end{pmatrix} \leftarrow qth\ row\ say; \quad \textbf{So that } a_k' \textbf{ will be the negative of the qth column of } B^{-1}$$

As un illustration consuider the first and last tableaux of example 2 of section 4.

It.	Basis	Value	X1	X2	X3	X4
0	X3	1700	3	4	1	.
	X4	1600	2	5*	.	1
	-Z	0	-2	-4	.	.
1	X3	420	7/5*	.	1	-5/4
	X2	320	2/5	1	.	1/5
	-Z	1280	-2/5	.	.	4/5
2	X1	300	1	.	5/7	-4/7
	X2	200	.	1	-2/7	3/7
	-Z	1400	.	.	2/7	4/7

First tableau (iteration 0); *Final tableau* (iteration 2)

The optimal basis is (x_1, x_2). The matrix of coefficients of the basis in the first form for the constraints is:

$$B = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$$

In the first canonical form the matrix of coefficients of the slack variables (x_3, x_4) is:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In the final tableau it is $B^{-1}I = B^{-1} = \begin{bmatrix} 5/7 & -4/7 \\ -2/7 & 3/7 \end{bmatrix}$

Example 1: ***For the example 4 of section 4, find the invers matrix B^{-1}.***

Solution: ***The coefficients matrix of the final basis (x_1, x_2, x_3, x_4) in the first tableau is:***

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \end{bmatrix}$$

Also, the coefficients matrix of the slack variable (x_3, x_4, x_5, x_6) in the first and last tableau is:

$$in\ first \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad in\ final \begin{bmatrix} 0 & 0 & 4/5 & -1/5 \\ 0 & 0 & 1/5 & 1/5 \\ 1 & 0 & 4/5 & -1/5 \\ 0 & 1 & 1/5 & 1/5 \end{bmatrix}$$

So that B^{-1} is : $$B^{-1} = \begin{bmatrix} 0 & 0 & 4/5 & -1/5 \\ 0 & 0 & 1/5 & 1/5 \\ -1 & 0 & 4/5 & -1/5 \\ 0 & -1 & 1/5 & 1/5 \end{bmatrix}$$

Note the sign change in the first two columns since they arose from a " ≥ " constraint

For checking note that:

$$B^{-1}B = \begin{bmatrix} 0 & 0 & 4/5 & -1/5 \\ 0 & 0 & 1/5 & 1/5 \\ -1 & 0 & 4/5 & -1/5 \\ 0 & -1 & 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Also, the value of the basic variables are of course is given by Eq. 7.4 as:

$$Value = b' = B^{-1}b = \begin{bmatrix} 0 & 0 & 4/5 & -1/5 \\ 0 & 0 & 1/5 & 1/5 \\ -1 & 0 & 4/5 & -1/5 \\ 0 & -1 & 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 2 \\ 3 \end{bmatrix}$$

7.2 Simplex Multipliers - π_i -

In each canonical form the basic variables appropriate to that form have been eliminated from the objective function Z. the simplex method does this in an iteratrive manner. It is possible to imagine it being done at each stage by using the first form for the constraints. For the general problem:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots\cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \cdots\cdots + a_{2n}x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \cdots\cdots + a_{mn}x_n = b_m$$

$$c_1x_1 + c_2x_2 + c_3x_3 \cdots\cdots + c_nx_n = Z_{Min}$$

If we multiply the constraints by numders $\pi_1, \pi_2, \dots, \pi_m$ and add to Z to obtain

$$x_1\left(c_1 + \sum_{i=1}^{m} a_{i1}\,\pi_i\right) + x_2\left(c_2 + \sum_{i=1}^{m} a_{i2}\,\pi_i\right) + \cdots + x_n\left(c_n + \sum_{i=1}^{m} a_{in}\,\pi_i\right) = Z + \sum_{i=1}^{m} b_i\,\pi_i \qquad \cdots 7.6$$

Now, for values of π_i so that the coefficients of the basic variabvles in Eq. 7.6 are zero. Such π_i are called as the Simplex Multipliers.

If $x_1, x_2, x_3, \dots\dots, x_m$ are basis (there is no loss of generality) the π_i are determined from:

$$a_{11}\pi_1 + a_{21}\pi_2 + a_{31}\pi_3 \cdots\cdots + a_{m1}\pi_m = -c_1$$

$$a_{12}\pi_1 + a_{22}\pi_2 + a_{32}\pi_3 \cdots\cdots + a_{m2}\pi_m = -c_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{1m}\pi_1 + a_{2m}\pi_2 + a_{3m}\pi_3 \cdots\cdots + a_{mm}\pi_m = -c_m$$

i.e. $\quad B^T\pi = -C_B \quad$ *-----------------7.7*

Where: ***B***: is the ***m x m*** matrix of the basis.

