University of Mustansiriyah

College of Engineering

Department of Highway and Transportation Engineering

Second Class (2^{nd} Class)

Mathematics

(2025 – 2026)

Differential Equations

Differential Equations

Differential equations are the equations which contain derivatives.

Classification of D.E.

1. Ordinary D.E.

Example: a) $\frac{dy}{dx} = e^x + \sin x$ b) $y^{``} - 2y^{`} + y = \cos x$

2. Partial D.E.

Example: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

Def: the order of a D.E. is the order of the highest order derivative which appears in the equation.

1st order ordinary D.E.

It can be written as $M(x,y)dx + N(x,y)dy = 0$

Example: $y^{`} + xy = \sin x$

$$\frac{dy}{dx} + xy = \sin x$$

$$dy + (xy - \sin x)dx = 0 \Rightarrow M(x,y) = xy - \sin x$$

$$N(x,y) = 1$$

1. **Separable variable**

This has the form of $M(x)dx + N(y)dy = 0$

Example: solve the D.E. $x(2y-3)dx + (x^2+1)dy = 0$
divided by $(2y-3)(x^2+1)$

$$\frac{x}{x^2+1}dx + \frac{dy}{2y-3} = 0$$

$$\frac{1}{2}\ln|x^2+1| + \frac{1}{2}\ln|2y-3| = c$$

Example: solve the D.E.

$(x+1)\frac{dy}{dx} = x(y^2+1) \quad \div (x+1)(y^2+1)$

$\frac{dy}{y^2+1} = \frac{xdx}{x+1}$

$\int \frac{dy}{y^2+1} = \int \frac{xdx}{x+1}$

$\tan^{-1} y = \int \frac{x+1-1}{x+1}\, dx$

$\tan^{-1} y = \int dx - \int \frac{dx}{x+1}$

$\tan^{-1} y = x - \ln|x+1| + c$

$y = \tan(x - \ln|x+1| + c)$

Example: solve $\frac{dy}{dx} = e^{x-y} = \frac{e^x}{e^y}$

$\frac{dy}{dx} = e^x \cdot e^{-y}$

$\frac{dy}{e^{-y}} = e^x dx$

$\int e^y dy = \int e^x dx \Rightarrow e^y = e^x + c \Rightarrow y = \ln(e^x + c)$

Example: solve $\ln x \frac{dy}{dx} = \frac{x}{y}$

$\int \frac{\ln x}{x}\, dx = \int \frac{dy}{y}$

$\frac{(\ln x)^2}{2} = \ln y + c$

$(\ln x)^2 = 2\ln y + 2c \Rightarrow (\ln x)^2 = \ln y^2 + 2c \Rightarrow \ln y^2 = (\ln x)^2 + 2c$

$y^2 = e^{(\ln x)^2 + 2c} \Rightarrow y = \sqrt{}$

Example: solve $xe^y dy = -\frac{x^2+1}{y}\, dx$

$\int ye^y dy = \int \frac{-(x^2+1)}{x}\, dx$

$ye^y - \int e^y dy = -\int \left(x + \frac{1}{x}\right)\, dx$

$\Rightarrow ye^y - e^y = -\left(\frac{x^2}{2} + \ln x\right) + c$

Example: solve $x^2 y \frac{dy}{dx} = (1+x)\csc y$

$\frac{y}{\csc y}\ dy = \frac{1+x}{x^2}\ dx$

$\int y \cdot \sin y\ \ dy = \int \left(x^{-2} + \frac{1}{x}\right) dx$

$y(-\cos y) - \int -\cos y dy = \frac{x^{-1}}{-1} + \ln x + c$

$-y\cos y + \sin y = -\frac{1}{x} + \ln x + c$.

