University of Mustansiriyah

College of Engineering

Department of Highway and Transportation Engineering

Second Class (2^{nd} Class)

Mathematics

(2025 – 2026)

Infinite Series

Infinite series

Def: infinite series are an expression of the form $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$ where $a_1, a_2, a_3, \ldots,$ are called the terms of series.

Sum of infinite series:

Let $s_1 = a_1$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

The seq. $\{s_n\}$ is called the seq. of partial sum of series.

If $\lim\limits_{n\to\infty} s_n = s$ then $\sum_{n=1}^{\infty} a_n$ is converge and s is called the sum of the series

and if s_n diverge then $\sum_{n=1}^{\infty} a_n$ is diverge and have no sum.

Example: find the sum of the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$.

$$s_1 = \frac{3}{10}$$

$$s_2 = \frac{3}{10} + \frac{3}{10^2}$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3}$$

$$\vdots$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} \quad \textit{(multiply both sides by } \tfrac{1}{10}\textit{)}$$

$$\frac{1}{10}\, s_n = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^n} + \frac{3}{10^{n+1}}$$

now we subtract this from s_n, we have:

$$s_n - \frac{1}{10}\, s_n = \frac{3}{10} - \frac{3}{10^{n+1}} = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

$\frac{9}{10}\ s_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$

$s_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$

$\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \frac{1}{3}$

So we say that sum of the infinite series $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}$ is $\frac{1}{3}$ and we write $\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{1}{3}$.

Example: find the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

$s_1 = 1$

$s_2 = 1 - 1 = 0$

$s_3 = 1 - 1 + 1 = 1$

$s_4 = 1 - 1 + 1 - 1 = 0$

The seq. of partial sums is 1,0,1,0,1,0, ...

Since this sequence is diverge so the series div. and have no sum.

Geometric series

Geometric series are series of the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = a(1 + r + r^2 + r^3 + \cdots + r^{n-1} + \cdots)$.

Example:

1) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots \left(a = 1,\ r = \frac{1}{2}\right)$
2) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + (-1)^n\ \frac{1}{3^n} + \cdots \left(a = 1,\ r = -\frac{1}{3}\right)$
3) $3 + \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \cdots \left(a = 3,\ r = \frac{1}{10}\right)$
4) $1 + 2 + 4 + 8 + 16 + \cdots + 2^n + \cdots\ (a = 1,\ r = 2)$

Test the convergency or divergency of geometric series

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots$ converge $\frac{a}{1-r}$

If $|r| \geq 1$, the series diverge.

Example: the series $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ are geometric series with $a = 1,\ r = \frac{1}{2}$

$|r| = \frac{1}{2} < 1$ converge and $s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$.

Example: the series $\sum_{n=0}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$

$|r| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1$ the series is diverge and have no sum.

Homework: Test the following series

1) $2 + \frac{2}{7} + \frac{2}{7^2} + \frac{2}{7^3} + \cdots + \frac{2}{7^n} + \cdots = \sum_{n=0}^{\infty} \frac{2}{7^n}$.
2) The geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{3}$, find its sum.
3) $3 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \cdots = \sum_{n=0}^{\infty} \frac{3}{5^n}$.

Convergency or divergency test of infinite series (series of positive term)

1. Comparison test:

Let $\sum a_n$ and $\sum b_n$ be series with positive terms, then:

a) The series $\sum a_n$ convergent if $a_n \leq b_n$ and $\sum b_n$ is convergent.

b) The series $\sum a_n$ is divergent if $a_n \geq b_n$ and $\sum b_n$ is divergent.

Example: test the series

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$\frac{1}{n^2+n} \leq \frac{1}{n^2}$

but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge

then $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is converge.

2. $\sum_{n=0}^{\infty} \frac{\sin n}{2^n}$

$\sin n \leq 1 \ \forall n$

$\frac{\sin n}{2^n} \leq \frac{1}{2^n}$

but $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ (geometric series)

$a = 1,\ r = \frac{1}{2} < 1$ converge

then $\sum \frac{sinn}{2^n}$ is converge (by comparison test).

Note: *P*-series is infinite series at the form $\sum_{n=1}^{\infty} \frac{1}{n^P}$ and it is converge if $P > 1$ and diverge if $0 < P \leq 1$.