$C_B^T = (C_1, C_2, \ldots\ldots, C_m)$ is the matrix of the coeficientys of the basic variables in the first form for Z.

$$\textit{Also,} \qquad \pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \vdots \\ \pi_m \end{pmatrix}$$

$$\textit{Thus} \qquad \pi = -(B^T)^{-1} C_B = -(B^{-1})^T C_B \qquad \text{------------------7.8}$$

Howevere, the values of π_i could be estimated by inspecting the Simplex tableaux, as follows:

1- If x_i is the slack variable in the ***pth*** constraint which is a "≤" constraints, its coefficients in the optimal form for ***Z*** will be (π_p).

2- If x_j is the slack variable in the ***qth*** constraint which is a "≥" constraints, its coefficients in the optimal form for ***Z*** will be ($-\pi_q$).

As un illustration consuider once again the first and last tableaux of example 2 of section 4. The first form of the constraints and ***Z*** is:

$$3x_1 + 4x_2 + x_3 = 1700 \qquad (\pi_1 = 2/7)$$
$$2x_1 + 5x_2 + x_4 = 1600 \qquad (\pi_2 = 4/7)$$
$$Z = -2x_1 - 4x_2$$

The coefficients of the slack variables (x_3, x_4) in the optimal tableau are:

(2/7 , 4/7)

Thus the Simplex multiplier are: $\pi_1 = 2/7$ & $\pi_2 = 4/7$

Multiplying the constraints by (π_1 & π_2) as shown and add to Z to obtain:

$$x_1\,[-2+3\,(2/7) + 2\,(4/7)] + x_2\,[-4 + 4\,(2/7) + 5\,(4/7)] + (2/7)\,x_3 + (4/7)\,x_4$$
$$= Z + 1700\,(2/7) + 1600\,(4/7)$$

i.e $$\frac{2}{7}x_3 + \frac{4}{7}x_4 = Z + 1400$$

Which is indeed the final form for Z

Now, from Eq.7.6, in the final form, since the coeficients of the basic variable are zero and the value of non basic variable are zero therefore each term on the **L.H.S**. is zero. Thus the optimal value of ***Z*** is given by:

$$Z = -\sum_{i=1}^{m} b_i\,\pi_i \qquad \text{------------------7.9}$$

Example 2: ***For the example 4 of section 4, find the Simplex Multipliers*** π_i.

Solution:

The coefficients of the slack variables (x_3, x_4, x_5, x_6) in the final tableau are:

$$(0,0,16/5,1/5)$$

Thus the Simplex multiplier are: $\pi_1 = -0$; $\pi_2 = -0$, $\pi_3 = 16/5$, $\pi_4 = 1/5$

Also; $$Z = -\sum_{i=1}^{m} b_i\,\pi_i = -(-0\times 10 + -0\times 5 + \frac{16}{5}\times 20 + \frac{1}{5}\times 20) = -68$$

7.3 The Effect of Change in the Problem

We shell consider the following cases:

i. ***Changes in the b_i (R.H.S. values).***
ii. ***Changes in the C_j (the coefficients in the objective function).***
iii. ***The inclusion of more variables.***
iv. ***The addition of extra constraints.***

i. Changes in the b_i.

The original constraints are assumed to be:

$$Ax = b \;\; with \; Z_{Min} = c^T x$$

Suppose the new problem is:

$$Ax = b + \Delta b \; where \;\; \Delta b = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \\ \vdots \\ \Delta b_n \end{pmatrix} \quad with\ same\ Z_{Min} = c^T x$$

For the original problem, the following equations are true:

For basic variables	$x_B = B^{-1} b = b'$	
For non – basic variables	$c_j + \sum_{i=1}^{m} a_{ij}\,\pi_i \geq 0$	------------------7.10
Value of optimum	$Z = -\sum_{i=1}^{m} b_i\,\pi_i$	

Now, if only the b_i change, ***Eq. 7.10*** will still be valid for new problem. Thus provided the same basic solution is also feasible for the new problem, it will be the optimal basic feasible solution for this problem with the following values:

$$x_B{}^* = B^{-1}(b + \Delta b) = b' + B^{-1}\Delta b \quad \text{------------------7.11}$$

$$\textit{Provided that } x_B{}^* \geq 0$$

Also, the new value of Z will be:

$$Z^* = -\sum_{i=1}^{m}(b_i + \Delta b_i)\,\pi_i \quad \text{------------------7.12}$$

Where π_i *are the Simplex multipliers of the* ***original problem***.

