



Stephen Prokf'yevich Timoshenko (1878–1972), a Russian-born engineer who emigrated to the United States, was one of the most widely known authors of books in the field of elasticity, strength of materials, and vibrations. He held the chair of mechanics at the university of Michigan and later at Stanford University, and he is regarded as the father of engineering mechanics in the United States. The improved theory he presented in 1921 for the vibration of beams has become known as the Timoshenko beam theory. (Courtesy of *Applied Mechanics Reviews*.)

CHAPTER 8

Continuous Systems

Chapter Outline

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The vibration analysis of continuous systems, which are also known as *distributed systems*, is considered in this chapter. The equations of motion of continuous systems will be partial differential equations. The equations of motion of several continuous systems, including the transverse vibration of a tightly stretched string or cable, longitudinal vibration of a bar, torsional vibration of a shaft or rod, the lateral vibration of beams, and transverse vibration of a membrane are derived by considering the free-body diagram of an infinitesimally small

element of the particular system and applying the Newton's second law of motion. The free-vibration solution of the system is found by assuming harmonic motion and applying the relevant boundary conditions. The solution gives infinite number of natural frequencies and the corresponding mode shapes. The free-vibration displacement of the system is found as a linear superposition of the mode shapes, the constants involved being determined from the known initial conditions of the system. In the case of transverse vibration of a string of infinite length, the traveling-wave solution is presented. In the case of the longitudinal vibration of a bar, the vibration response under an initial force is also found. In the case of the transverse vibration of beams, all the common boundary conditions are summarized and the orthogonality of normal modes is proved. The forced vibration of beams is presented using the mode superposition method. The effect of axial force on the natural frequencies and mode shapes of beams is considered. The *thick beam theory*, also called the *Timoshenko beam theory*, is presented by considering the effects of rotary inertia and shear deformation. The free vibration of rectangular membranes is presented. *Rayleigh's method*, based on Rayleigh's quotient, for finding the approximate fundamental frequencies of continuous systems is outlined. The extension of the method, known as the *Rayleigh-Ritz method*, is outlined for determining approximate values of several frequencies. Finally, MATLAB solutions are presented for the free and forced vibration of typical continuous systems.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

- Derive the equation of motion of a continuous system from the free-body diagram of an infinitesimally small element of the system and Newton's second law.
- Find the natural frequencies and mode shapes of the system using harmonic solution.
- Determine the free-vibration solution using a linear superposition of the mode shapes and the initial conditions.
- Find the free-vibration solutions of string, bar, shaft, beam, and membrane problems.
- Express the vibration of an infinite string in the form of traveling waves.
- Determine the forced-vibration solution of continuous systems using mode superposition method.
- Find the effects of axial force, rotary inertia, and shear deformation on the vibration of beams.
- Apply the Rayleigh and Rayleigh-Ritz methods to find the approximate natural frequencies of continuous systems.
- Use MATLAB to find the natural frequencies, mode shapes, and forced response of continuous systems.

8.1 Introduction

We have so far dealt with discrete systems where mass, damping, and elasticity were assumed to be present only at certain discrete points in the system. In many cases, known as *distributed* or *continuous systems*, it is not possible to identify discrete masses, dampers,

or springs. We must then consider the continuous distribution of the mass, damping, and elasticity and assume that each of the infinite number of points of the system can vibrate. This is why a continuous system is also called a *system of infinite degrees of freedom*.

If a system is modeled as a discrete one, the governing equations are ordinary differential equations, which are relatively easy to solve. On the other hand, if the system is modeled as a continuous one, the governing equations are partial differential equations, which are more difficult. However, the information obtained from a discrete model of a system may not be as accurate as that obtained from a continuous model. The choice between the two models must be made carefully, with due consideration of factors such as the purpose of the analysis, the influence of the analysis on design, and the computational time available.