2. **Homogeneous Equations**

A first order differential equation is homogeneous if it can be put into the following form $\left[\frac{dy}{dx} = F\left(\frac{y}{x}\right)\right] \dots \dots \dots \dots *$

Assume $v = \frac{y}{x} \Rightarrow y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

e.q ……* becomes $v + x\frac{dv}{dx} = F(v)$

$$v - F(v) = -x\frac{dv}{dx}$$

$$\frac{dx}{x} = -\frac{dv}{v-F(v)}$$

$$\left[\frac{dx}{x} + \frac{dv}{v-F(v)} = 0\right] \text{ separable D.E.}$$

Now, we can solve the equation by integrating with respect to x and v, we can then return to x and y by substituting $v = \frac{y}{x}$.

Example: solve $(x^2 + y^2)\ dx + xy\ dy = 0$

$(x^2 + y^2)\ dx + xy\ dy = 0$

$\frac{x^2+y^2}{xy} + \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\left(\frac{x^2+y^2}{xy}\right)$, assume $v = \frac{y}{x}$

$\frac{dy}{dx} = -\left(\frac{x}{y} + \frac{y}{x}\right)$

$\frac{dy}{dx} = -\left(\frac{1}{v} + v\right)$

$\frac{dy}{dx} = -\left(\frac{1+v^2}{v}\right) \quad = F(v)$

$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0 \Rightarrow \frac{dx}{x} + \frac{dv}{v + \frac{1+v^2}{v}}$

$\frac{dx}{x} + \frac{dv}{\frac{1+2v^2}{v}} = 0$

$\int \frac{dx}{x} + \int \frac{v\,dv}{1+2v^2} = 0$

$\ln|x| + \frac{1}{4}\ln|1 + 2v^2| = c$

$\ln|x| + \frac{1}{4}\ln\left|1 + 2\left(\frac{y}{x}\right)^2\right| = c$

Example: solve $\left(xe^{\frac{y}{x}} + y\right) dx = x\,dy$

$\left(xe^{\frac{y}{x}} + y\right) dx - x\,dy = 0$

$\frac{xe^{\frac{y}{x}} + y}{x} - \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{xe^{\frac{y}{x}}}{x} + \frac{y}{x}$, let $v = \frac{y}{x}$

$F(v) = e^v + v$

$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$

$\frac{dx}{x} + \frac{dv}{v - (e^v + v)} = 0$

$\frac{dx}{x} + \frac{dv}{-e^v} = 0 \Rightarrow \int \frac{dx}{x} + \int -e^{-v}\,dv = 0 \Rightarrow \ln|x| + e^{-v} = c \Rightarrow \ln|x| + e^{-\frac{y}{x}} = c$.

Example: solve $(x + y)dy + (x - y)dx = 0$.

$(x + y)dy + (x - y)dx = 0$

$\frac{dy}{dx} + \frac{(x-y)}{(x+y)} = 0 \Rightarrow \frac{dy}{dx} = -\frac{(x-y)}{(x+y)}$

$\frac{dy}{dx} = -\left(\frac{\frac{x}{x} - \frac{y}{x}}{\frac{x}{x} + \frac{y}{x}}\right) = -\left(\frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}\right)$

let $v = \frac{y}{x} \Rightarrow F(v) = -\left(\frac{1-v}{1+v}\right)$

$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$

$\frac{dx}{x} + \frac{dv}{v + \frac{1-v}{1+v}} = 0$

$$\frac{dx}{x}+\frac{dv}{\frac{v+v^2+1-v}{1+v}}=0$$

$$\frac{dx}{x}+\frac{1+v}{v^2+1}\,dv \Rightarrow \frac{dx}{x}+\frac{dv}{1+v^2}+\frac{v}{1+v^2}\,dv=0$$

$$\int\frac{dx}{x}+\int\frac{dv}{1+v^2}+\int\frac{vdv}{1+v^2}=0$$

$$\ln|x|+\tan^{-1}(v)+\frac{1}{2}\ln|1+v^2|=c \Rightarrow \text{put } v=\frac{y}{x}\,.$$

Example: solve $\frac{dy}{dx}=\frac{y}{x}+\cos\left(\frac{y}{x}-1\right)$

Let $v=\frac{y}{x} \Rightarrow F(v)=v+\cos(v-1)$

$$\frac{dx}{x}+\frac{dv}{v-(v+\cos(v-1))}=0$$

$$\frac{dx}{x}+\frac{dv}{-\cos(v-1)}=0$$

$$\int\frac{dx}{x}-\int\sec(v-1)\,dv=0$$

$$\ln|x|-\ln|\sec(v-1)+\tan(v-1)|=c$$

put $v=\frac{y}{x}$

3. **Exact D.E.**

A D.E. of the form $Mdx+Ndy=0$ is called the exact D.E. if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ and solution is of the form $f(x,y)=c$.

