University of Mustansiriyah

College of Engineering

Department of Highway and Transportation Engineering

Second Class (2^{nd} Class)

Mathematics

(2025 – 2026)

Multiple Integrals

Multiple integrals

Double integrals:

Let $f(x,y)$ be a cont. function inside and on a boundary of region R, then $\iint f(x,y)\,dR$ is called the double integral of $f(x,y)$ over R.

Theorem: let R be the rectangle defined by the inequalities
$a \le x \le b,\ c \le y \le d$ if $f(x,y)$ is continuous on this rectangle then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dxdy = \int_a^b \int_c^d f(x,y)\,dydx$$

Example: evaluate $\int_0^3 \int_1^2 (1+8xy)\,dydx$

$\int_0^3 \left[\int_1^2 (1+8xy)\,dy\right] dx = \int_0^3 \left[y + \frac{8}{2}xy^2\right]_1^2 dx = \int_0^3 [(2+16x) - (1+4x)]\,dx = \int_0^3 (1+12x)\,dx = [x+6x^2]_0^3 = 57$.

Or $\int_1^2 \int_0^3 (1+8xy)\,dxdy$ (H.W.)

Example: evaluate the double integral $\mathrm{I} = \int_0^1 \int_0^{x^2} (x+y)\,dydx$

$\int_0^1 \left[\int_0^{x^2} (x+y)\,dy\right] dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{x^2} dx = \int_0^1 \left[x^3 + \frac{x^4}{2}\right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{1}{4} + \frac{1}{10} = \frac{7}{20}$

Example: evaluate the double integral $\iint_R y^2x\,dA$ over the rectangle

$R = \{(x,y);\ -3 \le x \le 2\ ,\ 0 \le y \le 1\}$

$$\int_{-3}^{2} \int_0^1 y^2x\,dydx = \int_{-3}^{2} \left[\frac{1}{3}\,y^3x\right]_0^1 dx = \int_{-3}^{2} \frac{1}{3}x\,dx = \left[\frac{x^2}{6}\right]_{-3}^{2} = \frac{-5}{6}\ .$$

Homework: use a double integral to find the volume of the solid that is bounded above by the plane $f(x,y) = 4 - x - y$ and below by the rectangle $R = \{(x,y); 0 \le x \le 1\,,\ 0 \le y \le 2\}$

Example: evaluate $\int_0^{\pi} \int_0^{\cos y} x \sin y \; dxdy$

$= \int_0^{\pi} [\int_0^{\cos y} x \sin y \; dx\,]\, dy = \int_0^{\pi} \left[\frac{x^2}{2} \sin y\right]_0^{\cos y} dy = \int_0^{\pi} \frac{1}{2} \cos^2 y \sin y \; dy = \left[-\frac{1}{6}\cos^3 y\right]_0^{\pi} = \frac{1}{3}$.

Note: if $f(x,y) = 1$, then the double integral represents the area of the region R $\quad A = \iint_R \, dR = \iint_R \, dxdy = \iint_R \, dydx$

To evaluate the double integral, we have two cases:

1. Let $y_1 = g_1(x),\ y_2 = g_2(x)$, then $\iint_R f(x,y)\, dR = \int_a^b \int_{y_1}^{y_2} f(x,y)\, dy\, dx$

2. let $x_1 = g_1(y),\ x_2 = g_2(y)$, then $\iint_R f(x,y)\, dR = \int_c^d \int_{x_1}^{x_2} f(x,y)\, dx\, dy$

Example: find the volume of the solid whose base is the region in the $xy-$ plane is bounded by $y = 4 - x^2$ and $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

$y = 4 - x^2\ ,\ y = 3x$
$4 - x^2 = 3x \Rightarrow x^2 + 3x - 4 = 0 \Rightarrow (x + 4)(x - 1) = 0 \Rightarrow x = -4\ ,\ x = 1$
$V = \int_{-4}^{1}\int_{3x}^{4-x^2}(x + 4)\ dydx = \int_{-4}^{1}[(x + 4)y]_{3x}^{4-x^2}\,dx = \int_{-4}^{1}[(x + 4)(4 - x^2 - 3x)]\,dx = \int_{-4}^{1}(4x - x^3 - 3x^2 + 16 - 4x^2 - 12x)\,dx = \int_{-4}^{1}(-x^3 - 7x^2 - 8x + 16)\,dx = \left[\frac{-x^4}{4} - \frac{7x^3}{3} - \frac{8x^2}{2} + 16x\right]_{-4}^{1} = \left(\frac{-1}{4} - \frac{7}{3} - 4 + 16\right) - \left(\frac{-(-4)^4}{4} - \frac{7(-4)^3}{3} - 4(-4)^2 + 16(-4)\right) = 52.0833$.

