University of Mustansiriyah

College of Engineering

Department of Highway and Transportation Engineering

Second Class (2^{nd} Class)

Mathematics

(2025 – 2026)

Vectors

<u>Vectors</u>

Vector is defined to be a directed line segment.

$$V = (v_1, v_2)$$

Theorem: Let V = (v_1, v_2, v_3) and W = (w_1, w_2, w_3) are two vectors, then:

1. V + W = $(v_1 + w_1,\ v_2 + w_2,\ v_3 + w_3)$

2. kV = $(kv_1,\ kv_2,\ kv_3)$, k is any constant.

Example: If V = (1, -3, 2), W = (4, 2, 1) then

$\mathrm{V} + \mathrm{W} = (5, -1, 3), 2\mathrm{V} = (2, -6, 4)$

-W = (-4, -2, -1), V – W = V+(-W) = (-3, -5, 1)

<u>Remark:</u>

If P_1 (x_1, y_1, z_1), $P_1(x_2, y_2, z_2)$ be any two points in the space, then:

$P_1P_2 = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$

Length of the vector:

The length of the vector V is usually denoted by |V| such that:

If V = (v_1, v_2) then |V| = $\sqrt{v_1^2 + v_2^2}$

If V = (v_1, v_2, v_3) then |V| = $\sqrt{v_1^2 + v_2^2 + v_3^2}$

$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Properties of vectors:

Let v, u, w be any vectors and let k, l scalars, then

1. $u + v = v + u$

2. $(u + v) + w = u + (v + w)$

3. $u + o = o + u = u$

4. $u + (-u) = o$

5. $k\ (l\ u) = (kl)\ u$

6. $(k + l)\ u = k\ u + l\ u$

7. $k\ (u + v) = ku + kv$

Unit vectors:

Any vector u whose length is equal to the unit length is called a unit vector, the vector i, j and k are unit vectors

$i = (1, 0, 0), j = (0, 1, 0), K = (0, 0, 1)$

$|i| = \sqrt{1^2 + 0^2 + 0^2}$, $|j| = \sqrt{0^2 + 1^2 + 0^2} = 1$

$|k| = \sqrt{0^2 + 0^2 + 1^2} = 1$

Direction:

The direction of non a zero vector V to be the unit vector obtained by dividing v by its length

Direction of V = $\frac{V}{|V|}$

Example:

The direction of V = 3i – 4j is: direction of

$$V = \frac{V}{|V|} = \frac{3i-4j}{\sqrt{(3)^2+(-4)^2}} = \frac{3i-4j}{\sqrt{25}} = \frac{3}{5}i - \frac{4}{5}j$$

To check length

$$\left|\frac{3}{5}i - \frac{4}{5}j\right| = \sqrt{(\frac{3}{5})^2 + (\frac{-4}{5})^2} = \sqrt{\frac{25}{25}} = 1 \text{ (unit vector)}$$

Note: We can write any vector V in terms of i, j, k

$$V = (v_1 + v_2 + v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3)$$
$$= v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1\, i + v_2\, j + v_3\, k$$

Example: $V = (2, 1, 3) = 2i + j + 3k$

Definition: Two vectors are said to be parallel if they are either positive or negative scalar multiples of an another.

U and V are parallel if $U = kV$, (k any scalar)

Example: The vectors $U = 2i + 4j + 3k$ and $V = 6i + 12\, j + 9k$ are parallel since $V = 3U$ where $k = 3$

Dot product (Scalar product)

The scalar product (or dot product) of the two vectors u and v is the number $u.v = |u|\, |v|\, cos\theta$ where θ is the angle between them.