The solution is performed by the following steps:

1. Find the basic formula for solution by $f=\int Mdx=f(x,y)+\emptyset(y)$.
2. Find the value of $\emptyset(y)$ by:

   $$\frac{\partial f}{\partial y}=f`+\emptyset`(y)=N$$

3. Substitute value of $\emptyset(y)$ in basic formula.

Example: solve $\left(\frac{1}{x^2}+\frac{2y}{x^3}\right)dx-\frac{1}{x^2}dy=0$

$$M=\frac{1}{x^2}+\frac{2y}{x^3}\,,\ N=-\frac{1}{x^2}$$

$$\frac{\partial M}{\partial y}=\frac{2}{x^3}\,,\ \frac{\partial N}{\partial x}=\frac{2}{x^3}\quad \text{Exact}$$

$$f(x,y)=\int M\,dx=\int\left(\frac{1}{x^2}+\frac{2y}{x^3}\right)dx$$

$$f(x,y) = \int x^{-2} + 2yx^{-3}\, dx = -\frac{1}{x} - \frac{y}{x^2} + \emptyset(y)$$
$$\frac{\partial f}{\partial y} = -\frac{1}{x^2} + \emptyset'(y) = -\frac{1}{x^2}$$
$$\emptyset'(y) = 0 \Rightarrow \emptyset(y) = c$$
$$\therefore f(x,y) = -\frac{1}{x} - \frac{y}{x^2} + c$$

Example: solve $(x^2 + y^2)dx + (2xy + \cos y)dy = 0$

$$M = x^2 + y^2\ ,\ N = 2xy + \cos y$$
$\frac{\partial M}{\partial y} = 2y\ ,\ \frac{\partial N}{\partial x} = 2y$ Exact
$$f(x,y) = \int M\, dx = \int x^2 + y^2\, dx$$
$$f(x,y) = \frac{x^3}{3} + y^2x + \emptyset(y)$$
$$\frac{\partial f}{\partial y} = 2xy + \emptyset'(y) = 2xy + \cos y$$
$$\emptyset'(y) = \cos y \Rightarrow \emptyset(y) = \int \cos y\ dy = \sin y + c$$
$$\therefore f(x,y) = \frac{x^3}{3} + y^2x + \sin y + c$$

Example: solve $(x + y)\, dx + (x + y^2)\, dy = 0$

$$M = x + y\ ,\ N = x + y^2$$
$\frac{\partial M}{\partial y} = 1\ ,\ \frac{\partial N}{\partial x} = 1$ Exact
$$f(x,y) = \int M\, dx = \int (x + y)\, dx = \frac{x^2}{2} + yx + \emptyset(y)$$
$$\frac{\partial f}{\partial y} = x + \emptyset'(y) = x + y^2$$
$$\emptyset'(y) = y^2 \Rightarrow \emptyset(y) = \int y^2\, dy = \frac{y^3}{3} + c$$
$$\therefore f(x,y) = \frac{x^2}{2} + yx + \frac{y^3}{3} + c\ .$$

Example: solve $(2xy + y^2)\, dx + (x^2 + 2xy - y)\, dy = 0$

$$M = 2xy + y^2\ ,\ N = x^2 + 2xy - y$$
$\frac{\partial M}{\partial y} = 2x + 2y\ ,\ \frac{\partial N}{\partial x} = 2x + 2y$ Exact
$$f(x,y) = \int M\, dx = \int (2xy + y^2)\, dx$$
$$f(x,y) = x^2y + y^2x + \emptyset(y)$$

$$\frac{\partial f}{\partial y} = x^2 + 2xy + \emptyset'(y) = x^2 + 2xy - y$$
$$\emptyset'(y) = -y \Rightarrow \emptyset(y) = \int -y\, dy = -\frac{y^2}{2} + c$$
$$\therefore f(x,y) = x^2 y + y^2 x - \frac{y^2}{2} + c$$

H.W. solve the following D.E.