Example: find the area of the region bounded by the curve $y^3 + x = 0$ and the line $y = x + 2$.

$x = -y^2\ ,\ x = y - 2$

$-y^2 = y - 2$
$y^2 + y - 2 = 0$
$(y + 2)(y - 1) = 0$
$y = -2\ ,\ y = 1$
$A = \int_{-2}^{1}\int_{y-2}^{-y^2}dxdy = \int_{-2}^{1}\ [x]_{y-2}^{-y^2}\ dy = \int_{-2}^{1}[(-y^2) - (y - 2)]\,dy = \int_{-2}^{1}(-y^2 - y + 2)\,dy = \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1} = \left[\frac{-1}{3} - \frac{1}{2} + 2\right] - \left[\frac{-(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2)\right] = 4.5$.

Example: integrate the function $f(x, y) = xy$ over the region bounded by $x = y - y^2,\ x = -2$.

$y - y^2 = -2$
$y^2 - y - 2 = 0$
$(y + 1)(y - 2) = 0$
$y = -1,\ y = 2$

$I = \iint_R xy\ dR = \int_{-1}^{2}\left[\int_{-2}^{y-y^2} xy\,dx\right]dy = \int_{-1}^{2}\left[\frac{x^2y}{2}\right]_{-2}^{y-y^2} dy = \int_{-1}^{2}\left[\frac{(y-y^2)^2y}{2} - \frac{(-2)^2y}{2}\right] dy = \int_{-1}^{2}\left(\frac{y^3-2y^4+y^5}{2} - 2y\right) dy = \frac{1}{2}\left[\frac{y^4}{4} - \frac{2y^5}{5} + \frac{y^6}{6} - y^2\right]_{-1}^{2} = \cdots$

Example: evaluate $\iint_R (2x - y^2)\,dA$ over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$ and $y = 3$.

$\iint_R (2x - y^2)\,dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\,dxdy = \int_1^3 [x^2 - y^2x]_{1-y}^{y-1}\ dy =$
$\int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\,dy = \int_1^3 (2y^2 - 2y^3)\,dy =$
$\left(\frac{2y^3}{3} - \frac{y^4}{2}\right)_1^3 = \frac{-68}{3}$.

Example: find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

$V = \iint_R (4 - y)\,dA$

$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\,dydx = \int_{-2}^{2}\left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx =$
$\int_{-2}^{2} 8\sqrt{4 - x^2}\ dx = 16\,\pi$.

Example: reverse order, then integrate the double integral $\int_0^4 \int_{-\sqrt{4-y}}^{\frac{y-4}{2}} dxdy$.

$x = \frac{y-4}{2}$, $\quad x = -\sqrt{4-y}$
$2x = y - 4$, $x^2 = 4 - y$
$y = 2x + 4$, $y = 4 - x^2$
$\mathrm{I} = \int_{-2}^{0} \int_{2x+4}^{4-x^2} dydx = \int_{-2}^{0} [y]_{2x+4}^{4-x^2} dx = \int_{-2}^{0} (4 - x^2 - 2x - 4)\, dx =$
$\int_{-2}^{0} (-x^2 - 2x)\, dx = \left[\frac{-x^3}{3} - x^2\right]_{-2}^{0} = \cdots$

Example: reverse order and evaluate the integral $\mathrm{I} = \int_0^2 \int_{\frac{y}{2}}^{1} e^{x^2}\, dxdy$

$= \int_0^2 \int_{\frac{y}{2}}^{1} e^{x^2}\, dxdy = \int_0^1 \int_0^{2x} e^{x^2}\, dydx = \int_0^1 \left[e^{x^2} y\right]_0^{2x} dx = \int_0^1 2x\, e^{x^2}\, dx =$
$\left[e^{x^2}\right]_0^1 = e - 1$.

Double integral in polar coordinate:

$x = r\cos\theta$
$y = r\sin\theta$
$dR = rdrd\theta$

$$\iint_R f(x,y)\, dR = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, (rdrd\theta).$$

Example: evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dydx$

$y=\sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow y^2+x^2 = 1 \Rightarrow r^2 = 1 \Rightarrow r = 1$
$x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$
$dydx = rdrd\theta$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \int_0^1 r^2(rdrd\theta) = \int_0^{\frac{\pi}{2}} \int_0^1 r^3 drd\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}\, d\theta = \left[\frac{\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}.$$

Example: change the cartesian integral into an equivalent polar integral and evaluate the polar integral $\int_0^2 \int_0^x y\, dydx$.

