University of Mustansiriyah

College of Engineering

Department of Highway and Transportation Engineering

Second Class (2^{nd} Class)

Mathematics

(2025 – 2026)

Polar Coordinates

Polar coordinates

The relation between polar coordinates and Cartesian coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example:

<u>Polar equation</u>	<u>Cartesian equivalent:</u>
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2cos^2\theta - r^2sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$

Example: Find a Cartesian equation for the curve $r\cos\left(\theta - \frac{\pi}{3}\right) = 3.$

Sol: $r\cos\left(\theta - \frac{\pi}{3}\right) = 3$

$$r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 3$$

$$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 3$$

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3 \Rightarrow x + \sqrt{3}y = 6$$

Example: Replace the following polar equation to equivalent Cartesian equations and identify their graphs.

1. $r\cos\theta = -4 \Rightarrow x = -4$

The graph: vertical line through $x = -4$ on the x-axis.

2. $r^2 = 4r\cos\theta$

$$x^2 + y^2 = 4x \Rightarrow x^2 - 4x + y^2 = 0 \Rightarrow x^2 - 4x + 4 - 4 + y^2 = 0$$
$$\Rightarrow (x-2)^2 + y^2 = 4$$

The graph: circle with radius 2, center (2,0)

3. $r = \dfrac{4}{2\cos\theta - \sin\theta}$

$$2r\cos\theta - r\sin\theta = 4$$

$$2x - y = 4 \Rightarrow y = 2x - 4$$

The graph: line, Slope $m = 2$

Example: Find the polar coordinates of the Cartesian point $P(1,1)$.

Sol: $x = 1, y = 1$

$$r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$\therefore P(r,\theta) = P\left(\sqrt{2}, \frac{\pi}{4}\right).$$

Example: Find the Cartesian coordinates of the polar point $P(1, \frac{\pi}{2})$.

$$r = 1, \theta = \frac{\pi}{2}$$

$$x = r\cos\theta = 1 \cdot \cos\frac{\pi}{2} = 0$$

$$y = r\sin\theta = 1 \cdot \sin\frac{\pi}{2} = 1$$

$$\therefore P(x,y) = P(0,1)$$

Graphing in polar coordinates:

The graph of the equation

Three kinds of symmetry are easy to detect in the graph of an equation

$F(r,\theta) = 0$, the graph is:

1. Symmetric about the origin if the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$.

$$r = r^2$$

2. Symmetric about the x-axis if the equation unchanged when θ is replaced by $-\theta$, or the pair (r,θ) by the pair $(-r, \pi - \theta)$.

$$\cos\theta = \cos(-\theta)$$

3. Symmetric about the y-axis if the equation is unchanged when θ is replaced by $\pi - \theta$, or the pair (r,θ) by the pair $(-r, -\theta)$.

$$\sin\theta = \sin(\pi - \theta).$$

Example: $r = 3 + 2\sin\theta$.

θ	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	……
r	1	3	4	$3+\sqrt{3}$	5	3	……

Graph in polar coordinates:

1. $r = a\,(1 + \cos\theta)$

2. $r = a\,(1 - \cos\theta)$

3. $r = a\,(1 + \sin\theta)$

4. $r = a\,(1 - \sin\theta)$

1, 2, 3 and 4 are called cardioid.

5. Circle

   $r = a$

$$r^2 = a^2 \cos 2\theta$$

$$r^2 = -a^2 \cos 2\theta$$

Area in polar coordinate:

If the function $r = f(\theta)$ cont. and non-negative from $\alpha \le \theta \le \beta$, then the area A enclosed by the polar curve $r = f(\theta)$.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \; d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

Example: Find the area of the region enclosed by the cardioid $r = 2(1 + \cos\theta)$.

$$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} [2(1 + \cos\theta)]^2 \, d\theta.$$

$$= \frac{4}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta$$

$$= \int_0^{2\pi} \left[2 + 4\cos\theta + 2\frac{1 + \cos 2\theta}{2}\right] d\theta = \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi.$$

Example: Find the area of the region in the first quadrant of the cardioid $r = 1 - \cos\theta$.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right] d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{8}\pi - 1.$$

Area between curves:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (r_2^2 - r_1^2)\, d\theta$$

Example: Find the area of the region that lies inside the function $r = 3\sin\theta$ and outside the function $r = 2 - \sin\theta$.