1. $\left(x + \sqrt{y^2+1}\right) dx - \left(y - \frac{xy}{\sqrt{y^2+1}}\right) dy = 0$

2. $\left(e^x + \ln y + \frac{y}{x}\right) dx + \left(\frac{x}{y} + \ln x + \sin y\right) dy = 0$

Integrating factor

An equation which is not exact can be made exact by multiplying it by a suitable factor, which is called an integrating factor.

Case 1: if $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, then $I.F. = F(x) = e^{\int f(x)\, dx}$

Case 2: if $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, then $I.F. = F(y) = e^{-\int g(y)\, dy}$

Example: solve the D.E. $(2\cos y + 4x^2)\, dx - x \sin y \; dy = 0$

$$\frac{\partial M}{\partial y} = -2\sin y \; , \; \frac{\partial N}{\partial x} = -\sin y$$
$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\sin y + \sin y}{-x \sin y} = \frac{1}{x} = f(x)$$
$$I.F. = F(x) = e^{\int f(x)\, dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$
$$x\, [(2\cos y + 4x^2)\, dx - x\sin y \; dy] = 0$$
$$(2x \cos y + 4x^3)\, dx - x^2 \sin y \; dy = 0$$
$\frac{\partial M}{\partial y} = -2x \sin y \; , \; \frac{\partial M}{\partial x} = -2x \sin y$ exact.

$f(x,y) = \int M\, dx = \int 2x\cos y + 4x^3\, dx = x^2\cos y + x^4 + \emptyset(y)$
$\frac{\partial f}{\partial y} = -x^2 \sin y + \emptyset'(y) = -x^2 \sin y$
$\emptyset'(y) = 0 \Rightarrow \emptyset(y) = c$
Sol: $f(x,y) = x^2\, cos\, y + x^4 + c$

Example: solve the D.E. $(4y + 3x^2)\, dy + 2xy\, dx = 0$

$\frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = 6x$ not exact

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2x-6x}{2xy} = \frac{-2}{y} = g(y)$

$I.F. = F(y) = e^{-\int g(y)\, dy} = e^{2\int \frac{dy}{y}} = e^{2\, \ln y} = e^{\ln y^2} = y^2$
$y^2\, [(4y + 3x^2)\, dy + 2xy\, dx = 0]$
$(4y^3 + 3x^2y^2)\, dy + 2xy^3\, dx = 0$
$\frac{\partial M}{\partial y} = 6xy^2$, $\frac{\partial N}{\partial x} = 6xy^2$ exact
$f(x,y) = \int M\, dx = \int 2xy^3\, dx = x^2y^3 + \emptyset(y)$
$\frac{\partial f}{\partial y} = 3x^2y^2 + \emptyset(y) = 4y^3 + 3x^2y^2$
$\emptyset'(y) = 4y^3 \Rightarrow \emptyset(y) = y^4 + c$
$\therefore f(x,y) = x^2y^3 + y^4 + c$

4. **<u>Linear equations</u>**

A differential equation that can be written in the form $\left[\frac{dy}{dx} + P(x)y = Q(x)\right]$ where P and Q are function of x , is called a linear first order equation. The solution of this type of equations is performed by two steps:

Step 1: Find $I(x) = e^{\int P(x)\, dx}$

Step 2: $I(x) \cdot y = \int I(x) \cdot Q(x)\, dx$

Example: solve the D.E. $\frac{dy}{dx} + \frac{y}{x} = \frac{\sin x}{x}$ linear

$P(x) = \frac{1}{x}$, $Q(x) = \frac{\sin x}{x}$

$I(x) = e^{\int P(x)\,dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

$x \cdot y = \int x \cdot \frac{\sin x}{x}\, dx \Rightarrow xy = -\cos x + c$

Example: solve $x\frac{dy}{dx} - 3y = x^2$

$\frac{dy}{dx} - \frac{3}{x}y = x$, $P(x) = -\frac{3}{x}$

$Q(x) = x$

$I(x) = e^{\int P(x)\,dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3} = \frac{1}{x^3}$