In this chapter, we shall consider the vibration of simple continuous systems—strings, bars, shafts, beams, and membranes. A more specialized treatment of the vibration of continuous structural elements is given in references [8.1–8.3]. In general, the frequency equation of a continuous system is a transcendental equation that yields an infinite number of natural frequencies and normal modes. This is in contrast to the behavior of discrete systems, which yield a finite number of such frequencies and modes. We need to apply boundary conditions to find the natural frequencies of a continuous system. The question of boundary conditions does not arise in the case of discrete systems except in an indirect way, because the influence coefficients depend on the manner in which the system is supported.

8.2 Transverse Vibration of a String or Cable

8.2.1 Equation of Motion

Consider a tightly stretched elastic string or cable of length l subjected to a transverse force $f(x, t)$ per unit length, as shown in Fig. 8.1(a). The transverse displacement of the string, $w(x, t)$, is assumed to be small. Equilibrium of the forces in the z direction gives (see Fig. 8.1(b)):

The net force acting on an element is equal to the inertia force acting on the element, or

$$(P + dP) \sin(\theta + d\theta) + f dx - P \sin \theta = \rho dx \frac{\partial^2 w}{\partial t^2} \quad (8.1)$$

where P is the tension, ρ is the mass per unit length, and θ is the angle the deflected string makes with the x -axis. For an elemental length dx ,

$$dP = \frac{\partial P}{\partial x} dx \quad (8.2)$$

$$\sin \theta \simeq \tan \theta = \frac{\partial w}{\partial x} \quad (8.3)$$

and

$$\sin(\theta + d\theta) \simeq \tan(\theta + d\theta) = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \quad (8.4)$$

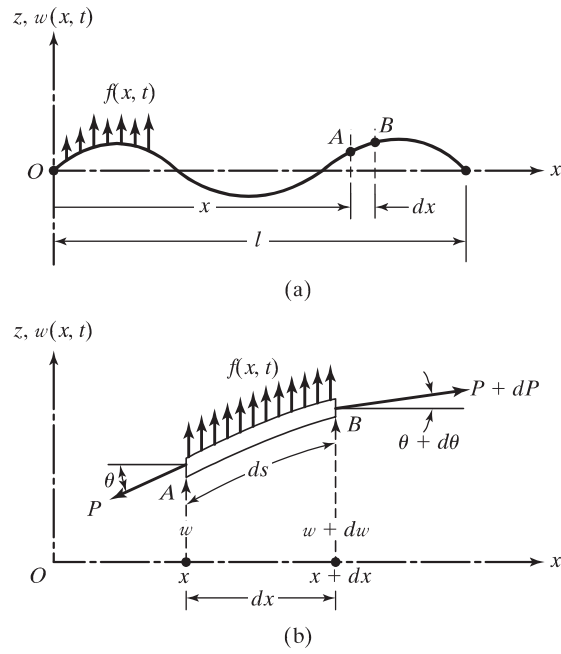


FIGURE 8.1 A vibrating string.

Hence the forced-vibration equation of the nonuniform string, Eq. (8.1), can be simplified to

$$\frac{\partial}{\partial x} \left[P \frac{\partial w(x, t)}{\partial x} \right] + f(x, t) = \rho(x) \frac{\partial^2 w(x, t)}{\partial t^2} \quad (8.5)$$

If the string is uniform and the tension is constant, Eq. (8.5) reduces to

$$P \frac{\partial^2 w(x, t)}{\partial x^2} + f(x, t) = \rho \frac{\partial^2 w(x, t)}{\partial t^2} \quad (8.6)$$

If $f(x, t) = 0$, we obtain the free-vibration equation

$$P \frac{\partial^2 w(x, t)}{\partial x^2} = \rho \frac{\partial^2 w(x, t)}{\partial t^2} \quad (8.7)$$

or

$$c^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2} \quad (8.8)$$

where

$$c = \left(\frac{P}{\rho} \right)^{1/2} \quad (8.9)$$

Equation (8.8) is also known as the *wave equation*.