***Note that**:* *from* **Eq. 7.9** *it could be obtain that:*

$$\frac{\partial Z}{\partial b_i} = -\pi_i \quad \text{------------------7.13}$$

Therefore, if b_i *change by too large values, there will come at which* $x_B{}^*$ *as given by Eq.711 is not feasible. Then we would have to start again.*

Example 2: ***For the example 2 of sec.4,***

A: suppose we can buy extra board from a second timber merchant. How much per square meter are we prepared to pay for it.

Solution: we suppose that 1700 in first constraint changes to 1701.

Thus the new vector of b's will be $\begin{pmatrix} 1701 \\ 1600 \end{pmatrix}$

Also, new values of the basic varables will be:

$$\begin{pmatrix} x_1{}^* \\ x_2{}^* \end{pmatrix} = B^{-1}(b + \Delta b) = \begin{pmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 1701 \\ 1600 \end{pmatrix} = \begin{pmatrix} 300 \\ 200 \end{pmatrix} + \begin{pmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 300 + \frac{5}{7} \\ 200 - \frac{2}{7} \end{pmatrix}$$

New optimal $Z^* = -\sum_{i=1}^{m}(b_i + \Delta b_i)\,\pi_i = -(\frac{2}{7} \times 1701 + \frac{4}{7} \times 1600) = -1400 - \frac{2}{7}$

Thus, the profit will increase by** $\frac{2}{7}$ \$ **which is the maximum price could be paied for extra 1 m^2 of board.

B: Suppose we can obtain extra machine time by working overtime. If the costs is \$7 per hour extra. Is it worth it?

Solution: The new vector of $\boldsymbol{b's}$ will be $\begin{pmatrix}1700\\1610\end{pmatrix}$

Also, new values of the basic varables will be:

$$\begin{pmatrix}x_1^*\\x_2^*\end{pmatrix} = B^{-1}(b+\Delta b) = \begin{pmatrix}\frac{5}{7} & \frac{-4}{7}\\ \frac{-2}{7} & \frac{3}{7}\end{pmatrix}\begin{pmatrix}1700\\1610\end{pmatrix} = \begin{pmatrix}300-\frac{40}{7}\\ 200+\frac{30}{7}\end{pmatrix}$$

$$New \quad Z^* = -\sum_{i=1}^{m}(b_i + \Delta b_i)\,\pi_i = -(\frac{2}{7} \times 1700 + \frac{4}{7} \times 1610) = -1400 - \frac{40}{7}$$

Thus, the profit will increase by $\frac{40}{7}$ $ ***for 10 hours (i.e 4/7per hour) . But since the hour's overtime costs $7 , so it is not worth.***

ii. Changes in the C_i .

Suppose that the values of $\boldsymbol{c's}$ in the objective function of tha same example were changed to $\$p_1$ **&** $\$p_2$. For what possible values of them so that the solution we have obtained is still optimal. In this case only the last line will change, why?.

So, the objective function in the first tableau is:

$$-P_1\,x1 - P_2x2 = Z + 0$$

Also, the canonical form for the constraints for the basis is:

$$x1 + 0x2 + \frac{5}{7}x3 - \frac{4}{7}x4 = 300$$

$$0\,x1 + x2 - \frac{2}{7}x3 + \frac{3}{7}x4 = 200$$

To obtain $\boldsymbol{Z}$ in this new canonical form, eliminate x_1 **&** x_2 from $\boldsymbol{Z}$ by multiplying the above constraints by p_1 **&** p_2 and adding to $\boldsymbol{Z}$ to obtain:

$$\left(\frac{5}{7}P_1 - \frac{2}{7}P_2\right)x3 + \left(\frac{-4}{7}P_1 + \frac{3}{7}P_2\right)x4 = Z + 300P_1 + 200P_2$$

Thus, the solution is optimal if the coefficient of non-basic variables x_3 **&** x_4 are positive. So,

$$\left(\frac{5}{7}P_1 - \frac{2}{7}P_2\right) \geq 0 \;\;\&\;\; \left(\frac{-4}{7}P_1 + \frac{3}{7}P_2\right) \geq 0$$

$$\therefore \quad \frac{P_1}{P_2} \geq \frac{2}{5} = 0.4 \quad \& \quad \frac{P_1}{P_2} \leq \frac{3}{4} = 0.75$$

Note that: If one or more of the coefficients of non-basic variables in the new objective function are nagative we shell bring this variable into basis and continue with further iterations of Simplex method. i.e. the earlier calculations will not be wasted

iii. The inclusion of More Variables.

Suppose a third type of kit (type C) can be made with a 4 m^2 of board and 20 minutes of machine time and a profit of $p. The problem is should we make it or not?

