University of Mustansiriyah

College of Engineering

Department of Highway and Transportation Engineering

Second Class (2^{nd} Class)

Mathematics

(2025 – 2026)

Partial Derivatives

Partial derivative:

The partial derivative of $f(x,y)$ w.r.t. its independent variables x and y given by:

1. $\frac{\partial f}{\partial x}=f_x=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$

The calculating by diff. $f(x,y)$ w.r.t. y and dealing y as a constant.

2. $\frac{\partial f}{\partial y}=f_y=\lim\limits_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$

The calculating by diff. $f(x,y)$ w.r.t. x and dealing x as a constant.

Example: Find $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ for $f(x,y)=x^2y+\sin(xy)$.

$$\frac{\partial f}{\partial x}=f_x=2xy+y\cos(xy)$$

$$\frac{\partial f}{\partial y}=f_y=x^2+x\cos(xy)$$

Example: Find $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ for $f(x,y)=3x^3+5x^2y-2y^2$.

$f_x=9x^2+10xy$

$f_y=5x^2-4y$

Example: Find $\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}$ for $w=\sqrt{e^{x^2}+e^{xy}}$

$$\frac{\partial w}{\partial x}=\frac{1}{2}\left(e^{x^2}+e^{xy}\right)^{-\frac{1}{2}}\cdot\left(e^{x^2}\cdot 2x+e^{xy}\cdot y\right)=\frac{\left(2x\,e^{x^2}+y\,e^{xy}\right)}{2\sqrt{e^{x^2}+e^{xy}}}$$

$$\frac{\partial w}{\partial y}=\frac{1}{2}\left(e^{x^2}+e^{xy}\right)^{-\frac{1}{2}}\cdot e^{xy}\cdot x=\frac{x\,e^{xy}}{2\sqrt{e^{x^2}+e^{xy}}}$$

Example: Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for $f(x, y) = \frac{\tan(x^3+4y)}{x^2y-6y^2}$

$$\frac{\partial f}{\partial x} = \frac{(x^2y - 6y^2) \cdot \sec^2(x^3 + 4y) \cdot 3x^2 - \tan(x^3 + 4y) \cdot 2xy}{(x^2y - 6y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2y - 6y^2) \cdot \sec^2(x^3 + 4y) \cdot 4 - \tan(x^3 + 4y) \cdot (x^2 - 12y)}{(x^2y - 6y^2)^2}$$

Example: Find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$ for $w = \sin(x^2 + 3y) \cdot (z^3 + 4xy)^2$

$$\frac{\partial w}{\partial x} = \sin(x^2 + 3y) \cdot 2(z^3 + 4xy) \cdot 4y + (z^3 + 4xy)^2 \cdot \cos(x^2 + 3y) \cdot 2x$$

$$\frac{\partial w}{\partial y} = \sin(x^2 + 3y) \cdot 2(z^3 + 4xy) \cdot 4x + (z^3 + 4xy)^2 \cdot \cos(x^2 + 3y) \cdot 3$$

$$\frac{\partial w}{\partial z} = \sin(x^2 + 3y) \cdot 2(z^3 + 4xy) \cdot 3z^2 + (z^3 + 4xy) \cdot \boldsymbol{zero}$$

Chain Rule:

1. If $w = f(x, y)$ and $x = g(t), y = h(t)$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

Example: Let $w = f(x, y) = x^2 y,\ x = \cos t\ , y = \sin t$, find $\frac{dw}{dt}$ at $t = \frac{\pi}{4}$.

$f_x = 2xy$, $\frac{dx}{dt} = -\sin t$

$f_y = x^2$, $\frac{dy}{dt} = \cos t$

$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= (2xy)(-\sin t) + (x^2)(\cos t)$$

$$= (2\cos t \cdot \sin t)(-\sin t) + (\cos^2 t)(\cos t)$$

At $t = \frac{\pi}{4}$

$$\frac{dw}{dt} = -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) = \frac{-1}{2\sqrt{2}}.$$

Example: Find $\frac{dw}{dt}$ for $w = x^2 + 2xy,\ x = 2t^2,\ y = 3t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x + 2y)\ 4t + 2x \cdot 3$$

$$= (2(2t^2) + 2(3t)) \cdot 4t + 2(2t^2) \cdot 3$$

$$= (4t^2 + 6t) \cdot 4t + 12t^2 = 16t^3 + 24t^2 + 12t^2 = 16t^3 + 36t^2$$

2. If $w = f(x, y, z)$ and $x = g(s, t),\ y = h(s, t),\ z = k(s, t)$

Then:

$$\frac{\partial w}{\partial s} = f_x \cdot x_s + f_y \cdot y_s + f_z \cdot z_s$$
$$= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = f_x \cdot x_t + f_y \cdot y_t + f_z \cdot z_t$$
$$= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