8.2.2 Initial and Boundary Conditions

The equation of motion, Eq. (8.5) or its special forms (8.6) and (8.7), is a partial differential equation of the second order. Since the order of the highest derivative of w with respect to x and t in this equation is two, we need to specify two boundary and two initial conditions in finding the solution $w(x, t)$. If the string has a known deflection $w_0(x)$ and velocity $\dot{w}_0(x)$ at time $t = 0$, the initial conditions are specified as

$$w(x, t = 0) = w_0(x)$$

$$\frac{\partial w}{\partial t}(x, t = 0) = \dot{w}_0(x) \quad (8.10)$$

If the string is fixed at an end, say $x = 0$, the displacement w must always be zero, and so the boundary condition is

$$w(x = 0, t) = 0, \quad t \geq 0 \quad (8.11)$$

If the string or cable is connected to a pin that can move in a perpendicular direction as shown in Fig. 8.2, the end cannot support a transverse force. Hence the boundary condition becomes

$$P(x) \frac{\partial w(x, t)}{\partial x} = 0 \quad (8.12)$$

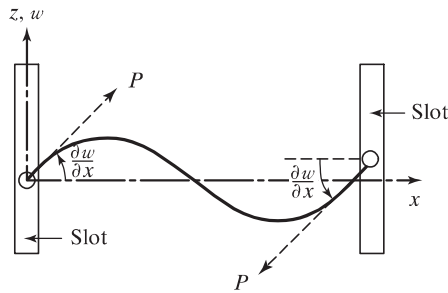


FIGURE 8.2 String connected to pins at the ends.

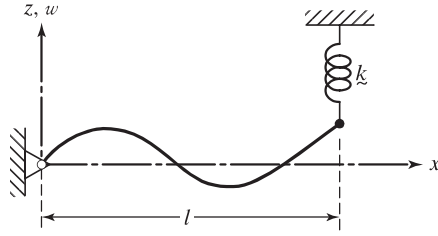


FIGURE 8.3 String with elastic constraint.

If the end $x = 0$ is free and P is a constant, then Eq. (8.12) becomes

$$\frac{\partial w(0, t)}{\partial x} = 0, \quad t \geq 0 \quad (8.13)$$

If the end $x = l$ is constrained elastically as shown in Fig. 8.3, the boundary condition becomes

$$P(x) \frac{\partial w(x, t)}{\partial x} \bigg|_{x=l} = -k w(x, t) \big|_{x=l}, \quad t \geq 0 \quad (8.14)$$

where k is the spring constant.

8.2.3 Free Vibration of a Uniform String

The free-vibration equation, Eq. (8.8), can be solved by the method of separation of variables. In this method, the solution is written as the product of a function $W(x)$ (which depends only on x) and a function $T(t)$ (which depends only on t) [8.4]:

$$w(x, t) = W(x)T(t) \quad (8.15)$$

Substitution of Eq. (8.15) into Eq. (8.8) leads to

$$\frac{c^2}{W} \frac{d^2 W}{dx^2} = \frac{1}{T} \frac{d^2 T}{dt^2} \quad (8.16)$$

Since the left-hand side of this equation depends only on x and the right-hand side depends only on t , their common value must be a constant—say, a —so that

$$\frac{c^2}{W} \frac{d^2 W}{dx^2} = \frac{1}{T} \frac{d^2 T}{dt^2} = a \quad (8.17)$$

The equations implied in Eq. (8.17) can be written as

$$\frac{d^2 W}{dx^2} - \frac{a}{c^2} W = 0 \quad (8.18)$$

$$\frac{d^2T}{dt^2} - aT = 0 \quad (8.19)$$

Since the constant a is generally negative (see Problem 8.9), we can set $a = -\omega^2$ and write Eqs. (8.18) and (8.19) as

$$\frac{d^2W}{dx^2} + \frac{\omega^2}{c^2}W = 0 \quad (8.20)$$

$$\frac{d^2T}{dt^2} + \omega^2T = 0 \quad (8.21)$$

The solutions of these equations are given by

$$W(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \quad (8.22)$$

$$T(t) = C \cos \omega t + D \sin \omega t \quad (8.23)$$

where ω is the frequency of vibration and the constants A , B , C , and D can be evaluated from the boundary and initial conditions.