Example: the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is converge since $P = 2 > 1$.

Example: the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is diverge since $P = \frac{1}{2} < 1$.

Example: the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is diverge since $P = 1$. *(called harmonic series)*

2. Limit comparison test

Let $\sum a_n$, $\sum b_n$ be a positive series, then both series are con. or div. .

If $\rho = \lim_{n\to\infty} \frac{a_n}{b_n}$ (is finite) $\neq 0$

Example: test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Let $\sum a_n = \sum \frac{1}{n^2+n}$ by comparison with $\sum b_n = \sum \frac{1}{n^2}$ (which converge $\begin{matrix} P - series \\ P = 2 > 1 \end{matrix}$)

$$\rho = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0$$

$\therefore \sum \frac{1}{n^2+n}$ is converge.

Example: test the series $\sum_{n=1}^{\infty} \frac{1}{n-\frac{1}{4}}$

Let $\sum a_n = \sum \frac{1}{n-\frac{1}{4}}$ comparison with $\sum b_n = \sum \frac{1}{n}$ (which is diverge P-series)

$\rho = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n-\frac{1}{4}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{4n}} = 1 \neq 0$ then $\sum \frac{1}{n-\frac{1}{4}}$ is diverge.

3. <u>Ration test</u>

Let $\sum a_n$ be series with positive terms and $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

If $\rho < 1$ then $\sum a_n$ is converge.

If $\rho > 1$ then $\sum a_n$ is diverge.

If $\rho = 1$ fails (another test must be tried)

Example: test the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$a_n = \frac{n}{2^n}$, $a_{n+1} = \frac{n+1}{2^{n+1}}$

$$\rho = \lim_{n\to\infty} \frac{a_n+1}{a_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2^n \cdot 2} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2} < 1$$

then the series $\sum \frac{n}{2^n}$ is converge.

Example: test the series $\sum_{n=1}^{\infty} \frac{n}{10^n}$

$a_n = \frac{n}{10^n}$, $a_{n+1} = \frac{n+1}{10^{n+1}}$

$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{10^{n+1}} \cdot \frac{10^n}{n} = \lim_{n\to\infty} \frac{n+1}{10^n \cdot 10} \cdot \frac{10^n}{n} = \lim_{n\to\infty} \frac{n+1}{10n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{10} = \frac{1}{10} < 1$ then the series $\sum \frac{n}{10^n}$ is converge.

Example: test the series $\sum_{n=1}^{\infty} \frac{n!\,n!}{(2n)!}$

$a_n = \frac{n!\,n!}{(2n)!}$, $a_{n+1} = \frac{(n+1)!\,(n+1)!}{(2(n+1))!}$

$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{n!n!} = \lim_{n\to\infty} \frac{(n+1)n!\,(n+1)n!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!}$

$\lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)2n!} \cdot 2n! = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{6}{n}+\frac{2}{n^2}} = \frac{1}{4} < 1$ is converge.

4. <u>n^th root test</u>

Let $\sum a_n$ be series with positive term and $\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$

If $\rho < 1$ then $\sum a_n$ converge.
If $\rho > 1$ then $\sum a_n$ diverge.
If $\rho = 1$ fails (try another test)

Example: test the series $\sum_{n=1}^{\infty} \frac{1}{n^n} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$

$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{1}{n}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$ then the series is converge.

Example: test the series $\sum_{n=1}^{\infty} \left(\frac{4n-5}{2n+1}\right)^n$

$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{4n-5}{2n+1}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{4n-5}{2n+1} = \lim_{n\to\infty} \frac{4-\frac{5}{n}}{2+\frac{1}{n}} = 2 > 1$

then the series is diverge.

Theorem:

1. Let $\sum a_n$, $\sum b_n$ are con. series then $\sum a_n \pm b_n$ are con.
2. $\sum c a_n = c \sum a_n$ (whether $\sum a_n$ con. or div.)
3. Let $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ then $\sum_{n=m}^{\infty} a_n = a_m + a_{m+1} + a_{m+2} + \cdots$ doesn't affected to the con. or div.