$\frac{1}{x^3} \cdot y = \int \frac{1}{x^3} \cdot x\, dx$

$\frac{1}{x^3}\, y = \int x^{-2}\, dx \Rightarrow \frac{1}{x^3}\, y = \frac{x^{-1}}{-1} + c \Rightarrow \frac{1}{x^3}\, y = -\frac{1}{x} + c \Rightarrow y = -x^2 + cx^3.$

H.W. solve the following D.E.

1. $\frac{dy}{dx} + 2y = e^{-x}$
2. $\cosh x\, dy + (y \sinh x + e^x)\, dx = 0$

<u>Bernoulli's equation:</u>

$y' + P(x)y = Q(x)y^n$

To solve Bernoulli eq. multiply both sides by y^{-n} .

$y^{-n}y' + P(x)y^{1-n} = Q(x)$

let $u = y^{1-n} \Rightarrow u' = 1 - n\, y^{-n} \cdot y' \Rightarrow \frac{u'}{1-n} = y^{-n}y'$

$\frac{u'}{1-n} + P(x)\, u = Q(x) \Rightarrow u' + (1-n)\, P(x)\, u = (1-n)\, Q(x) \quad (\boldsymbol{linear})$

Example: $y' - 5y = -\frac{5}{2}\ xy^3$

$y^{-3} \cdot y' - 5y^{-2} = \frac{-5}{2}\ x$ Let $u = y^{-2}$

$u' = -2y^{-3}y'$

$-\frac{u'}{2} - 5u = -\frac{5}{2}\ x$

$u' + 10\,u = 5x \qquad (linear)$

$I = e^{\int P(x)\,dx} = e^{\int 10\,dx} = e^{10x}$

$e^{10x}\,u = \int e^{10\,x} \cdot 5x\,dx$

$e^{10x} \cdot u = \frac{5x}{10}\ e^{10x} - \frac{5}{100}\ e^{10x} + c$

$e^{10x}\,u = \frac{x}{2} e^{10x} - \frac{1}{20} e^{10x} + c$

$u = \frac{x}{2} - \frac{1}{20} + c\ e^{-10x}$, $u = y^{-2}$

Example: solve $y' + \frac{1}{x}\ y = x^2y^6$

$y^{-6}y' + \frac{1}{x}\ y^{-5} = x^2$

Let $u = y^{-5} \Rightarrow u' = -5y^{-6}y'$

$\left(\frac{u'}{-5} + \frac{1}{x}\ u = x^2\right) * -5$

$u' + \frac{-5}{x}\ u = -5x^2$

$P(x) = \frac{-5}{x}$, $I = e^{\int -\frac{5}{x} dx} = e^{-5\ln x} = e^{\ln x^{-5}} = x^{-5}$

$x^{-5} \cdot u = \int x^{-5} \cdot -5x^2\ dx$

$x^{-5}\ u = \int -5x^{-3}\ dx$

$x^{-5}\ u = \frac{-5}{-2}\ x^{-2} + c$

$u = \frac{5}{2}x^3 + c\ x^5 \qquad [u = y^{-5}]$

2^{nd} order D.E. with a constant coefficient

$y'' + ay' + by = f(x)$, where a and b are constant if $f(x) = 0$, then D.E. is homo.

$y'' + ay' + by = 0 \dots\dots\dots \mathbf{1}$
$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$
$D^2y + a\,Dy + by = 0$
$(D^2 + a\,D + b)\,y = 0$
$m^2 + am + b = 0$ (characteristic equation)
m_1 , m_2 roots of this equation

1. If m_1, m_2 are different real roots, the solution of $\dots\dots\dots \mathbf{1}$
   $y = c_1\,e^{m_1x} + c_2\,e^{m_2x}$
2. If $m_1 = m_2$ real, the solution of $\dots\dots\dots \mathbf{1}$
   $y = (c_1 + c_2x)e^{mx}$
3. If m_1, m_2 are complex conjugate
   $m_1, m_2 = \alpha \pm i\beta$
   then the solution of $\dots\dots\dots \mathbf{1}$
   $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

Example: solve the D.E.