8.2.4 Free Vibration of a String with Both Ends Fixed

If the string is fixed at both ends, the boundary conditions are $w(0, t) = w(l, t) = 0$ for all time $t \geq 0$. Hence, from Eq. (8.15), we obtain

$$W(0) = 0 \quad (8.24)$$

$$W(l) = 0 \quad (8.25)$$

In order to satisfy Eq. (8.24), A must be zero in Eq. (8.22). Equation (8.25) requires that

$$B \sin \frac{\omega l}{c} = 0 \quad (8.26)$$

Since B cannot be zero for a nontrivial solution, we have

$$\sin \frac{\omega l}{c} = 0 \quad (8.27)$$

Equation (8.27) is called the *frequency or characteristic equation* and is satisfied by several values of ω . The values of ω are called the *eigenvalues* (or *natural frequencies* or *characteristic values*) of the problem. The n th natural frequency is given by

$$\frac{\omega_n l}{c} = n\pi, \quad n = 1, 2, \dots$$

or

$$\omega_n = \frac{nc\pi}{l}, \quad n = 1, 2, \dots \quad (8.28)$$

The solution $w_n(x, t)$ corresponding to ω_n can be expressed as

$$w_n(x, t) = W_n(x)T_n(t) = \sin \frac{n\pi x}{l} \left[C_n \cos \frac{nc\pi t}{l} + D_n \sin \frac{nc\pi t}{l} \right] \quad (8.29)$$

where C_n and D_n are arbitrary constants. The solution $w_n(x, t)$ is called the *n*th mode of vibration or *n*th harmonic or *n*th normal mode of the string. In this mode, each point of the string vibrates with an amplitude proportional to the value of W_n at that point, with the circular frequency $\omega_n = (nc\pi)/l$. The function $W_n(x)$ is called the *n*th normal mode, or characteristic function. The first three modes of vibration are shown in Fig. 8.4. The mode corresponding to $n = 1$ is called the *fundamental mode*, and ω_1 is called the *fundamental frequency*. The fundamental period is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2l}{c}$$

The points at which $w_n = 0$ for all times are called *nodes*. Thus the fundamental mode has two nodes, at $x = 0$ and $x = l$; the second mode has three nodes, at $x = 0$, $x = l/2$, and $x = l$; etc.

The general solution of Eq. (8.8), which satisfies the boundary conditions of Eqs. (8.24) and (8.25), is given by the superposition of all $w_n(x, t)$:

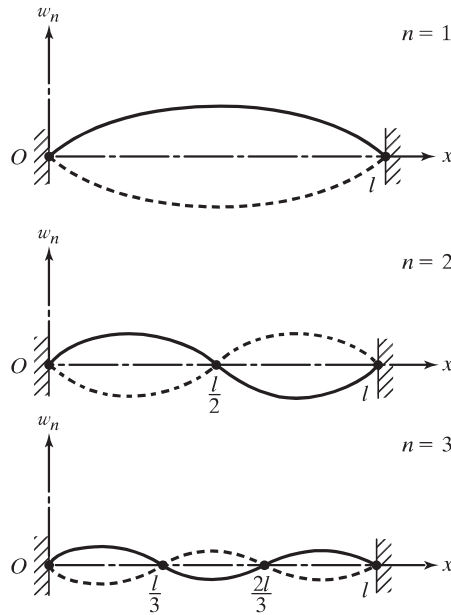


FIGURE 8.4 Mode shapes of a string.

$$\begin{aligned}
 w(x, t) &= \sum_{n=1}^{\infty} w_n(x, t) \\
 &= \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left[C_n \cos \frac{nc\pi t}{l} + D_n \sin \frac{nc\pi t}{l} \right] \quad (8.30)
 \end{aligned}$$