Example: Find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ of $f(x, y, z) = x + 2y + z^2$

$$x = \frac{t}{s}, \qquad y = t^2 + \ln s, \qquad z = 2t$$

$$\frac{\partial f}{\partial t} = f_x \cdot x_t + f_y \cdot y_t + f_z \cdot z_t$$
$$= (1)\left(\frac{1}{s}\right) + (2)(2t) + (2z)(2) = \frac{1}{s} + 12t$$

$$\frac{\partial f}{\partial s} = f_x \cdot x_s + f_y \cdot y_s + f_z \cdot z_s$$
$$= (1)\left(\frac{-t}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{-t}{s^2} + \frac{2}{s}$$

Example: Let $w = \cos(u\,v)$, $u = xyz$, $v = \frac{\pi}{4(x^2+y^2)}$, find $\frac{\partial w}{\partial x}$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{x}$$

$$\frac{\partial w}{\partial u} = -v\,\sin(u\,v)$$

$$\frac{\partial w}{\partial v} = -u\,\sin(u\,v)$$

$$\frac{\partial u}{\partial x} = yz\,, \qquad \frac{\partial v}{\partial x} = \frac{-\pi(2x)}{4(x^2+y^2)^2} = \frac{-\pi x}{[2(x^2+y^2)^2]}$$

$$\frac{\partial w}{\partial x} = -v\ \sin(u\,v)\cdot yz + u\ \sin(u\,v)\cdot \frac{\pi x}{2(x^2+y^2)^2}$$

Example: If $w = f(u,v),\ u = x + y,\ v = x - y$ prove that $\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} = (\frac{\partial w}{\partial u})^2 - (\frac{\partial w}{\partial v})^2$.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial w}{\partial u}\cdot 1 + \frac{\partial w}{\partial v}\cdot 1 = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial w}{\partial u}\cdot 1 + \frac{\partial w}{\partial v}\cdot -1 = \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}$$

$$\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} = (\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v})\cdot(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v})$$

$$= (\frac{\partial w}{\partial u})^2 - \frac{\partial w}{\partial u}\cdot\frac{\partial w}{\partial v} + \frac{\partial w}{\partial v}\cdot\frac{\partial w}{\partial u} - (\frac{\partial w}{\partial v})^2$$

$$= (\frac{\partial w}{\partial u})^2 - (\frac{\partial w}{\partial v})^2 .$$

Total differential:

The total differential of $w = f(x,y)$ is given by:

$$dw = df = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy$$

Example: Let $f(x,y) = x^2y + y^2$ and $x = t^2,\ y = 2t + 1$ find df.

$$\frac{\partial f}{\partial x} = 2xy, \qquad \frac{\partial f}{\partial y} = x^2 + 2y$$

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$$

$$df = 2xy\, dx + (x^2 + 2y)\, dy$$

But:

$$dx = 2t\, dt$$

$$dy = 2dt$$

$$df = 2xy(2t\, dt) + (x^2 + 2y)\,(2dt)$$

$$= (10t^4 + 4t^3 + 8t + 4)\, dt\,.$$

Remark: Let $w = f(x,y,z),\ x = g(s,t),\ y = h(s,t),\ z = k(s,t)$ then

$$dw = df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$$

$$dx = \frac{\partial x}{\partial s}\, ds + \frac{\partial x}{\partial t}\, dt$$

$$dy = \frac{\partial y}{\partial s}\, ds + \frac{\partial y}{\partial t}\, dt$$

$$dz = \frac{\partial z}{\partial s}\, ds + \frac{\partial z}{\partial t}\, dt$$

Example: Let $f(x,y,z) = x^2y + y^2 + z^2,\ x = rt^3,\ y = rt,\ z = r^2$ find the total differential of f .

$$df = f_x\, d_x + f_y\, d_y + f_z\, d_z$$

$$= (2xy)\, dx + (x^2 + 2y)\, dy + (2z)\, dz$$

But: $dx = x_r\, d_r + x_t\, d_t = t^3dr + 3rt^2\, dt$

$$dy = y_r\, d_r + y_t\, dt = tdr + rdt$$

$$dz = z_r d_r = 2rdr$$

$$\therefore df = (2rt^3)(rt)\,(t^3dr + 3rt^2dt) + (rt^3)^2 + (2rt)(tdr + rdt)$$
$$+ (2r^2)(2rdr)\ .$$

Homework: Find the total differentiation dw for the function $w = \ln(x^2 + y^2 + 2z)$ where $x = r + s,\ y = r - s,\ z = 2rs$.