$x = 2 \Rightarrow r\cos\theta = 2 \Rightarrow r = \frac{2}{\cos\theta} = 2\sec\theta$.
$y = x \Rightarrow r\sin\theta = r\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$
$y = r\sin\theta\,, dydx = rdrd\theta$

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} (r\sin\theta)\, rdrd\theta = \int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} r^2 \sin\theta\;\, drd\theta =$$
$$\int_0^{\frac{\pi}{4}} \left[\sin\theta\, \frac{r^3}{3}\right]_0^{2\sec\theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{8}{3}\sec^3\theta \sin\theta\;\, d\theta = \frac{8}{3} \int_0^{\frac{\pi}{4}} \sec^2\theta \cdot \frac{1}{\cos\theta} \cdot \sin\theta\;\, d\theta =$$
$$\frac{8}{3} \int_0^{\frac{\pi}{4}} \sec^2\theta \cdot \tan\theta\;\, d\theta = \frac{8}{3} \left[\frac{\tan^2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{4}{3}\ .$$

Example: use polar coordinates to evaluate $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)^{\frac{3}{2}}\; dydx$.

$$\int_0^{\pi} \int_0^1 (r^2)^{\frac{3}{2}}\; rdrd\theta = \int_0^{\pi} \int_0^1 r^4\;\, drd\theta = \int_0^{\pi} \left[\frac{r^5}{5}\right]_0^1 d\theta = \int_0^{\pi} \frac{1}{5}\;\, d\theta = \frac{1}{5}\theta\Big]_0^{\pi} = \frac{1}{5}\pi = \frac{\pi}{5}\ .$$

Example: evaluate $\iint_R \sin\theta \; dA$, where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1+\cos\theta)$.

$\iint_R \sin\theta \; dA$

$= \int_0^{\frac{\pi}{2}} \int_2^{2(1+\cos\theta)} \sin\theta \; rdrd\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \; r^2 \sin\theta \Big]_2^{2(1+\cos\theta)} \; d\theta$

$= 2\int_0^{\frac{\pi}{2}} \; [(1+\cos\theta)^2 \sin\theta - \sin\theta] \; d\theta$

$= 2\left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta\right]_0^{\frac{\pi}{2}} = \frac{8}{3}$.

Triple integral:

The integral $\iiint_G \; f(x,y,z) \; dV$ is called the triple integral of $f(x,y,z)$ over solid G bounded by the surface:

$$\boldsymbol{dV = dzdydx \text{ or } dzdxdy \text{ or } dzdR}$$

Note: if $f(x,y,z) = 1$, then $\iiint_G \; dV =$ Volume of the solid $G, V = \iiint_G \; dV = \iiint_G \; dzdR$ where R is the projected region of G in $xy -$ plane .

Example: find $\int_0^1 \int_0^{1-x} \int_0^{2-x} (xyz) \; dzdydx$.

$\int_0^1 \int_0^{1-x} \left[xy\frac{z^2}{2}\right]_0^{2-x} \; dydx$

$= \int_0^1 \left(\int_0^{1-x} xy\frac{(2-x)^2}{2} \; dy\right) \; dx$

$= \int_0^1 \left[\frac{x(2-x)^2}{2} \cdot \frac{y^2}{2}\right]_0^{1-x} \; dx$

$= \frac{1}{4} \int_0^1 (4x - 12x^2 + 13x^3 - 6x^4 + x^6)dx$

$= \frac{1}{4}\left(4\frac{x^2}{2} - 12\frac{x^3}{3} + 13\frac{x^4}{4} - 6\frac{x^5}{5} + \frac{x^7}{7}\right)_0^1$

$= \cdots\cdots$

Example: find the volume bounded by the coordinate plane $y + z = 2,\ x + y^2 = 4$

$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dzdydx = \int_0^4 \int_0^{\sqrt{4-x}} (2-y)\, dydx = \int_0^4 \left(2y - \frac{y^2}{2}\right)_0^{\sqrt{4-x}} dx =$$

$$\int_0^4 2\sqrt{4-x} - \frac{1}{2}(4-x)\, dx = \left(-2\frac{(4-x)^{\frac{3}{2}}}{\frac{3}{2}} - 2x + \frac{x^2}{4}\right)_0^4 = \cdots\cdots$$

Example: find the volume bounded by $z = 0$, laterally by $x^2 + 4y^2 = 4$ and above by $z = x + 2$.