Solution: Let x_5 be unit of type C, our problem in standard form is:

$$3x_1 + 4x_2 + x_3 + 4x_5 = 1700$$
$$2x_1 + 5x_2 + x_4 + 10/3\, x_5 = 1600$$
$$-Z - 2x_1 - 4x_2 - p\, x_5 = 0$$

In the final tableau the first two rows of x_5 column will be, by ***Eq. 7.5***

$$a_5' = B^{-1} a_5 = \begin{pmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} 4 \\ \frac{10}{3} \end{pmatrix} = \begin{pmatrix} \frac{20}{21} \\ \frac{6}{21} \end{pmatrix}$$

The Simplex multipliers are $\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} \\ \frac{4}{7} \end{pmatrix}$

Then by Eq. 7.6 the coefficient of x_5 in the canonical form for Z will be:

$$C_5 + \sum_{i=1}^{m} a_{i5}\, \pi_i = -P + 4 \times \frac{2}{7} + \frac{10}{3} \times \frac{4}{7} = -P + \frac{64}{21}$$

So, the final tableau will be (the only change is the x_5 column)

It.	Basis	Value	X1	X2	X3	X4	X5
	X1	300	1	.	5/7	-4/7	20/21
	X2	200	.	1	-2/7	3/7	6/21
	-Z	1400	.	.	2/7	4/7	-P + 64/21

Final tableau

Now, if (-P + 64/21) ≥ 0 then this solution is optimum and x_5 remains non-basic and we do not make a new model . While if (-P + 64/21) < 0 or (P > 64/21) it should be made x_5 as a basic, then try to "pick up" the compution from the canonical form just given and continue with the Simplex method.

iv The Addition of Extra Constraints

For the same example consider the following addinal constraint:

$$x_1 + x_2 \leq 550$$

this constraint has to be included in the problem. However, in this case for our optimum solution of $x_1 = 300$ & $x_2 = 200$ (i.e $x_1 + x_2 = 500 \leq 550$), this constraint has no effect on the optimal solution. *But, let the constraint be as:*

$$x_1 + x_2 \leq 450$$

So, let $x_1 + x_2 + x_5 = 450$;

Where x_5 is a slack variable.

Now, include this constraint in the final canonical form as:

$$x1 + 0\,x2 + \frac{5}{7}x3 - \frac{4}{7}x4 = 300$$

$$0\,x1 + x2 - \frac{2}{7}x3 + \frac{3}{7}x4 = 200$$

$$x_1 + x_2 + x_5 = 450$$

Eliminate x_1 & x_2 from the new **3**rd constraint yield:

$$\frac{-3}{7}\,x3 + \frac{1}{7}\,x4 + x5 = -50$$

The last tableau would have been:

It.	Basis	Value	X1	X2	X3	X4	X5
	X1	300	1	.	5/7	-4/7	.
	X2	200	.	1	-2/7	3/7	.
	X5	-50	.	.	-3/7	1/7	1
	-Z	1400	.	.	2/7	4/7	.

Final tableau

*At this point we encounter a problem. Where the **Z** function is in the optimal form since all the coefficients of non-basic variables are positive while the basic variable x_5 is negative . in such case we must use the **dual Simplex method.***

7.4 The Dual Simplex Method

If the objective function has all positive coefficients ($C_j \geq 0$), the steps are as follows:

1- Find a negative basic variable. If there is non, we have the optimal solution. If there is more than one find the most negative. Suppose this variable is the basic variable in the ***rth*** constraint. This gives the variable to come out of the basis.

2- In this ***r***-row look for negative coefficients a_{rj}' . if there are non ***there is no feasible soution to the problem***. For negative coefficients a_{rj}' in this row find that:

$$min_j \left| \frac{c_j'}{a_{rj}'} \right|$$

If this arises in column ***s*** , variable ***s*** is the variable to enter the basis.

3- Carry out the usual Simplex transformation with a_{rs}' as pivot and as follows:

$$c_j^+ = c_j' - \frac{c_s' \, a_{rj}'}{a_{rs}'}$$

And since a_{rs}' is negative, and all c_j' are positive, this is positive, since ***s*** arose from:

$$min_j \left| \frac{c_j'}{a_{rj}'} \right| = \left| \frac{c_s'}{a_{rs}'} \right|$$

So, the results are as indicated in the following table.