<u>Higher order partial derivatives:</u>

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = f_{xy}$$

Example: If $f(x,y) = xy^2 + \sinh(xy)$, find f_{xx} , f_{xy} , f_{yy} , f_{yx} .

$f_x = y^2 + y\ \cosh(xy)$

$f_y = 2xy + x\ \cosh(xy)$

$f_{xx} = 0 + y^2\ \sinh(xy)$

$f_{yy} = 2x + x^2\ \sinh(xy)$

$f_{xy} = 2y + xy\ \sinh(xy) + \cosh(xy)$

$f_{yx} = 2y + xy\ \sinh(xy) + \cosh(xy)$

Example: Find f_{xx} , f_{yy} , f_{xy} , f_{yx} of function $f(x, y) = e^x \sin y + y \cos x$.

$f_x = e^x \sin y - y \sin x$

$f_y = e^x \cos y + \cos x$

$f_{xx} = e^x \sin y - y \cos x$

$f_{xy} = e^x \cos y - \sin x$

$f_{yx} = e^x \cos y - \sin x$

$f_{yy} = - e^x \sin y$

Example: Show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$ if $v = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$

$$\frac{\partial v}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot 2x = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 v}{\partial x^2} = -x \cdot \frac{-3}{2}(x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2x + (x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot -1$$

$$= 3x^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot 2y = -y(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 v}{\partial y^2} = -y \cdot -\frac{3}{2}(x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2y + (x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot -1$$

$$= 3y^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial v}{\partial z} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot 2z = -z(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 v}{\partial z^2} = -z \cdot \frac{-3}{2}(x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2z + (x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot -1$$

$$= 3z^2(x^2 + y^2 + z^2)^{-\frac{5}{2}} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$

$$= 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$
$$+ 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$
$$+ 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$
$$= 3(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot (x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$
$$= 3(x^2+y^2+z^2)^{-\frac{3}{2}} - 3(x^2+y^2+z^2)^{-\frac{3}{2}} = 0$$

Example: If $w = f(x, y)$ and $x = e^t$, $y = 2t - 1$ then find $\frac{d^2w}{dt^2}$

$$\frac{d^2w}{dt^2} = \frac{d}{dt}\left(\frac{dw}{dt}\right)$$

$$\frac{dw}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

$$= e^t f_x + 2f_y$$

$$\frac{d^2w}{dt^2} = \frac{d}{dt}\left(\frac{dw}{dt}\right) = \frac{d}{dt}(e^t f_x + 2f_y) = \frac{d}{dt}(e^t f_x) + \frac{d}{dt}(2f_y)$$
$$= e^t \frac{d}{dt}(f_x) + f_x e^t + 2\frac{d}{dt}(f_y)$$
$$= e^t\left[f_{xx}\frac{dx}{dt} + f_{xy}\frac{dy}{dt}\right] + f_x e^t + 2\left[f_{yx}\frac{dx}{dt} + f_{yy}\frac{dt}{dt}\right]$$
$$= e^t\left[f_{xx} e^t + f_{xy}(2)\right] + f_x e^t + 2\left[f_{yx} e^t + f_{yy}(2)\right]$$

$$\therefore \frac{d^2w}{dt^2} = e^{2t} f_{xx} + 4e^t f_{xy} + 4f_{yy} + e^t f_x .$$

Jacobian determinate:

The condition where the independent variables x and y can be regarded as $x = x(u, v),\ \ y = y(u, v)$ is:

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

$J\left(\frac{x,y}{u,v}\right)$ is called the Jacobian of x, y with respect to u, v

Example: $x = r\ \cos\theta\,,\quad y = r\ \sin\theta$

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\ \sin\theta \\ \sin\theta & r\ \cos\theta \end{vmatrix} = r\ \cos^2\theta + r\ \sin^2\theta = r \neq 0$$

Therefore, we can regard r, θ as a function of x, y .

$$r = \sqrt{x^2 + y^2}\,, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example: Let $x = u^2 + v + 3,\ \ y = 3u^2 + 3v - 10$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 1 \\ 6u & 3 \end{vmatrix} = 6u - 6u = 0$$

Therefore, we can't regard u, v as a function of x, y .