Now from above we can find the angle between u and v which is:

$$cos\theta = \frac{u.v}{|u||v|}$$

Note: $\boldsymbol{i \cdot i = |i||i|\, cos0 = 1 \cdot 1 \cdot 1 = 1}$

$\boldsymbol{i \cdot i = j \cdot j = k \cdot k = 1}$

$\boldsymbol{i \cdot j = j \cdot k = k \cdot i = 0}$

$$u \cdot v = (u_1 i + u_2 j + u_3 k) \cdot (v_1 i + v_2 j + v_3 k) = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example: Find the θ between $u = i - 2j - 2k$ and $v = 6i + 3j + 2k$

$u \cdot v = (1)(6) + (-2)(3) + (-2)(2) = -4$

$u.v = |4|\ |v|\ cos\theta$

$|u| = \sqrt{1 + 4 + 4} = 3$ and $|v| = \sqrt{36 + 9 + 4} = 7$

$$cos\theta = \frac{u \cdot v}{|u||v|} = \frac{-4}{21}$$

$$\boldsymbol{\theta = \cos^{-1}(\frac{-4}{21}) = 101}$$

Definition: Two vectors whose scalar product is zero are said to be **orthogonal**.

u and v are orthogonal (perpendicular) if $u \cdot v = 0$

Example: The vectors $A = 3i - 2j + k$ and $B = 2j + 4k$ are orthogonal because $A \cdot B = (3)(0) + (-2)(2) + (1)(4) = 0$

Vector projections and scalar components

1. Vector projections of B onto A is denoted by $V = proj_A^B = \frac{A \cdot B}{A \cdot A} A$
2. Scalar projection of B onto $A = \frac{A \cdot B}{|A|}$

$$|Proj_A^B| = \frac{A \cdot B}{|A|}$$

Note: $\boldsymbol{A \cdot A = |A^2|}$

Example: If $A = 2i - j + 3k$ and $B = 4i - j + 2k$ then find $Proj_A^B$ and $|Proj_A^B|$.

$$Proj_A^B = \frac{A \cdot B}{A \cdot A} A = \frac{8+1+6}{4+1+9}(2i - j - 3k) = \frac{15}{14}(2i - j + 3k)$$

$$|Proj_A^B| = \frac{A \cdot B}{|A|} = \frac{15}{\sqrt{14}}$$

Cross product:

$A \times B = |A|\,|B| \sin\theta \;\, n$

Where n is a unit normal vector to the plane containing A and B

Note: $\boldsymbol{B \times A = -(A \times B)}$

Note: $\boldsymbol{i \times i = j \times j = k \times k = 0}$

$\boldsymbol{i \times j = k}$
$\boldsymbol{j \times k = i}$
$\boldsymbol{k \times i = j}$

If $A = a_1 i + a_2 j + a_3 k$ and $B = b_1 i + b_2 j + b_3 k$ then

$A \times B = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k) = a_1 b_1\, i \times i + a_1 b_2\, i \times j + a_1 b_3\, i \times k + a_2 b_1\, j \times i + a_2 b_2\, j \times j + a_2 b_3\, j \times k + a_3 b_1\, k \times i + a_3 b_2\, k \times j + a_3 b_3\, k \times k = (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k$

From above we can get that

$$\boldsymbol{A \times B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix}$$

Example: Find $A \times B$ and $B \times A$ if $A = 2i + j + k, B = -4i + 3j + k$.

Sol: $A \times B = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} =$

$$i\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - j\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = -2i - 6j + 10k$$

$$B \times A = -(A \times B) = 2i + 6j - 10k.$$

Note: The area of the parallelogram is $|A \times B|$ of base $|A|$ and $|B|\,|\sin\theta\,|$ the height.

Area $= |A \times B| = |A||B|\,|\sin\theta||n| = |A||B|\,|\sin\theta\,|$

and the area of the triangle $= \frac{1}{2}\,|A \times B|$

Example: Find the area of the triangle whose vertices are A (1, -1,0), B (2,1, -1) and C (-1,1,2).

The area of the triangle is $= \frac{1}{2}\,|AB \times AC|$

$$AB = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$AC = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$AB \times AC = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = i\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} = 6i + 6k.$$

Thus, the triangle's area is $\frac{1}{2}|6i + 6k| = \frac{1}{2}\sqrt{72} = 3\sqrt{2}$.