This equation gives all possible vibrations of the string; the particular vibration that occurs is uniquely determined by the specified initial conditions. The initial conditions give unique values of the constants C_n and D_n . If the initial conditions are specified as in Eq. (8.10), we obtain

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = w_0(x) \quad (8.31)$$

$$\sum_{n=1}^{\infty} \frac{nc\pi}{l} D_n \sin \frac{n\pi x}{l} = \dot{w}_0(x) \quad (8.32)$$

which can be seen to be Fourier sine series expansions of $w_0(x)$ and $\dot{w}_0(x)$ in the interval $0 \leq x \leq l$. The values of C_n and D_n can be determined by multiplying Eqs. (8.31) and (8.32) by $\sin(n\pi x/l)$ and integrating with respect to x from 0 to l :

$$C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx \quad (8.33)$$

$$D_n = \frac{2}{nc\pi} \int_0^l \dot{w}_0(x) \sin \frac{n\pi x}{l} dx \quad (8.34)$$

Note: The solution given by Eq. (8.30) can be identified as the *mode superposition method* since the response is expressed as a superposition of the normal modes. The procedure is applicable in finding not only the free-vibration solution but also the forced-vibration solution of continuous systems.

EXAMPLE 8.1

Dynamic Response of a Plucked String

If a string of length l , fixed at both ends, is plucked at its midpoint as shown in Fig. 8.5 and then released, determine its subsequent motion.

Solution: The solution is given by Eq. (8.30) with C_n and D_n given by Eqs. (8.33) and (8.34), respectively. Since there is no initial velocity, $\dot{w}_0(x) = 0$, and so $D_n = 0$. Thus the solution of Eq. (8.30) reduces to

$$w(x, t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{nc\pi t}{l} \quad (E.1)$$

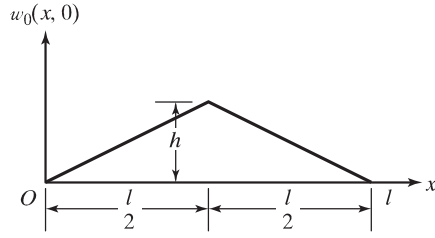


FIGURE 8.5 Initial deflection of the string.

where

$$C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx \quad (\text{E.2})$$

The initial deflection $w_0(x)$ is given by

$$w_0(x) = \begin{cases} \frac{2hx}{l} & \text{for } 0 \leq x \leq \frac{l}{2} \\ \frac{2h(l-x)}{l} & \text{for } \frac{l}{2} \leq x \leq l \end{cases} \quad (\text{E.3})$$

By substituting Eq. (E.3) into Eq. (E.2), C_n can be evaluated:

$$\begin{aligned} C_n &= \frac{2}{l} \left\{ \int_0^{l/2} \frac{2hx}{l} \sin \frac{n\pi x}{l} dx + \int_{l/2}^l \frac{2h}{l} (l-x) \sin \frac{n\pi x}{l} dx \right\} \\ &= \begin{cases} \frac{8h}{\pi^2 n^2} \sin \frac{n\pi}{2} & \text{for } n = 1, 3, 5, \dots \\ 0 & \text{for } n = 2, 4, 6, \dots \end{cases} \quad (\text{E.4}) \end{aligned}$$

By using the relation

$$\sin \frac{n\pi}{2} = (-1)^{(n-1)/2}, \quad n = 1, 3, 5, \dots \quad (\text{E.5})$$

the desired solution can be expressed as

$$w(x, t) = \frac{8h}{\pi^2} \left\{ \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{1}{9} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} + \dots \right\} \quad (\text{E.6})$$

In this case, no even harmonics are excited.