<u>Maximum, minimum and saddle points</u>

To find points of local max. and local min. and saddle point for the function $w = f(x, y)$ we have two cases.

Case (1): If the region over whole $xy -$ plane.

Case (2): If the region R is specified in the $xy -$ plane.

For case (1) to find the points of local max. , local min. and saddle point of $w = f(x, y)$, then:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 0 \ldots\ldots\ldots (1) \\ \frac{\partial f}{\partial y} &= 0 \ldots\ldots\ldots (2) \end{aligned}\right\} \text{ solving equation (1) and (2) to get the critical points}$$

say (a, b) .

1. If $f_{xx} \Big| > 0$ and $\left(f_{xx} f_{yy} - (f_{xy})^2\right)\Big|_{(a,b)} > 0$ then f has local min. at (a, b) .

2. If $f_{xx} \Big| < 0$ and $\left(f_{xx} f_{yy} - (f_{xy})^2\right)\Big|_{(a,b)} > 0$ then f has local max. at (a, b) .

3. If $|(f_{xx} f_{yy} - (f_{xy})^2)|_{(a,b)} < 0$ then f has a saddle point at (a, b) .

and if $\left|\left(f_{xx} f_{yy} - (f_{xy})^2\right)\right|_{(a,b)} = 0$ then this test fails.

Note $f_{xx} f_{yy} - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}$ is called the discriminate D of the function $f(x, y)$.

Example: Find local max., local min. and saddle point of $f(x, y) = x^3 + y^3 - 3xy + 15$.

$$\frac{\partial f}{\partial x} = f_x = 3x^2 - 3y = 0 \dots\dots\dots (1)$$

$$\frac{\partial f}{\partial y} = f_y = 3y^2 - 3x = 0 \dots\dots\dots (2)$$

From $(1) \Rightarrow y = x^2$ and sub. in (2) .

$$y^2 - x = 0 \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0$$

Either $x = 0$, $y = 0$ or $x = 1$, $y = 1$

Then $(0,0), (1,1)$ are critical points

$$f_{xx} = 6x, \qquad f_{yy} = 6y, \qquad f_{xy} = -3$$

$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2 = 36xy - (-3)^2 = 36xy - 9$$

at $(0,0) \Rightarrow D|_{(0,0)} = -9 < 0$

Then f has a saddle point at $(0,0)$

at $(1,1) \Rightarrow D|_{(1,1)} = 36 - 9 = 27 > 0$

$f_{xx}|_{(1,1)} = 6 > 0$

$\therefore f$ has a local min. at $(1,1)$

$w = f(x, y) = x^3 + y^3 - 3xy + 15$

at $(1,1) = (1)^3 + (1)^3 - 3(1)(1) + 15 = 14$

Example: Find local max., local min. and saddle point of

$f(x,y) = x^2 - 4xy + y^2 + 5x - 2y$.

$\frac{\partial f}{\partial x} = 2x - 4y + 5 = 0 \dots\dots\dots (1)$

$\frac{\partial f}{\partial y} = -4x + 2y - 2 = 0 \dots\dots\dots (2)$

$$\left.\begin{array}{r} 4x - 8y + 10 = 0 \\ -4x + 2y - 2 = 0 \end{array}\right\}$$

$-6y + 8 = 0 \Rightarrow y = \frac{8}{6} = \frac{4}{3}$, $\qquad x = \frac{1}{6}$

$\therefore \left(\frac{1}{6}, \frac{4}{3}\right)$ is critical point

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = -4$

$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 4 - (-4)^2 = -12 < 0$

$\therefore f$ has a saddle point at $(\frac{1}{6}, \frac{4}{3})$

Example: Test the following surface for minima, maxima and saddle points

$z = x^2 + xy + 3x + 2y + 5$.

$z_x = 2x + y + 3 = 0$

$z_y = x + 2 = 0 \Rightarrow x = -2$

$2(-2) + y + 3 = 0 \Rightarrow y = 1 \Rightarrow$ point $(-2,1)$

$z_{xx} = 2, \qquad z_{yy} = 0, \qquad z_{xy} = 1$

$z_{xx} \cdot z_{xx} - {z_{xy}}^2 = (2)(0) - 1^2 = -1 < 0$ saddle point.

<u>**Homework:**</u> Locate all relative extrema and saddle points of

$f(x,y) = 4xy - x^4 - y^4$.

Case (2): in this case the region R in $xy -$ plane is specified to find absolute max. and absolute min. , the solution is divided into two steps.

Step (1): testing the point inside the region R this is done a similar manner as in case (1) keeping in mind the point obtained must be inside R, otherwise any point outside R is ignored.