$x^2 + 4y^2 = 4$

$\frac{x^2}{4} + y^2 = 1$

$\Rightarrow y = \pm\sqrt{1 - \frac{x^2}{4}}$

$$V = \int_{-2}^{2} \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} \int_0^{x+2} dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} (x+2)\, dydx$$

$$= \int_{-2}^{2} \big((x+2)y\big)_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} dx$$

$$= \int_{-2}^{2} (x+2)\, 2\sqrt{1 - \frac{x^2}{4}}\, dx$$

$$= \int_{-2}^{2} 2x\sqrt{1 - \frac{x^2}{4}}\, dx + \int_{-2}^{2} 4\sqrt{1 - \frac{x^2}{4}}\, dx$$

$= \cdots\cdots$

Example: use a triple integral to find the volume of the solid enclosed between the cylinder $x^2 + y^2 = 9$ and the planes $z = 1$ and $x + z = 5$.

$$V = \iiint_G dv$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-x} dzdydx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [z]_1^{5-x}\, dydx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\, dydx$$
$$= \int_{-3}^{3} (4-x)(y)_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\, dx = \int_{-3}^{3} (4-x)\left(2\sqrt{9-x^2}\right) dx$$
$$= \int_{-3}^{3} (8-2x)\sqrt{9-x^2}\; dx$$
$$= 8\int_{-3}^{3} \sqrt{9-x^2}\; dx - \int_{-3}^{3} 2x\sqrt{9-x^2}\; dx$$
$$= \cdots\cdots$$

Example: find the volume enclosed between the two surfaces $z = 8 - x^2 - y^2$, $z = x^2 + 3y^2$.

$$V = \iiint_G \, dv$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{8-2x^2}{4}}}^{\sqrt{\frac{8-2x^2}{4}}} \int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$$
$$= \int_{-2}^{2} \int_{-\sqrt{\frac{8-2x^2}{4}}}^{\sqrt{\frac{8-2x^2}{4}}} \left((8-x^2-y^2)-(x^2+3y^2)\right) dydx$$
$$= \int_{-2}^{2} \int_{-\sqrt{\frac{8-2x^2}{4}}}^{\sqrt{\frac{8-2x^2}{4}}} (8-2x^2-4y^2)\, dydx$$
$$= \cdots$$

Example: find the volume of a solid whose height is the surface $z = x + y + 10$ and whose base is the region in $xy -$ plane shown.

$$V = \iint_R \, f(x,y)\, dR$$
$$= \iint (x+y+10)\, dR$$

in polar coordinate $x = r\cos\theta$, $y = r\sin\theta$
$dR = rdrd\theta$

$$= \int_0^{\pi} \int_1^2 [r\cos\theta + r\sin\theta + 10]\,(rdrd\theta)$$
$$= \int_0^{\pi} \left[\frac{r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta + 5r^2\right]_1^2 d\theta$$
$$= \int_0^{\pi} \frac{8}{3}\cos\theta + \frac{8}{3}\sin\theta + 2\theta - \left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + 5\right) d\theta$$

$$= \int_0^{\pi} \frac{7}{3}[\cos\theta + \sin\theta] + 15\, d\theta$$
$$= \left[\frac{7}{3}\sin\theta - \frac{7}{3}\cos\theta + 15\theta\right]_0^{\pi} = \cdots$$

Triple integral in cylindrical and spherical coordinates:

Cylindrical coordinates

the triple integral by using cylindrical coordinates can be as following

I $= \iiint dz\, r dr d\theta$

Example: find the volume bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2y$.