■

8.2.5 Traveling-Wave Solution

The solution of the wave equation, Eq. (8.8), for a string of infinite length can be expressed as [8.5]

$$w(x, t) = w_1(x - ct) + w_2(x + ct) \quad (8.35)$$

where w_1 and w_2 are arbitrary functions of $(x - ct)$ and $(x + ct)$, respectively. To show that Eq. (8.35) is the correct solution of Eq. (8.8), we first differentiate Eq. (8.35):

$$\frac{\partial^2 w(x, t)}{\partial x^2} = w_1''(x - ct) + w_2''(x + ct) \quad (8.36)$$

$$\frac{\partial^2 w(x, t)}{\partial t^2} = c^2 w_1''(x - ct) + c^2 w_2''(x + ct) \quad (8.37)$$

Substitution of these equations into Eq. (8.8) reveals that the wave equation is satisfied. In Eq. (8.35), $w_1(x - ct)$ and $w_2(x + ct)$ represent waves that propagate in the positive and negative directions of the x -axis, respectively, with a velocity c .

For a given problem, the arbitrary functions w_1 and w_2 are determined from the initial conditions, Eq. (8.10). Substitution of Eq. (8.35) into Eq. (8.10) gives, at $t = 0$,

$$w_1(x) + w_2(x) = w_0(x) \quad (8.38)$$

$$-cw_1'(x) + cw_2'(x) = \dot{w}_0(x) \quad (8.39)$$

where the prime indicates differentiation with respect to the respective argument at $t = 0$ (that is, with respect to x). Integration of Eq. (8.39) yields

$$-w_1(x) + w_2(x) = \frac{1}{c} \int_{x_0}^x \dot{w}_0(x') dx' \quad (8.40)$$

where x_0 is a constant. Solution of Eqs. (8.38) and (8.40) gives w_1 and w_2 :

$$w_1(x) = \frac{1}{2} \left[w_0(x) - \frac{1}{c} \int_{x_0}^x \dot{w}_0(x') dx' \right] \quad (8.41)$$

$$w_2(x) = \frac{1}{2} \left[w_0(x) + \frac{1}{c} \int_{x_0}^x \dot{w}_0(x') dx' \right] \quad (8.42)$$

By replacing x by $(x - ct)$ and $(x + ct)$, respectively, in Eqs. (8.41) and (8.42), we obtain the total solution:

$$\begin{aligned} w(x, t) &= w_1(x - ct) + w_2(x + ct) \\ &= \frac{1}{2} [w_0(x - ct) + w_0(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \dot{w}_0(x') dx' \end{aligned} \quad (8.43)$$

The following points should be noted:

1. As can be seen from Eq. (8.43), there is no need to apply boundary conditions to the problem.
2. The solution given by Eq. (8.43) can be expressed as

$$w(x, t) = w_D(x, t) + w_V(x, t) \quad (8.44)$$

where $w_D(x, t)$ denotes the waves propagating due to the known initial displacement $w_0(x)$ with zero initial velocity, and $w_V(x, t)$ represents waves traveling due only to the known initial velocity $\dot{w}_0(x)$ with zero initial displacement.

The transverse vibration of a string fixed at both ends excited by the transverse impact of an elastic load at an intermediate point was considered in [8.6]. A review of the literature on the dynamics of cables and chains was given by Triantafyllou [8.7].

8.3 Longitudinal Vibration of a Bar or Rod

8.3.1 Equation of Motion and Solution

Consider an elastic bar of length l with varying cross-sectional area $A(x)$, as shown in Fig. 8.6. The forces acting on the cross sections of a small element of the bar are given by P and $P + dP$ with

$$P = \sigma A = EA \frac{\partial u}{\partial x} \quad (8.45)$$

where σ is the axial stress, E is Young's modulus, u is the axial displacement, and $\partial u / \partial x$ is the axial strain. If $f(x, t)$ denotes the external force per unit length, the summation of the forces in the x direction gives the equation of motion

$$(P + dP) + f dx - P = \rho A dx \frac{\partial^2 u}{\partial t^2} \quad (8.46)$$