Step (2): testing the points on the boundary of R by using the equation specifying the bounding and substituting its inside the equation $f(x, y)$, then maxim. or mini. is done again any point not a bounding must be ignored.

Example: Find the absolute maximum and minimum of the function $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0$, $y = 0$, $y + 2x = 2$ in the first quadrant.

$f(x, y) = x^2 + y^2$

a. Interior point

$f_x = 2x = 0 \Rightarrow x = 0$

$f_y = 2y = 0 \Rightarrow y = 0$

$y + 2x = 2$

$x = 0 \Rightarrow y = 2$

$y = 0 \Rightarrow x = 1$

Point $(0,0)$ $f = 0$

$f_{xx} = 2, \qquad f_{yy} = 2, \qquad f_{xy} = 0$

$f_{xx} f_{yy} - (f_{xy})^2 = (2)(2) - 0^2 = 4 > 0$

$f_{xx} = 2 > 0$ minima.

b. At boundary points

1. $x = 0 \Rightarrow f = x^2 + y^2 = 0^2 + y^2 \Rightarrow f = y^2$

$f_y = 2y = 0 \Rightarrow y = 0$ $\qquad \therefore f = 0^2 + 0^2 = 0$

at $y = 2 \Rightarrow f = y^2 = 2^2 = 4$

2. $y = 0 \Rightarrow f = x^2 + 0^2 \Rightarrow f = x^2$

$f_x = 2x = 0$ $\therefore x = 0$ $f = 0^2 + 0^2 = 0$

at $x = 1 \Rightarrow f = x^2 = 1^2 = 1$

3. at $y + 2x = 2 \Rightarrow y = 2 - 2x$

$f = x^2 + (2 - 2x^2) - x^2 + 4 - 8x + 4x^2$

$f = 5x^2 - 8x + 4$

$f_x = 10x - 8 = 0 \Rightarrow x = \frac{8}{10} = \frac{4}{5}$ $\therefore y = 2 - 2\ (\frac{4}{5}) = \frac{2}{5}$

at $p\ (\frac{4}{5}, \frac{2}{5})$, $f = (\frac{4}{5})^2 + (\frac{2}{5})^2 = \frac{20}{25} = \frac{4}{5}$.

$f = 0,4,1,\frac{4}{5}$

Minima $f = 0$ at $p(0,0)$

Maxima $f = 4$ at $p(0,2)$

Example: Find the absolute maxima and minima of the function $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \le x \le 5, -3 \le y \le 0$.

a. Interior point

$T_x = 2x + y - 6 = 0$
$T_y = x + 2y = 0$

Subtraction

$0 - 3y - 6 = 0$
$-3y = 6 \Rightarrow y = -2 \Rightarrow x = 4$

$T = (4)^2 + (4)(-2) + (-2)^2 - 6(4) = -12$.
$T_{xx} = 2,\quad T_{yy} = 2,\quad T_{xy} = 1$
$(2)(2) - 1^2 = 3 > 0,\quad T_{xx} = 2 > 0$
$\therefore$ the point $(4, -2)$, $T = -12$ is minima

b. Boundary points

1. $y = 0 \Rightarrow T = x^2 - 6x$

   $T_x = 2x - 6 = 0 \Rightarrow x = 3 \Rightarrow T = (3)^2 - 6(3) = -9$

   at $x = 5 \Rightarrow T = (5)^2 - 6(5) = -5$

   at $x = 0 \Rightarrow T = (0)^2 - 6(0) = 0$

2. $y = -3 \Rightarrow T = x^2 - 9x + 9$

   $T_x = 2x - 9 = 0 \Rightarrow x = 4.5$

   $\therefore T = (4.5)^2 - 9(4.5) + 9 = -11.25$

   at $x = 5 \Rightarrow T = (5)^2 - 9(5) + 9 = -11$

   at $x = 0 \Rightarrow T = (0)^2 - 9(0) + 9 = 9$

3. $x = 0 \Rightarrow T = y^2$

   $T_y = 2y = 0 \qquad \therefore y = 0 \Rightarrow T = (0)^2 = 0$

4. $x = 5 \Rightarrow T = y^2 + 5y - 5$

   $T_y = 2y + 5 = 0 \Rightarrow y = -2.5$

   $\therefore T = (-2.5)^2 + 5(-2.5) - 5 = -11.25$

   $\therefore T = -12, -9, -5, 0, -11.25, -11, 9$

   $\therefore T = -12$ at $p(4, -2)$ is minima

   $T = 9$ at $p(0, -3)$ is maxima

Gradient Vector

Let $f(x, y, z)$ be a scalar function, then the gradient of this scalar function is denoted and given by:

$$\nabla f = grad\ f = f_x\, \hat{i} + f_y\, \hat{j} + f_z\, \hat{k} \quad \text{where} \quad \hat{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Example: The temperature at any point (x, y, z) is given by $T(x, y, z) = xye^{-3z}$, find the gradient of T at $p(2,1,0)$.