Alternating series

The series of the form $\sum_{n=1}^{\infty} (-1)^{n+1}\ a_n$ or $\sum_{n=1}^{\infty} (-1)^{n}\ a_n$ are called alternating series, and it is converge if:

1. $\{a_n\}$ is dec.
2. $\lim\limits_{n\to\infty} a_n = 0$

Example: test the series $\sum_{n=1}^{\infty} (-1)^{n+1}\ \frac{1}{n}$

$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$

since $\{a_n\}$ dec. and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

then $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is converge.

Example: the series $\sum_{n=1}^{\infty} (-1)^{n+1}\ \frac{n+3}{n(n+1)}$

$\frac{a_{n+1}}{a_n} = \frac{n+4}{(n+1)(n+2)} \cdot \frac{n(n+1)}{n+3} = \frac{n^2+4n}{n^2+5n+6} < 1$

$\therefore \{a_n\}$ dec.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n+3}{n(n+1)} = \lim\limits_{n\to\infty} \frac{n+3}{n^2+n} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}+\frac{3}{n^2}}{1+\frac{1}{n}} = \frac{0}{1} = 0$

then $\sum (-1)^{n+1}\ \frac{n+3}{n(n+1)}$ is converge.

Absolute and conditional converge of alternating series

Def: if $\sum(-1)^n a_n$ is con. and $\sum|(-1)^n a_n|$ is con. then $\sum(-1)^n a_n$ is said to be absolutely convergent.

Example: the series $\sum(-1)^n \frac{1}{n^2}$ is converge since $a_n = \frac{1}{n^2} > \frac{1}{(n+1)^2} = a_{n+1}$, $\{a_n\}$ dec. and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$.

$\sum\left|(-1)^n \frac{1}{n^2}\right| = \sum \frac{1}{n^2}$ which is con. $\binom{P-series}{P=2>1}$

$\therefore \sum(-1)^n \frac{1}{n^2}$ is absolute converge.

Def: if $\sum (-1)^n\ a_n$ is conv. but $\sum|(-1)^n a_n|$ is div. then $\sum(-1)^n a_n$ is said to be conditionally convergent.

Example: the series $\sum(-1)^n \frac{1}{n}$ is con. since $a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$ dec.

$\lim_{n\to\infty} \frac{1}{n} = 0$ con.

but $\sum\left|(-1)^n \frac{1}{n}\right| = \sum \frac{1}{n}$ is div.

then $\sum(-1)^n \frac{1}{n}$ is conditionally con.

(Power series)

Taylor's and Maclaurin series

Def: if the function $f(x)$ is differentiable $n-$times at $x = a,$ then this function can be written as:

$$\boldsymbol{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a^2) + \cdots + \frac{f^{(n)}_{(a)}}{n!}(x-a)^n + \cdots}$$

$$\boldsymbol{= \sum_{n=0}^{\infty} f^{(n)}_{(a)}\ \frac{(x-a)^n}{n!}}$$

this series is called **Taylor series**, and special case if $a = 0,$ then it is **Maclaurin series**, which is:

$$\boldsymbol{f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + \frac{f^{(n)}_{(0)}}{n!}x^n + \cdots = \sum_{n=0}^{\infty} f^{(n)}_{(0)}\ \frac{x^n}{n!}}$$

Example: find Maclaurin series for $f(x) = e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\begin{array}{ll} f(x) = e^x & f(0) = 1 \\ f'(x) = e^x & f'(0) = 1 \\ f''(x) = e^x & f''(0) = 1 \end{array}$$

Example: find Maclaurin series for $f(x) = \sin x$

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \end{array}$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Note: similarity to above we can get that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Homework: Find Maclaurin series for

1. $f(x) = e^{ax}$ 2. $f(x) = \ln(1 + x)$

Example: find Taylor series for $f(x) = \ln x$ at $x = 1$.

$$\begin{array}{ll} f(x) = \text{In}\, x & f(1) = \ln 1 = 0 \\ f'(x) = \frac{1}{x} & f'(1) = 1 \\ f''(x) = \frac{-1}{x^2} & f''(1) = -1 \\ f'''(x) = \frac{2}{x^3} & f'''(1) = 2 \end{array}$$

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\,\frac{(x-a)^2}{2!} + f'''(a)\,\frac{(x-a)^3}{3!} + \cdots$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^{n+1}}{(n+1)!}\ n!$$

Homework: Find Taylor series for $f(x) = \sqrt{x}$ at $x = 4$.