It.	Basis	Value	X1	X2	X3	X4	X5	Row	Remark
2	X1	300	1	.	5/7	-4/7	.	1	
	X2	200	.	1	-2/7	3/7	.	2	
	X5	-50	.	.	-3/7*	1/7	1	3	
	-Z	1400	.	.	2/7	4/7	.	4	
3	X1	650/3	1	.	.	-1/3	5/3	5	Pivot*(-5/7) + R1
	X2	700/3	.	1	.	1/3	-2/3	6	Pivot*(2/7) + R2
	X3	350/3	.	.	1	-1/3	-7/3	7	Pivot = R3/(-3/7)
	- Z	4100/3	.	.	.	2/3	2/3	8	Pivot*(-2/7) + R4

Note that, ***in some times the dual Simplex method allows us to avoid the use of artifucial variables, as explaned in the following example .***

Example: *Find non-negative x_1 , x_2 such that:*

$$Z_{min} = x_1 + x_2$$

Sub. to: $x_1 + 2x_2 \geq 6$

$$2x_1 + x_2 \geq 6$$

$$7x_1 + 8x_2 \leq 56$$

Solution: *in standard form the problem is:*

1: $x_1 + 2x_2 - x_3 = 6$

$2x_1 + x_2 - x_4 = 6$

$7x_1 + 8x_2 + x_5 = 56$

For which

$-Z + x_1 + x_2 = 0$

2: *Multiply the first two constraints by (-1) to have:*

$-x_1 - 2x_2 + x_3 = -6$

$-2x_1 - x_2 + x_4 = -6$

$7x_1 + 8x_2 + x_5 = 56$

For which $-Z + x_1 + x_2 = 0$

Now, the solution is rpresnted in the following table avoiding the use of artificial variables and artificial function W.

It.	Basis	Value	X1	X2	X3	X4	X5	Row	Remark
0	X3	- 6	- 1	- 2	1	.	.	1	
	X4	- 6	- 2*	- 1	.	1	.	2	
	X5	56	7	8	.	.	1	3	
	- Z	0	1	1	.	.	.	4	
1	X3	-3	.	-3/2*	1	-1/2	.	5	Pv*(1) + R1
	X1	3	1	1/2	.	-1/2	.	6	Pv = Pivot = R2/(-2)
	X5	35	.	9/2	.	7/2	1	7	Pv*(-7) + R3
	- Z	-3	.	1/2	.	1/2	.	8	Pv*(-1) + R4
2	X2	2	.	1	-2/3	1/3	.	9	Pv = Pivot = R5/(-3/2)
	X1	2	1	.	1/3	-2/3	.	10	Pv*(-1/2) + R6
	X5	26	.	.	3	2	1	11	Pv*(-9/2) + R7
	- Z	- 4	.	.	1/3	1/3	.	12	Pv*(-1/2) + R8

The optimal results are:

$$x_2 = 2 \, , x_1 = 2 \; \& \; x_5 = 26 \quad \text{and } Z = 4$$

Also, note that:

The coefficients matrix of the final basis (x_2, x_1, x_5) in the first tableau is:

$$B = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ +8 & +7 & 1 \end{bmatrix}$$

Also, the coefficients matrix of the slack variable (x_3 , x_4, x_5) in the first and last tableau is:

$$in\ first \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad in\ final \begin{bmatrix} -2/3 & 1/3 & 0 \\ 1/3 & -2/3 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

So that B^{-1} is : $$B^{-1} = \begin{bmatrix} -2/3 & 1/3 & 0 \\ 1/3 & -2/3 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

For checking note that:

$$B^{-1}B = \begin{bmatrix} -2/3 & 1/3 & 0 \\ 1/3 & -2/3 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ +8 & +7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Also, the value of the basic variables are of course is given by Eq. 7.4 as:

$$Value = b' = B^{-1}b = \begin{bmatrix} -2/3 & 1/3 & 0 \\ 1/3 & -2/3 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ -6 \\ 56 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 26 \end{bmatrix}$$

The coefficients of the slack variables (x3, x4, x5) in the final tableau are:

(1/3, 1/3, 0)

Thus the Simplex multiplier are: $\pi_1 = 1/3$; $\pi_2 = 1/3$, $\pi_3 = 0$

So, $$Z = -\sum_{i=1}^{m} b_i\,\pi_i = -\left(\frac{1}{3} \times (-6) + \frac{1}{3} \times (-6) + 0 \times 56\right) = 4$$

Note that, the values of ($\pi_1 = 1/3$ & $\pi_2 = 1/3$) was not multyplied by (-1) inspite of they were brought from "≥" sign, why?

8- Transportation Problem

8.1 The Nature of the Problem and its Solution

Example 1

A company which runs a chain of Depaetment Stores wishes to transport some beds from its 3 warehouse to 5 of its retail outlets. There are 15, 25, 20 beds respectively at the warehouse and the 5 stores need 20, 12, 5, 8 & 15 beds respectively. The costs ($'s) of moving 1 bed from warehouse to store are given in the following table. How should the distribution be planned so as to minimise the costs ?