$$\left.\begin{aligned} T_x &= ye^{-3z} \\ T_y &= xe^{-3z} \\ T_z &= -3xye^{-3z} \end{aligned}\right\} \text{ at } p(2,1,0) \qquad \begin{aligned} T_x &= 1 \\ T_y &= 2 \\ T_z &= -6 \end{aligned}$$

$$\nabla T = T_x\, i + T_y\, j + T_z\, k$$

$$\nabla T|_p = i + 2j - 6k$$

Example: Show that $\nabla(fg) = f\nabla g + g\nabla f$ where f, g are scalar functions.

$$\nabla(fg) = \frac{\partial}{\partial x}(fg)i + \frac{\partial}{\partial y}(fg)j + \frac{\partial}{\partial z}(fg)k \,.$$

$$= (f_x g + g_x f)i + (f_y g + g_y f)j + (f_z g + g_z f)k$$

$$= f(g_x i + g_y j + g_z k) + g(f_x i + f_y j + f_z k)$$

$$\therefore\ \nabla(fg) = f\nabla g + g\nabla f \,.$$

<u>Divergency of a vector field</u>

Let $F(x, y, z)$ be a vector field, such that:

$$F(x, y, z) = M(x, y, z)i + N(x, y, z)j + P(x, y, z)k \,.$$

Then the divergence of F denoted by

$\nabla \cdot F = (\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k) \cdot (Mi + Nj + Pk) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$(scalar)

Then $\text{div}(F) = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

Homework: Let $F = (xyz)i + \cos(xyz)\, j + (xy^2z^3)k$, then calculate $\text{div}(F)$

Directional derivative

The directional derivative of the function $w = f(x,y)$ at $P_\circ(x_\circ, y_\circ)$ towards $P_1(x_1, y_1)$ is given by:

$[Du\, f = (\nabla f)_{P_\circ} \cdot u]$, where u is a unit vector

$u = \frac{P_\circ P_1}{|P_\circ P_1|}$, and $(\nabla f)_{P_\circ}$ is grad f at $P_\circ$.

Example: Find the directional derivative of $f(x,y) = x^2y + y^3$ at $P_\circ(-1,-1)$ in the direction of $P_1(1,1)$.

$$\left.\begin{matrix} f_x = 2xy \\ f_y = x^2 + 3y^2 \end{matrix}\right\} \text{ at } P_\circ(-1,-1) \Rightarrow \begin{matrix} f_x|_{P_\circ} = 2 \\ f_y|_{P_\circ} = 4 \end{matrix}$$

$\nabla f|_{P_\circ} = 2i + 4j$.

$P_\circ P_1 = 2i + 2j$, $\qquad |P_\circ P_1| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$$u = \frac{P_\circ P_1}{|P_\circ P_1|} = \frac{1}{2\sqrt{2}}(2i + 2j) = \frac{1}{\sqrt{2}}(i + j)$$

$$(Du\, f)_{P_\circ} = \nabla f|_{P_\circ} \cdot u = (2i + 4j) \cdot \frac{1}{\sqrt{2}}(i + j) = \frac{1}{\sqrt{2}}(2 + 4) = \frac{6}{\sqrt{2}} \quad .$$

Example: Find the directional derivative of the function $f(x,y,z) = x^3y + xyz$ at $P_\circ(-1,1,1)$ in the direction of the vector $A = 2i - 3j + 6k$.

$$\left.\begin{matrix} f_x = 3x^2y + yz^2 \\ f_y = x^3 + xz^2 \\ f_z = 2xyz \end{matrix}\right\} \text{ at } P_\circ(-1,1,1) \Rightarrow \begin{matrix} f_x|_{P_\circ} = 4 \\ f_y|_{P_\circ} = -2 \\ f_z|_{P_\circ} = -2 \end{matrix}$$