The triple scalar or box product:

The product $(A \times B) \cdot C$ is called the triple scalar product of A, B and C, such that.

$(A \times B) \cdot C = |A \times B|\,|C|\ \cos\theta$

The above product is the volume of the parallelepiped, the number $|A \times B|$ is the area of the base parallelogram and $|C| \cos\theta$ is the height.

Now we can see that $(A \times B) \cdot C = (B \times C) \cdot A = (C \times A) \cdot B$ and $(A \times B) \cdot C = A \cdot (B \times C)$

Note: $A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Example: Find the volume of the box (Parallelepiped) determined by

$A = i + 2j - k, B = -2i + 3k$ and $C = 7j - 4k$.

Sol: The volume is the absolute value of $A \cdot (B \times C) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} =$

$\begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} = -21 - 16 + 14 = -23. \qquad \therefore V = 23$

Triple vector product:

$(A \times B) \times C \neq A \times (B \times C)$

$[(A \times B) \times C = (A \cdot C)B - (B \cdot C)A]$ and

$[A \times (B \times C)] = (A \cdot C)B - (A \cdot B)C]$

Example: Let $A = i - j + 2k, B = 2i + j + k, C = i + 2j - k$, find $(A \times B) \times C$

$(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$

$$A \cdot C = -3, \qquad B \cdot C = 3$$

$(A \cdot C)B - (B \cdot C)A = -3B - 3A = -9i - 9k.$

The direction cosines:

$$\cos\alpha = \frac{u_1}{|u|}$$

$$\cos\beta = \frac{u_2}{|u|}$$

$$\cos\gamma = \frac{u_3}{|u|}$$

Example: Find the direction cosine of $u = 2i - 4j + 4k$.

$$|u| = \sqrt{4 + 16 + 16} = 6$$

$$\cos\alpha = \frac{u_1}{|u|} = \frac{2}{6} \Rightarrow \alpha = \cos^{-1}\left(\frac{2}{6}\right) = 70.5$$

$$\cos\beta = \frac{u_2}{|u|} = \frac{-4}{6} \Rightarrow \beta = \cos^{-1}\left(\frac{-4}{6}\right) = 131.8$$

$$\cos\gamma = \frac{u_3}{|u|} = \frac{4}{6} \Rightarrow \gamma = \cos^{-1}\left(\frac{4}{6}\right) = 48.1$$

Equation for lines in 3-spaces:

A point P lies on the line through $P_{\circ}$ parallel to v if and only if $P_{\circ}P$ is a scalar multiple of V.

We find $P_{\circ}P = (x - x_{\circ})i + (y - y_{\circ})j + (z - z_{\circ})k$

$\therefore\ P_{\circ}P // v$ then $P_{\circ}P = tv$

$(x - x_{\circ})i + (y - y_{\circ})j + (z - z_{\circ})k = t(ai + bj + ck)$

$(x - x_{\circ})i + (y - y_{\circ})j + (z - z_{\circ})k = tai + tbj + tck$

$x - x_\circ = ta$

$y - y_\circ = tb \Rightarrow$

$z - z_\circ = tc$

$$\left.\begin{matrix} x = x_\circ + ta \\ y = y_\circ + tb \\ z = z_\circ + tc \end{matrix}\right\}$$ Are the parametric equations of the line in 3-space

$$(\frac{x - x_\circ}{a} = \frac{y - y_\circ}{b} = \frac{z - z_\circ}{c} = t)$$

Example: Find the parametric equations for the line through the point (-2,0,4) parallel to the vector $V = 2i + 4j - 2k$.

Sol: with $P_\circ(x_\circ, y_\circ, z_\circ) = (-2,0,4)$ and $ai + bj + ck = 2i + 4j - 2k$

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t$$

$$\frac{x+2}{2} = \frac{y}{4} = \frac{z-4}{-2} = t$$

Example: Find the parametric equations for the line through the points $P_1(-3,2,-3)$ and $P_2(1,-1,4)$.