		To (Demands)					
		S_1	S_2	S_3	S_4	S_5	
From (Sources)	W_1	1	0	3	4	2	$a_1=15$
	W_2	5	1	2	3	3	$a_2=25$
	W_3	4	8	1	4	3	$a_3=20$
		$b_1=20$	$b_2=12$	$b_3=5$	$b_4=8$	$b_5=15$	

Solution: Let x_{ij} be the number of beds sent from warehouse i to store j.

So, for supply or sources:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 15$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 25$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 20$$

For demand:

$$x_{11} + x_{21} + x_{31} = 20$$

$$x_{12} + x_{22} + x_{32} = 12$$

$$x_{13} + x_{23} + x_{33} = 5$$

$$x_{14} + x_{24} + x_{34} = 8$$

$$x_{15} + x_{25} + x_{35} = 15$$

Subject to these constraints, the cost is:

$C = 1\,X_{11} + 0\,X_{12} + 3\,X_{13} + 4\,X_{14} + \ldots\ldots\ldots\ldots + 4\,X_{34} + 3\,X_{35}$ *is to be minimized*

The above results will generalise to transportation problem with ***m-supply*** points ***(a_i , $i = 1, 2, \ldots, m$)*** and ***n-demands*** points ***(b_j , $j= 1, 2, \ldots, n$)*** where:

$$\boldsymbol{Supply = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = Demand} \qquad \text{------------8.1}$$

If c_{ij} is the cost of transportaion of 1 unit from supply point ***i*** to demand point ***j***, then the problem will be to find $x_{ij} \geq 0$ which satisfy that:

For supply

$$\begin{array}{l}
x_{11} + x_{12} + \ldots\ldots\ldots + x_{1n} = a_1 \\
\qquad\qquad x_{21} + x_{22} + \ldots\ldots\ldots + x_{2n} = a_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad\qquad x_{m1} + x_{m2} + \ldots\ldots\ldots + x_{mn} = a_m
\end{array}$$

For demand: ----8.2

$$\begin{array}{l}
x_{11} + x_{21} + \ldots\ldots\ldots + \; + x_{m1} = b_1 \\
\quad x_{12} + x_{22} + \ldots\ldots\ldots + \; + x_{m2} = b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad x_{1n} + x_{2n} + \ldots\ldots\ldots + \; + x_{mn} = b_n
\end{array}$$

With minimizes

$$C = c_{11} X_{11} + c_{12} X_{12} + c_{13} X_{13} + c_{ij} X_{ij} + \ldots\ldots\ldots + c_{mn} X_{mn}$$

More briefly the problem is to find x_{ij} for which

$$\sum_{j=1}^{n} x_{ij} = a_i > 0 \; (i = 1, \cdots\cdots, m) \qquad \text{------------8.3}$$

$$\sum_{i=1}^{m} x_{ij} = b_j > 0 \; (j = 1, \cdots\cdots, n) \qquad \text{------------8.4}$$

With minimizes $$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} * x_{ij} \qquad \text{------------8.5}$$

Since $\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$

So***, by Eq.8.1***, there are only ***(m + n - 1)*** independent constraints

Hence there are only ***(m + n -1) basic variables in a basic feasible solution.***

8.2 The "Stepping Stones" Algorithm

It could be solved the transportation problem by Simplex method directly. Such an approach would be inefficient and would fail to take account of the special structure of the constraints. We shall use an algorithm first developed by ***F. L. Hitchcock*** which is called ***"Stepping Stones"*** algorithm.

Now, suppose the ***($-u_i$) and ($-v_j$)*** are the Simplex multipiers for the ***i-th row and j-th column*** constraint respectively in the the system of ***Eq. 8.2.*** Therefore multiplication ***Eq. 8.2*** by ***($-u_i$) and ($-v_j$)*** and addition to ***C*** to obtain:

For supply

$x_{11} + x_{12} + \ldots\ldots\ldots + x_{1n} = a_1 \quad \times(-u_1)$

$x_{21} + x_{22} + \ldots\ldots\ldots + x_{2n} = a_2 \quad \times(-u_2)$

……………………………………………………………………………………

$x_{m1} + x_{m2} + \ldots\ldots\ldots + x_{mn} = a_m \quad \times(-u_m)$

For demand:

$x_{11} + x_{21} + \ldots\ldots\ldots + x_{m1} = b_1 \quad \times(-v_1)$

$x_{12} + x_{22} + \ldots\ldots\ldots + x_{m2} = b_2 \quad \times(-v_2)$

……………………………………………………………………………………

$x_{1n} + x_{2n} + \ldots\ldots\ldots + x_{mn} = b_n \quad \times(-v_n)$

$$\therefore \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij} - u_i - v_j)x_{ij} = C - \sum_{i=1}^{m} u_i\, a_i - \sum_{j=1}^{n} v_j b_j \qquad \text{------------8.6}$$

The coefficients of x_{ij} in ***Eq. 8.6*** is simply $c_{ij} - u_i - v_j$ results from the occurrence of x_{ij} in just two constraints for the ***i-th*** row and ***j-th*** column.