$(\nabla f)_{P_\circ} = 4i - 2j - 2k$

$A = 2i - 3j + 6k$

$|A| = \sqrt{4 + 9 + 36} = 7$

$$u = \frac{A}{|A|} = \frac{1}{7}(2i - 3j + 6k)$$

$$(Du\, f)_{P_\circ} = (\nabla f)_{P_\circ} \cdot u = (4i - 2j - 2k) \cdot \frac{1}{7}(2i - 3j + 6k) = \frac{1}{7}(8 + 6 - 12)$$
$$= \frac{2}{7} .$$

Example: The directional derivative of $w = f(x, y)$ at $P_\circ(1,2)$ in the direction of $P_1(2,3)$ is $2\sqrt{2}$, the directional derivative of f at $P_\circ$ in the direction of $P_2(1,0)$ is -3, find the directional derivative of f at $P_\circ$ in the direction of the origin.

$$u_1 = \frac{\overline{P_\circ P_1}}{|P_\circ P_1|} = \frac{i + j}{\sqrt{2}}$$

$$u_1 = \frac{1}{\sqrt{2}}(i + j)$$

$$(Du_1\, f)_{P_\circ} = (\nabla f)_{P_\circ} \cdot u_1$$

$$2\sqrt{2} = (f_x\, i + f_y\, j) \cdot \frac{1}{\sqrt{2}}(i + j)$$

$$4 = f_x + f_y \ldots\ldots\ldots\ldots\ldots (1)$$

$$u_2 = \frac{P_\circ P_2}{|P_\circ P_2|} = \frac{-2j}{\sqrt{4}} = -j$$

$$(Du_2\, f)_{P_\circ} = (\nabla f)_{P_\circ} \cdot u_2$$

$$-3 = (f_x\, i + f_y\, j) \cdot (-j)$$

$$-3 = -f_y \Rightarrow f_y = 3 \ldots\ldots\ldots (2)$$

Sub. in …… (1) we get $f_x = 1$

$$\therefore (\nabla f)_{P_\circ} = i + 3j$$

$$u_3 = \frac{P_\circ P_3}{|P_\circ P_3|} = \frac{-i - 2j}{\sqrt{5}} = \frac{1}{\sqrt{5}}(-i - 2j)$$

$$(Du_3\, f)_{P_\circ} = (\nabla f)_{P_\circ} \cdot u_3 = (i + 3j) \cdot \frac{1}{\sqrt{5}}(-i - 2j) = \frac{1}{\sqrt{5}}(-1 - 6) = \frac{-7}{\sqrt{5}}$$

is the directional derivative at $P_\circ$ in the direction of the origin.

Method of Lagrange multipliers

Suppose that $f(x,y,z)$ and $g(x,y,z)$ and their first partial derivatives are continuous. To find the local maximum and minimum values of f subject to the constraint $g(x,y,z)=0$, find the values of x,y,z and λ that satisfy the equations.

$$\nabla f = \lambda \, \nabla g \quad \text{and} \quad g(x,y,z) = 0$$

Where $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$ and λ is a variable called Lagrange multipliers

Example: Find the greatest and smallest values that function $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$, using Lagrange multipliers.

$\nabla f = \lambda \nabla g$ we find x, y and λ

$g(x,y) = 0$

$$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j = \lambda \left(\frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j\right)$$

$$yi + xj = \lambda \left(\frac{x}{4} i + yj\right)$$

$$\left.\begin{aligned} y &= \lambda \frac{x}{4} \\ x &= \lambda y \end{aligned}\right\} \quad \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$y = \lambda \left(\frac{\lambda y}{4}\right) = \lambda^2 \frac{y}{4} \Rightarrow y = \lambda^2 \frac{y}{4} \Rightarrow y - \lambda^2 \frac{y}{4} = 0 \Rightarrow y\left(1 - \frac{\lambda^2}{4}\right) = 0$$

$$\Rightarrow y\left(\frac{4-\lambda^2}{4}\right) = 0$$

Either $y = 0$ or $\lambda = \pm 2$

Case (1): if $y = 0 \Rightarrow x = 0$ but the point $(0,0)$ is not on the ellipse $0 + 0 \neq 1$ hence $y \neq 0$.

Case (2): if $y \neq 0$ then $\lambda = \pm 2$
sub. $x = \lambda y$
$x = \pm 2y$ and sub. in $g(x,y) = 0$

$$\frac{(2y)^2}{8} + \frac{y^2}{2} - 1 = 0 \Rightarrow \frac{4y^2}{8} + \frac{y^2}{2} - 1 = 0 \Rightarrow \frac{4y^2 + 4y^2}{8} = 1$$

$4y^2 + 4y^2 = 8 \Rightarrow 8y^2 = 8 \Rightarrow y = \pm 1$ at $y = 1 \Rightarrow x = \pm 2$

$y = -1 \Rightarrow x = \pm 2$

The function $f(x,y) = xy$ take extreme values on the ellipse at the points $(2,1), (-2,1), (2,-1), (-2,-1)$ and the extreme values are:

$xy = 2, \qquad xy = -2$

Max. at $(2,1), (-2,-1)$ and min. at $(-2,1), (2,-1)$.