Example: using the series of e^x to find the series of $\cosh x$.

$\cosh x = \frac{e^x + e^{-x}}{2}$

since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$ and $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$ then $\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

similarly, we get that $\sinh x = \frac{e^x + e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

Remark: the interval (a, b) of x which the power series $\sum a_n \ x^n$ converges absolutely is called interval of converge and R is called radius of convergency. To find this interval, we use ratio test, n^{th} root test $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ and radius of convergence $R = \frac{1}{2} \ |b - a|$.

Example: find the interval of converge and radius of the power series $\sum \frac{(-1)^n \ x^n}{3^n \ (n+1)}$

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \qquad \therefore \ |(-1)^n| = |(-1)^{n+1}| = 1$

$\lim_{n \to \infty} \left| \frac{x^{n+1}}{3^{n+1} \ (n+2)} \cdot \frac{3^n \ (n+1)}{x^n} \right| < 1$

$\lim_{n \to \infty} \left| \frac{x^n \cdot x}{3^n \cdot 3 \ (n+2)} \cdot \frac{3^n \ (n+1)}{x^n} \right| < 1$

$\Rightarrow \ \frac{|x|}{3} \lim_{n \to \infty} \frac{n+1}{n+2} < 1$

$\Rightarrow \ \frac{|x|}{3} < 1 \Rightarrow \ |x| < 3 \Rightarrow \ -3 < x < 3$

at $x = 3$

$\sum \frac{(-1)^n \ 3^n}{3^n \ (n+1)} = \sum (-1)^n \ \frac{1}{n+1}$ (is alt. series)

$\left\{ \frac{1}{n+1} \right\}$ is dec. , $\lim_{n \to \infty} \frac{1}{n+1} = 0$

$\therefore$ conv. by alt. test

at $x = -3$

$$\sum \frac{(-1)^n\,(-3)^n}{3^n\,(n+1)} = \sum \frac{(-1)^n\,(-1)^n\,(3)^n}{3^n\,(n+1)} = \sum \frac{(-1)^{2n}}{(n+1)} = \sum \frac{1}{n+1} \text{ div.}$$

the interval of con. $-3 < x \le 3$ and $R = 3$

Example: find the interval of converge of the series $\sum (-1)^n \frac{x^{2n}}{(2n)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}\right| =$$

$$\lim_{n\to\infty} \left|\frac{x^{2n} \cdot x^2\,(2n)!}{(2n+2)(2n+1)(2n)!\,x^{2n}}\right| = |x^2| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} < 1$$

thus, the interval of con. $-\infty < x < \infty$

Example: find the interval of conv. and radius of the series $\sum \frac{(x-5)^n}{n^2}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-5)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x-5)^n}\right| = \lim_{n\to\infty} \left|\frac{(x-5)^n\,(x-5)\,n^2}{(n^2+2n+1)\,(x-5)^n}\right| =$$

$$|x-5| \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} < 1 \Rightarrow |x-5| < 1 \Rightarrow -1 < x-5 < 1 \Rightarrow 4 < x < 6$$

at $x = 4 \Rightarrow \sum \frac{(-1)^n}{n^2}$ con. (alt. series)

at $x = 6 \Rightarrow \sum \frac{1}{n^2}$ con. ($P -$ series)

then the interval of conv. $[4,6]$ and $R = 1$.

Home Work

1. Determine whether the following series conv. or div.

   a. $\sum_{n=1}^{\infty} \frac{(n+4)!}{4!\, n!\, 4^n}$

   b. $\sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{4^n}$

   c. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$

2. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$.

3. Find Taylor series for $\tan^{-1} x$ about $x = 1$.

4. Find Maclaurin series for $f(x) = xe^x$.

5. Use series to evaluate $\int_0^1 \frac{\sin x}{x}\, dx$.

6. Evaluate $\lim_{x\to\infty} x^2 \left(e^{-\frac{1}{x^2}} - 1\right)$ using series .