Eq. 8.6 is the canonical form for the objective function appropriate to the basis. Therefore the coefficients of the basic variable will be zero.

For the basis $\qquad c_{ij} - u_i - v_j = 0 \qquad$ ------------8.7

There are ***m of u_i's and n of v_j's*** but since there are ***(m + n - 1)*** basic variable ***Eq. 8.7*** will give ***(m + n - 1)*** equations for ***(m + n)*** unknowns of ***(m of u_i and n of v_j)***. Therefore, the solution will be provided by giving one of them an arbitrary value says zero and solving for the others.

For the non − basic $\qquad c_{ij}{}' = c_{ij} - u_i - v_j \qquad$ ------------8.8

So, if $c'_{ij} \geq 0$, ***the solution is optimal.***

If $c'_{ij} < 0$ $(\boldsymbol{negative})$, indicates that these x_{ij} can be brought into basis and minimize the cost ***C***.

Now, return to example 1, using ***<u>the lowest cost first procedure</u>*** the first table is provided as follows:

3+w ; 1	12-w ; 0	; 3	; 4	; 2	15 (-4)
2- w ; 5	w ; -3 ; 1	0 ; 2	8 ; 3	15 ; 3	25 (0)
15 ; 4	; 8	5 ; 1	; 4	; 3	20 (-1)
20 (5)	12 (4)	5 (2)	8 (3)	15 (3)	vi \ ui

$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} X_{ij}$

$= 147$

$C = 3 \times 1 + 12 \times 0 + 2 \times 5 + 8 \times 3 + 15 \times 3 + 15 \times 4 + 5 \times 1 = 147\ \$$

Or

$C = \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j$

$= 15(-4) + 25(0) + 20(-1) + 20(5) + 12(4) + 5(2) + 8(3) + 15(3)$

$= 147\ \$$

X_{22} enters basis with Max. w =2 and X_{21} leaves basis

5+w ; 1	10-w ; 0	; 3	; 4	0 ; 2	15 (-1)
; 5	2+w ; 1	; 2	8 ; 3	15-w ; 3	25 (0)
15-w ; 4	; 8	5 ; 1	-1 ; 4	w ; -2 ; 3	20 (2)
20 (2)	12 (1)	5 (-1)	8 (3)	15 (3)	vi \ ui

$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} X_{ij}$

$= 141$

$= 147 - 3(2) = 141$

$= \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j$

X_{35} enters basis with Max. w =10 and X_{12} leaves basis

15 ; 1	; 0	; 3	; 4	; 2	15 (-3)
; 5	12 ; 1	; 2	8 ; 3	5 ; 3	25 (0)
5 ; 4	; 8	5 ; 1	; 4	10 ; 3	20 (0)
20 (4)	12 (1)	5 (1)	8 (3)	15 (3)	vi \ ui

Optimal

$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} X_{ij}$

$= 121$

$= 141 - 2(10) = 121$

$= \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j$

8.3 Unbalance in the Transportation Problem

The condition of ***Eq. 8.1*** plays an important role in the transportation problem. For an $m \times n$ array it means that there are $(m + n - 1)$ basic variables in a basic feasible solution. Suppose this balance between supply and demand does not hold.

Example 2 ***Suppose the 15, 25 and 20 beds at the warehouse W1, W2, W3 are to be used to supply 4 stores whose requirements are for 20, 12, 5 and 9 beds. Suppose the cost of moving 1 bed from warehouse to store is as given. How should the distribution be planned to minimise the cost?***

		To (Demands)				
		S_1	S_2	S_3	S_4	
From (Sources)	W_1	2	2	2	4	$a_1=15$
	W_2	3	1	1	3	$a_2=25$
	W_3	3	6	3	4	$a_3=20$
		$b_1=20$	$b_2=12$	$b_3=5$	$b_4=9$	

Solution: Since the $Supply = \sum_{i=1}^{m} a_i = 60\ beds$
While the $Demand = \sum_{j=1}^{n} b_j = 46\ beds$