Sol: $P_1P_2 = 4i - 3j + 7k$ with the points $(-3,2,-3)$

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t$$

<u>The distance from a point to a line:</u>

To find the distance from a point P to a line L, we carry out the following steps:

1. Find the point Q on L closest to P.
2. Calculate the distance from P to Q.

Example: Find the distance from the point $P(1,1,5)$ to the line $x = 1 + t$,

$y = 3 - t, z = 2t$.

Sol: Assume $Q(x, t, z)$ as a point so that ($PQ \perp$ line)

$PQ = (1 + t - 1)i + (3 - t - 1)j + (2t - 5)k = ti + (2 - t)j + (2t - 5)k$

$$\frac{x-1}{1} = \frac{y-3}{-1} = \frac{z}{2} = t$$

$$A = 1, \quad B = -1, \quad C = 2, \quad V = i - j + 2k$$

$V \cdot PQ = (t \times 1) + ((2 - t) \times -1) + ((2t - 5) \times 2)$

$$0 = t - 2 + t + 4t - 10$$

$$0 = 6t - 12 \Rightarrow 6t = 12 \Rightarrow t = 2$$

$PQ = 2i + (2 - 2)j + (2 \times 2 - 5)k = 2i - k$

$|PQ| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$

<u>Or</u>

$$\left.\begin{aligned} x &= 1 + 2 = 3 \\ y &= 3 - 2 = 1 \\ z &= 2 \times 2 = 4 \end{aligned}\right\} \text{Coordinate of Q}$$

$$d(P, Q) = \sqrt{(3-1)^2 + (1-1)^2 + (4-5)^2} = \sqrt{5}$$

<u>Equations for planes:</u>

If we have a plane through $P_\circ(x_\circ, y_\circ, z_\circ)$ and perpendicular to the vector

$N = Ai + Bj + Ck$ then the equation of this plane can be written as the following formula:

$[Ax + By + Cz = D]$ where $D = Ax_\circ + By_\circ + Cz_\circ$.

Example: Find an equation for plane through the point $P_\circ(0,2,-1)$ and perpendicular to vector $N = 3i - 2j - k$.

Sol: $D = Ax_\circ + By_\circ + Cz_\circ = 3 \times 0 + (-2) \times (2) + (-1) \times (-1) = -3$

$$A = 3, \quad B = -2, \quad C = -1$$

$Ax + By + Cz = D$, $3x - 2y - z = -3$. equation of the plane

Example: Find an equation for the plane through (1, -1,3) and parallel to the plane $3x + y + z = 7$.

Sol: $N = 3i + j + k$ (A=3, B=1, C=1)

$D = 3 \times 1 + 1 \times -1 + 1 \times 3 = 5$

$3x + y + z = 5.$ equation of the plane

Example: Find an equation for the plane through the points $P_1(1,1,-1)$, $P_2(2,0,2)$, $P_3(0,-2,1)$.

Sol: $P_1P_2 = (2-1)i + (0-1)j + (2-(-1))k = i - j + 3k$

$P_1P_3 = (0-1)i + (-2-1)j + (1-(-1))k = -i - 3j + 2k$

$$N = P_1P_2 \times P_1P_3 = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -3 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} k$$

$$N = 7i - 5j - 4k.$$

$P_1(1,1,-1),\quad A = 7,\quad B = -5,\quad C = -4$

$7x - 5y - 4z = 6.$ equation of the plane

Example: Find an equation for the plane through $P_\circ(2,4,5)$ and perpendicular to the line $\frac{x-5}{1} = \frac{y-1}{3} = \frac{z}{4}$

Sol: $A = 1, B = 3, C = 4$ $N = i + 3j + 4k$

$D = 1 \times 2 + 3 \times 4 + 4 \times 5 = 34$

$x + 3y + 4z = 34.$ equation of the plane

Example: Find a plane through $P_\circ(2,1,-1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.