$$z = x^2 + y^2 = r^2$$
$$z = 2y = 2r\sin\theta$$
$$r^2 = 2r\sin\theta \Rightarrow r^2 - 2r\sin\theta = 0 \Rightarrow r(r - 2\sin\theta) = 0 \Rightarrow r = 0,\ r = 2\sin\theta$$
$$2\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0,\ \pi$$

$$V = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} dz r dr d\theta$$
$$= \int_0^{\pi}\int_0^{2\sin\theta} [z]_{r^2}^{2r\sin\theta}\, r dr d\theta$$
$$= \int_0^{\pi}\int_0^{2\sin\theta} (2r\sin\theta - r^2)\, r dr d\theta$$
$$= \int_0^{\pi}\int_0^{2\sin\theta} (2r^2\sin\theta - r^3)\, dr d\theta$$
$$= \int_0^{\pi} \left[\frac{2}{3}\, r^3\sin\theta - \frac{r^4}{4}\right]_0^{2\sin\theta}$$
$$= \int_0^{\pi} \frac{2}{3}\,(2\sin\theta)^3\sin\theta - \frac{(2\sin\theta)^4}{4}\, d\theta$$
$$= \int_0^{\pi} \frac{16}{3}\sin^4\theta - 4\sin^4\theta\, d\theta$$
$$= \int_0^{\pi} \sin^4\theta \left(\frac{16}{3} - 4\right) d\theta$$
$$= \frac{4}{3}\int_0^{\pi} \sin^4\theta\, d\theta = \frac{4}{3}\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$
$$= \frac{4}{3}\cdot\frac{1}{4}\int_0^{\pi} (1 - 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$
$$= \frac{1}{3}\int_0^{\pi} \left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$
$$= \frac{1}{3}\int_0^{\pi} \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{3}\left[\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi}$$
$$= \frac{1}{3}\left[\left(\frac{3}{2}\pi - \sin 2\pi + \frac{1}{8}\sin 4\pi\right) - \left(0 - \sin 0 + \frac{1}{8}\sin 0\right)\right] = \frac{\pi}{2} \ .$$

Example: find the volume bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.

$z = 5 - x^2 - y^2 = 5 - (x^2 + y^2) = 5 - r^2$
$z = 4x^2 + 4y^2 = 4(x^2 + y^2) = 4r^2$
$5 - r^2 = 4r^2 \Rightarrow 5 - 5r^2 = 0 \Rightarrow r^2 = 1$
$\theta: 0 \to 2\pi$

$$V = \int_0^{2\pi}\int_0^1\int_{4r^2}^{5-r^2} dzrdrd\theta$$
$$= \int_0^{2\pi}\int_0^1 [z]_{4r^2}^{5-r^2}\, rdrd\theta = \int_0^{2\pi}\int_0^1 [5 - r^2 - 4r^2]\, rdrd\theta$$
$$= \int_0^{2\pi}\int_0^1 (5r - 5r^3)\, drd\theta$$
$$= 5\int_0^{2\pi}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)_0^1 d\theta = 5\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{4}\right) d\theta$$
$$= 5\int_0^{2\pi}\frac{1}{4}\, d\theta = \frac{5}{4}\,\theta\Big]_0^{2\pi} = \frac{5}{4}(2\pi) = \frac{5}{2}\,\pi \ .$$

Spherical coordinates:

The triple integral by using spherical coordinates can be as following:

$$\mathbf{I} = \iiint \boldsymbol{\rho^2 \sin \emptyset \ d\rho\, d\emptyset\, d\theta}$$

Where ρ: is the radius of sphere
$\emptyset$: is the angle between radius and positive $z -$ axis .
θ: is the angle between radius and positive $x -$ axis .

Also:
$x = \rho \cos\theta \sin\emptyset$
$y = \rho \sin\theta \sin\emptyset$
$z = \rho \cos\emptyset$
$dxdydz = \rho^2 \sin\emptyset \ d\rho\, d\emptyset\, d\theta = dV$

Example: find the volume of the upper region cut from the solid sphere $\rho = 1$ by the cone $\emptyset = \frac{\pi}{3}$.

$\rho: 0 \rightarrow 1$
$\emptyset: 0 \rightarrow \frac{\pi}{3}$
$\theta: 0 \rightarrow 2\pi$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin\emptyset \; d\rho \, d\emptyset \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left[\frac{\rho^3}{3}\right]_0^1 \sin\emptyset \; d\emptyset \, d\theta$$
$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left[\frac{1}{3} - \frac{0}{3}\right] \sin\emptyset \, d\emptyset \, d\theta = \frac{1}{3}\int_0^{2\pi} [-\cos\emptyset]_0^{\frac{\pi}{3}} \, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi} \left[-\cos\frac{\pi}{3} - (-\cos 0)\right] \, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi} \frac{1}{2} \, d\theta = \frac{1}{3}\left[\frac{1}{2}\theta\right]_0^{2\pi}$$
$$= \frac{1}{6}(2\pi - 0) = \frac{\pi}{3} \; .$$