Let $r_1 = r_2$

$3\sin\theta = 2 - \sin\theta$

$4\sin\theta = 2 \Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6} + k\pi.$

$$A = \frac{1}{2}\cdot 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} [(3\sin\theta)^2 - (2 - \sin\theta)^2]\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (9\sin^2\theta - 4 + 4\sin\theta - \sin^2\theta)\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (8\sin^2\theta - 4 + 4\sin\theta)\, d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\overset{\frac{1-\cos 2\theta}{2}}{\sin^2\theta} + \sin\theta - 1)\, d\theta = 4\left[\theta - \frac{\sin 2\theta}{2} - \cos\theta - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}.$$

$= 3\sqrt{3}.$

Example: Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (r_2^2 - r_1^2)\, d\theta$$

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [(1)^2 - (1 - \cos\theta)^2]\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 - 1 + 2\cos\theta - \cos^2\theta]\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [2\cos\theta - \cos^2\theta]\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[2\cos\theta - \frac{1 + \cos 2\theta}{2}\right] d\theta$$

$$= \frac{1}{2}\left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 - \frac{\pi}{4}.$$

Example: Find the area inside $r = 3a\cos\theta$, outside $r = a(1+\cos\theta)$.

$$A = 2\left[\frac{1}{2}\int_0^{\frac{\pi}{3}} \left((3a\cos\theta)^2 - (a(1+\cos\theta))^2\right)\, d\theta\right]$$

$$= \int_0^{\frac{\pi}{3}} (9a^2\cos^2\theta - a^2 - 2a^2\cos\theta - a^2cos^2\theta)\, d\theta$$

$$\frac{1+\cos 2\theta}{2}$$

$$= \int_0^{\frac{\pi}{3}} [8a^2\cos^2\theta - 2a^2\cos\theta - a^2]\, d\theta$$

$$= a^2\left[8\cdot\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - 2\sin\theta - \theta\right]_0^{\frac{\pi}{3}}$$

$$= \cdots$$

$$\left[\begin{array}{c} r_1 = r_2 \\ 3a\cos\theta = a(1+\cos\theta) \\ 1+\cos\theta - 3\cos\theta = 0 \\ 2\cos\theta = 1 \\ \cos\theta = \frac{1}{2} \\ \theta = \frac{\pi}{3} \end{array}\right.$$

Arc length of a polar curve

If the curve has the polar equation $r = f(\theta)$ where $f(\theta)$ is cont. for $\alpha \le \theta \le \beta$, then the arc length L from $\theta = \alpha$ to $\theta = \beta$ is:

$$\left[L = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta \right]$$

$$\left[L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta \right]$$

Example: Find the total arc length of the cardioid $r = 1 + \cos\theta$.

Sol: $r = 1 + \cos\theta$

$$\frac{dr}{d\theta} = -\sin\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta = \int_{0}^{2\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\ d\theta$$

$$= \int_{0}^{2\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\ d\theta$$

$$= \int_{0}^{2\pi} \sqrt{2 + 2\cos\theta} = \sqrt{2} \int_{0}^{2\pi} \sqrt{1 + \cos\theta}\ d\theta$$

Since $\cos^2\theta = \frac{1+\cos 2\theta}{2} \Rightarrow \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \Rightarrow 1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

$$= \sqrt{2} \int_{0}^{2\pi} \sqrt{2\cos^2\frac{\theta}{2}}\ d\theta = 2 \int_{0}^{2\pi} \cos\frac{\theta}{2}\ d\theta = 8.$$

<u>Home Work</u>

1. Find the length of the cardioid $r = a(1 - \cos\theta)$.

2. Find the area enclosed by $r^2 = 2a^2 \cos 2\theta$.

3. Find the area shared (common) by $r_1 = a$ and $r_2 = 2a \sin\theta$.

4. Find the area inside $r^2 = 2a^2 \cos 2\theta$ and outside $r = a$.

5. Find the area of the region that lies outside the circle $r = \sin\theta$ and inside the circle $r = 2 \sin\theta$.

6. Find the area of the region that lies inside $r = 1 + \sin\theta$ and outside $r = 1$.