$y'' + 3y' + 2y = 0$
$(D^2 + 3D + 2)\,y = 0$
$m^2 + 3m + 2 = 0$ (ch. Eq.)
$(m+1)\,(m+2) \Rightarrow m_1 = -1$, $m_2 = -2$
then the sol. is $y = c_1\,e^{-x} + c_2\,e^{-2x}$

Example: solve the D.E.

$y'' - 6y' + 9y = 0$
$(D^2 - 6D + 9)\,y = 0$
$m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m_1 = m_2 = 3$ equal real roots

The solution is $y = (c_1 + c_2\,x)\,e^{3x}$.

Example: solve the D.E.

$y'' - 6y' + 10y = 0$

The ch. eq. $m^2 - 6m + 10 = 0$

$m = \frac{6\mp\sqrt{36-40}}{2} = \frac{6\mp\sqrt{-4}}{2} = \frac{6\mp 2\sqrt{-1}}{2} = 3 \mp i\,,\quad \alpha = 3$

$\beta = 1$

The solution is $y = e^{3x}(c_1 \cos x + c_2 \sin x)$

2^{nd} order linear D.E. with a constant coefficient non homo

$y'' + ay' + by = f(x)$

The solution of this D.E. consists of two parts. The 1st part is homo (y_h) and the 2nd part is called the particular (y_p) and the general solution will be
$y = y_h + y_p$.

To find y_p , the following method may be used:

1. **Variation of parameter:**

Let $y_p = u_1 v_1 + u_2 v_2$ where u_1 , u_2 are the solution of homo particular and v_1 , v_2 can be found from the equations.

$u'_1\, v'_1 + u'_2\, v'_2 = f(x)$
$u_1\, v'_1 + u_2\, v'_2 = 0$

Example: solve the D.E. $y'' + 3y' + 2y = e^x$

to find y_h

$y'' + 3y' + 2y = 0$
$m^2 + 3m + 2 = 0$
$(m+2)\,(m+1) = 0 \Rightarrow m_1 = -2\,,\ m_2 = -1$
$y_h = c_1\, e^{-2x} + c_2\, e^{-x}$
$y_p = u_1 v_1 + u_2 v_2$, where $\quad u_1 = e^{-2x}$

$u_2 = e^{-x}$

are solution of homo. Part and v_1 , v_2 can be found from the equation.

$u'_1\, v'_1 + u'_2\, v'_2 = f(x)$
$u_1\, v'_1 + u_2\, v'_2 = 0$
$-2e^{-2x}\, v'_1 - e^{-x} v'_2 = e^{x}$ … … **1**
$e^{-2x} v'_1 + e^{-x} v'_2 = 0$ … … **2**

$-e^{-2x}\, v'_1 = e^{x} \Rightarrow v'_1 = -e^{3x}$

$$v_1 = -\frac{1}{3}\, e^{3x} + k_1$$

$-e^{x} + e^{-x} v'_2 = 0 \Rightarrow v'_2 = e^{2x}$

$$v_2 = \frac{1}{2}\, e^{2x} + k_2$$

$$\therefore\ y_p = e^{-2x}\left(-\frac{1}{3}\, e^{3x} + k_1\right) + e^{-x}\left(\frac{1}{2}\, e^{2x} + k_2\right)$$

the general solution is $y = y_h + y_p$

Example: solve the D.E.