Example: Use Lagrange multipliers to find maximum and minimum value of $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$.

$\nabla f = \lambda \nabla g \, , g(x,y) = 0$

$yi + xj = \lambda\,(2xi + 2yj)$

$yi + xj = 2\lambda\, xi + 2\lambda\, yj$

$y = 2\lambda\, x \,, \qquad x = 2\lambda\, y$

$\therefore\; x = 2\lambda\,(2\lambda\, x) = 4\lambda^2\, x$

$\Rightarrow x - 4\lambda^2\, x = 0$

$x(1 - 4\lambda^2) = 0$

Either $x = 0$ or $1 - 4\lambda^2 = 0$

If $x = 0 \Rightarrow y = 0$ but $0 + 0 \neq 1$ then $x \neq 0$

$1 - 4\lambda^2 = 0 \Rightarrow 4\lambda^2 = 1 \Rightarrow \lambda = \pm \frac{1}{2}$

$x = \pm\, y$ sub. In $g(x,y) = 0$

$y^2 + y^2 - 1 = 0 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm \frac{1}{\sqrt{2}}$ and $x = \pm \frac{1}{\sqrt{2}}$

$$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right),\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$$

$$\therefore f(x,y) = xy = \left(\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\cdot-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \text{ max.}$$

$$f(x,y) = xy = \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} \text{ min.}$$

Homework: Use the Lagrange multipliers to find maximum and minimum values of $f(x,y) = xy$ subject to the constraint $g(x,y) = 4x^2 + 8y^2 = 16$.

Tangent plane and normal line to a level surface $f(x,y,z) = C$

The tangent plane to the surface $f(x,y,z) = C$ at $P_\circ(x_\circ, y_\circ, z_\circ)$ where normal vector to this tangent plane $N = \nabla f|_{P_\circ}$ given by:

$$f_x|_{P_\circ}(x - x_\circ) + f_y|_{P_\circ}(y - y_\circ) + f_z|_{P_\circ}(z - z_\circ) = 0$$

the equation of the normal line given by:

$$x = x_\circ + f_x|_{P_\circ}\, t$$

$$y = y_\circ + f_y|_{P_\circ}\, t \qquad \text{or} \qquad \frac{x-x_\circ}{f_x|_{P_\circ}} = \frac{y-y_\circ}{f_y|_{P_\circ}} = \frac{z-z_\circ}{f_z|_{P_\circ}}$$

$$z = z_\circ + f_z|_{P_\circ}\, t$$

Example: Find the tangent plane and normal line to the surface $f(x,y,z) = x^2 + 2xy - z^2 = 0$ at the point $P_\circ(1,1,\sqrt{3})$.

$$\left.\begin{aligned} f_x &= 2x + 2y \\ f_y &= 2x \\ f_z &= -2z \end{aligned}\right\} \text{ at } (1,1,\sqrt{3}) \Rightarrow \qquad \begin{aligned} f_x &= 4 \\ f_y &= 2 \\ f_z &= -2\sqrt{3} \end{aligned}$$

$\nabla f|_{P_\circ} = 4i + 2j - 2\sqrt{3}\,k$, the tangent plane

$$4(x-1) + 2(y-1) - 2\sqrt{3}(z - \sqrt{3}) = 0$$

The normal line $\frac{x-1}{4} = \frac{y-1}{2} = \frac{z-\sqrt{3}}{-2\sqrt{3}} = t$.

Home Work

1. Given that $z = e^{xy}$, $x = 2u + v$, $y = \frac{u}{v}$, find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ (using chain rule).

2. At what rate is the area of a rectangle changing if its length is 15 ft. and increasing at 3 ft/sec. while its width is 6 ft. and increasing at 2 ft/sec.

3. Find equations of the tangent plane and normal line to the surface $z = xy^2$ at the point $(2,1,4)$.

4. Find the absolute maximum and minimum values of $f(x, y) = 3xy - 6x - 3y + 7$ on the closed triangular region R with vertices $(0,0)$, $(3,0)$ and $(0,5)$.

5. Find the dimensions of a rectangle having perimeter P and maximum area.