Sol: eq. of P_1 $2x + y - z = 3 \Rightarrow N_1 = 2i + j - k$

eq. of P_2 $x + 2y + z = 2 \Rightarrow N_2 = i + 2j + k$

$$N = N_1 \times N_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3i - 3j + 3k,$$ with $P_\circ(2,1,-1)$ then the equation of the plane is $3x - 3y + 3z = 0$

Example: Find a plane through $P_1(1,2,3)$ and $P_2(3,2,1)$ and perpendicular to the $4x - y + 2z = 5$.

Sol: $P_1P_2 = (3-1)i + (2-2)j + (1-3)k = 2i - 2k$

$V = 4i - j + 2k$

$$N = P_1P_2 \times V = \begin{vmatrix} i & j & k \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = i + 6j + k$$

$P_1(1,2,3), \quad A = 1, B = 6, C = 1$

$D = 1 \times 1 + 6 \times 2 + 1 \times 3 = 16$

$x + 6y + z = 16$ eq. of the plane

<u>The distance from a point to a plane:</u>

The normal distance from a point P to plane contain a point Q can be calculated as following:

$$d = \left| \frac{PQ \cdot N}{|N|} \right|$$

PQ is the vector from P to Q

N is the vector perpendicular on the plane, $|N|$ length of vector N

Example: Find the distance from the point $P(1,1,1)$ to the plane

$2x - 3y + 7z = 1.$

Sol: $N = 2i - 3j + 7k, |N| = \sqrt{(2)^2 + (-3)^2 + (7)^2} = \sqrt{62}$

If $x = y = 0$ then $2(0) - 3(0) + 7z = 1 \Rightarrow z = \frac{1}{7}$

$\therefore Q(0,0,\frac{1}{7})$

$PQ = (0-1)i + (0-1)j + (\frac{1}{7} - 1)\,k = -i - j - \frac{6}{7}k$

$PQ \cdot N = (-1)(2) + (-1)(-3) + (-\frac{6}{7})\,(7) = -5$

$$d = \left|\frac{PQ \cdot N}{|N|}\right| = \left|\frac{-5}{\sqrt{62}}\right| = \frac{5}{\sqrt{62}}.$$

Angles between planes:

The angle (θ) between intersecting planes can be calculated by the following formula:

$$[\theta = \cos^{-1} \frac{N_1 \cdot N_2}{|N_1|\;|N_2|}]$$

Where: N_1, N_2 are the vectors perpendicular on intersecting planes.

$|N_1|, |N_2|$ are the length of vectors N_1 and N_2.

Example: Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Sol: $N_1 = 3i - 6j - 2k, \quad N_2 = 2i + j - 2k$

$|N_1| = \sqrt{3^2 + (-6)^2 + (-2)^2} = \sqrt{49} = 7$

$|N_2| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$

$N_1 \cdot N_2 = (3 \times 2) + (-6 \times 1) + (-2 \times -2) = 4$

$$\theta = \cos^{-1}(\frac{4}{7 \times 3}) = \cos^{-1}\frac{4}{21} = 79°.$$

<u>Home Work</u>

1. Find an equation for the plane through the points $P_1(0,0,1), P_2(2,0,0)$ and $P_3(0,3,0)$.

2. Find the point in which the line $x = \frac{8}{3} + 2t, y = -2t, z = 1 + t$ meets the plane $3x + 2y + 6z - 6 = 0$.

3. Find the distance from the point $P(1,1,3)$ to the plane $3x + 2y + 6z = 6$.

4. Find a vector parallel to the line of intersection of the planes

$3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

5. Find the point in which the line $x = 1 + 2t, y = -1 - t, z = 3t$ meets the three coordinate-planes.

6. Find an equation of the plane through the point (2, -7,6) which is parallel to the plane $5x - 2y + z - 9 = 0$.

7. Show that the line $x - 4 = 2t, y = -t, z + 1 = -4t$ is parallel to the plane $3x + 2y + z = 7$.

8. The line $L_1: \frac{1-2x}{4} = \frac{2y-1}{6} = 0.5z$ intersect the plane

$x + 3y - z + 3 = 0$, find the point of intersection P and the equation of the line L_2 in this plane passes through P and $\perp$ to L_1.