$y'' + 2y' - 3y = 6$
$m^2 + 2m - 3 = 0$
$(m+3)\,(m-1) = 0$
$m_1 = -3$, $m_2 = 1$
$y_h = c_1\, e^{-3x} + c_2\, e^{x}$
$u_1 = e^{-3x}$, $u_2 = e^{x}$
$y_p = u_1\, v_1 + u_2\, v_2$
$u'_1\, v'_1 + u'_2\, v'_2 = f(x)$
$u_1\, v'_1 + u_2\, v'_2 = 0$
$-3e^{-3x}\, v'_1 + e^{x}\, v'_2 = 6$
$e^{-3x}\, v'_1 + e^{x}\, v'_2 = 0$

$-4e^{-3x}\, v'_1 = 6 \Rightarrow v'_1 = -\frac{3}{2}\, e^{3x}$
$-\frac{3}{2} + e^{x} v'_2 = 0 \Rightarrow v'_2 = \frac{3}{2}\, e^{-x}$
$v_1 = \int -\frac{3}{2}\, e^{3x}\, dx = -\frac{1}{2}\, e^{3x} + k_1$
$v_2 = \int \frac{3}{2}\, e^{-x}\, dx = -\frac{3}{2}\, e^{-x} + k_2$
$y_p = e^{-3x}\left(-\frac{1}{2}\, e^{3x} + k_1\right) + e^{x}\left(-\frac{3}{2}\, e^{-x} + k_2\right)$
$y = y_h + y_p$.

H.W. solve the D.E.

$\frac{d}{dx}(y\,y') - [(y')^2 - 4y^2] = y \sec 2x$.

2. <u>**Method of undetermined coefficients**</u>

Let $y'' + ay' + by = f(x)$

To find y_p , then chosen y_p according to the type of $f(x)$ given:

$\boldsymbol{f(x)}$	**chosen** $\boldsymbol{y_p}$
1. e^{ax}	$k\,e^{ax}$
2. $\sin ax$ or $\cos ax$	C $\cos ax + D \sin ax$
3. $ax^2 + bx + c$	$Ax^2 + Bx + C$

Example: solve the D.E. $y'' + 3y' + 2y = 2e^{2x} + 3x + 1$

to find y_h

$m^2 + 3m + 2 = 0$

$(m+2)\,(m+1) = 0 \Rightarrow m_1 = -2\,,\ m_2 = -1$

$y_h = c_1 e^{-2x} + c_2 e^{-x}$

$y_p = k\,e^{2x} + Bx + C$

$y'_p = 2k\,e^{2x} + B$

$y''_p = 4k\,e^{2x}$

$4k\,e^{2x} + 3(2k\,e^{2x} + B) + 2(k\,e^{2x} + Bx + C) = 2\,e^{2x} + 3x + 1$

$4k\,e^{2x} + 6k\,e^{2x} + 3B + 2k\,e^{2x} + 2Bx + 2C = 2\,e^{2x} + 3x + 1$

$12k\,e^{2x} = 2\,e^{2x} \Rightarrow 12k = 2 \Rightarrow k = \frac{1}{6}$

$2Bx = 3x \Rightarrow 2B = 3 \Rightarrow B = \frac{3}{2}$

$3B + 2C = 1 \Rightarrow \frac{9}{2} + 2C = 1 \Rightarrow C = \frac{-7}{4}$

$\therefore y_p = \frac{1}{6}\,e^{2x} + \frac{3}{2}x - \frac{7}{4}$ and $y = y_h + y_p$

Example: $y'' - y' = 2 \sin x$

$m^2 - m = 0 \Rightarrow m(m-1) = 0$
$y_h = c_1 + c_2\, e^x$
$y_p = C \cos x + D \sin x$
$y'_p = -C \sin x + D \cos x$
$y''_p = -C \cos x - D \sin x$

$(-C \cos x - D \sin x) - (-C \sin x + D \cos x) = 2 \sin x$
$-C \cos x - D \sin x + C \sin x - D \cos x = 2 \sin x$

$C - D = 2$
$-C - D = 0$

$-2D = 2 \Rightarrow \ D = -1$

$C = 1$

$y_p = \cos x - \sin x$
$y = y_h + y_p$

H.W. solve the D.E.

1. $y'' - y' = 5\, e^x - \sin 2x$

2. $y'' - 3y' + 2y = 5\, e^x$