The "trick" here is to introduce a dummy store which required 14 beds with a zero transportation cost C_{i5} . In the final solution if any beds are to be transported to this dummy store, shall be ignored. Those beds will remain in the warehouse. So, in this way, the proplem is created to a transportation problem for which equation 8.1 is true. The solution is represented in the following tables:

15 / 2	2	2	4	0	15 (-1)	X_{31} enters basis
5-w / 3	12 / 1	5 / 1	3+w / 3	0	25 (0)	X_{21} leaves basis
w (-1) / 3	6	3	6-w / 4	14 / 0	20 (1)	Max. w =5
20 (3)	12 (1)	5 (1)	9 (3)	14 (-1)	vi \ ui	C = 95 $

15 / 2	2	2	4	0	15 (0)	Optimal
3	12 / 1	5 / 1	8 / 3	0	25 (0)	
5 / 3	6	3	1 / 4	14 / 0	20 (1)	C = 90 $
20 (2)	12 (1)	5 (1)	9 (3)	14 (-1)	vi \ ui	

8.4 Degeneracy in the Transportation Problem

Degeneracy arises in a transportation problem if one or more of the basic variables becomes zero. Also, a degenerate solution might arise if partial sums of the row totals are equal to partial sums of the column totals. The problem may starts at the initial assumption of the basic variable or through improving the results. Howevere, it is important to remember that the numer of basis should always equals to ***(m + n -1).***

Example 3

A government department has received the following tenders from three firms F_1, F_2 and F_3 for three sizes S_1, S_2 and S_3 of service overcoat as shown in the following table. How should the orders be transported in a minimum cost?

		Price per coat in dollars To (Demands)			
		S_1	S_2	S_3	
From (Sources)	F_1	110	115	126	a_1=1000
	F_2	107	115	130	a_2=1500
	F_3	104	109	116	a_3=2500
		b_1=1000	b_2=1500	b_3=1200	

Solution: Since the $Supply = \sum_{i=1}^{m} a_i = 5000 \ coats$

While the $Demand = \sum_{j=1}^{n} b_j = 3700 \ coats$

Thus we introduce a ***"fictitious"*** category coat with demand of ***1300 of zero cost*** in order to make our problem in a transportation form. Also, it is convenient to work in unit of ***100 coats.*** Thus our array becomes:

110	115	10 126	-2 0	10 (-4)	C = 4273 X_{33} enters X_{23} leaves Max. w=2
-3 107	13+w 115	2-w 130	-6 0	15 (0)	
10 104	2-w 109	w -8 116	13 0	25 (-6)	
10 (110)	15 (115)	12 (130)	13 (6)	vi \ ui	

-4 110	-4 115	10-w 126	w -10 0	10 (10)	C = 4257
-3 107	15 115	130	-6 0	15 (6)	X_{14} enters X_{13} leaves
10 104	0 109	2+w 116	13-w 0	25 (0)	Max. w=10
10 (104)	15 (109)	12 (116)	13 (0)	vi \ ui	

110	115	126	10 0	10 (0)	C = 4157
-3 107	15-w 115	130	w -6 0	15 (6)	X_{24} enters X_{34} leaves
10 104	0+w 109	12 116	3-w 0	25 (0)	Max. w=3
10 (104)	15 (109)	12 (116)	13 (0)	vi \ ui	

0 110	0 115	126	10 0	10 (6)	C = 4139
w -3 107	12-w 115	130	3 0	15 (6)	X_{21} enters X_{31} leaves
10-w 104	3+w 109	12 116	0	25 (0)	Max. w=10
10 (104)	15 (109)	12 (116)	13 (-6)	vi \ ui	

110	w 0 115	126	10-w 0	10 (0)	**Optimal 1**
10 107	2-w 115	130	3+w 0	15 (0)	C = 4109
104	13 109	12 116	0	25 (-6)	X_{12} enters X_{22} leaves
10 (107)	15 (115)	12 (122)	13 (0)	vi \ ui	Max. w=2

This is optimal. However, since c_{12}' is zero , x_{12} could enter without changing the cost C

110	2 115	126	8 0	10 (6)	**Optimal 2**
10 107	115	130	5 0	15 (6)	C = 4109
104	13 109	12 116	0	25 (0)	
10 (101)	15 (109)	12 (116)	13 (-6)	vi ui	

Thus there are two optimal solutions each with a total cost of 410, 900 $

The first one is:

F_2 supplies 1000 coats to S_1 and 200 coat to S_2

F_3 supplies 1300 coats to S_2 and 1200 coat to S_3

The second is:

F_1 supplies 200 coats to S_2

F_2 supplies 1000 coats to S_1

F_3 supplies 1300 coats to S_2 and 1200 coat to S_3

Note, Wich is the most suitable one of these two solutions?

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