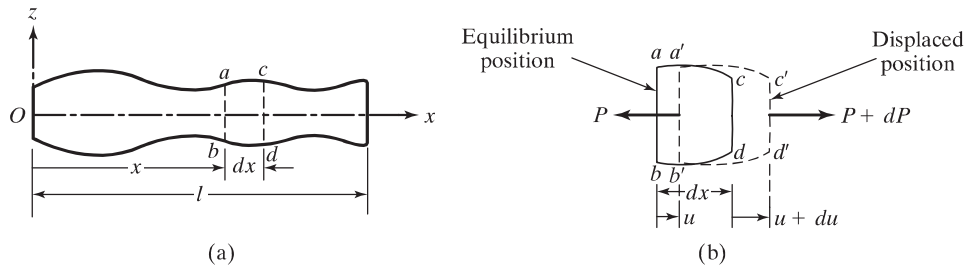


FIGURE 8.6 Longitudinal vibration of a bar.

where ρ is the mass density of the bar. By using the relation $dP = (\partial P / \partial x) dx$ and Eq. (8.45), the equation of motion for the forced longitudinal vibration of a nonuniform bar, Eq. (8.46), can be expressed as

$$\frac{\partial}{\partial x} \left[EA(x) \frac{\partial u(x, t)}{\partial x} \right] + f(x, t) = \rho(x) A(x) \frac{\partial^2 u}{\partial t^2}(x, t) \quad (8.47)$$

For a uniform bar, Eq. (8.47) reduces to

$$EA \frac{\partial^2 u}{\partial x^2}(x, t) + f(x, t) = \rho A \frac{\partial^2 u}{\partial t^2}(x, t) \quad (8.48)$$

The free-vibration equation can be obtained from Eq. (8.48), by setting $f = 0$, as

$$c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{\partial^2 u}{\partial t^2}(x, t) \quad (8.49)$$

where

$$c = \sqrt{\frac{E}{\rho}} \quad (8.50)$$

Note that Eqs. (8.47) to (8.50) can be seen to be similar to Eqs. (8.5), (8.6), (8.8), and (8.9), respectively. The solution of Eq. (8.49), which can be obtained as in the case of Eq. (8.8), can thus be written as

$$u(x, t) = U(x)T(t) \equiv \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)^1 \quad (8.51)$$

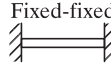
End Conditions of Bar	Boundary Conditions	Frequency Equation	Mode Shape (Normal Function)	Natural Frequencies
 Fixed-free	$u(0, t) = 0$ $\frac{\partial u}{\partial x}(l, t) = 0$	$\cos \frac{\omega l}{c} = 0$	$U_n(x) = C_n \sin \frac{(2n+1)\pi x}{2l}$	$\omega_n = \frac{(2n+1)\pi c}{2l};$ $n = 0, 1, 2, \dots$
 Free-free	$\frac{\partial u}{\partial x}(0, t) = 0$ $\frac{\partial u}{\partial x}(l, t) = 0$	$\sin \frac{\omega l}{c} = 0$	$U_n(x) = C_n \cos \frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l};$ $n = 0, 1, 2, \dots$
 Fixed-fixed	$u(0, t) = 0$ $u(l, t) = 0$	$\sin \frac{\omega l}{c} = 0$	$U_n(x) = C_n \cos \frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l};$ $n = 1, 2, 3, \dots$

FIGURE 8.7 Common boundary conditions for a bar in longitudinal vibration.

¹We use $\tilde{A}$ and $\tilde{B}$ in this section; A is used to denote the cross-sectional area of the bar.

where the function $U(x)$ represents the normal mode and depends only on x and the function $T(t)$ depends only on t . If the bar has known initial axial displacement $u_0(x)$ and initial velocity $\dot{u}_0(x)$, the initial conditions can be stated as

$$\begin{aligned} u(x, t = 0) &= u_0(x) \\ \frac{\partial u}{\partial t}(x, t = 0) &= \dot{u}_0(x) \end{aligned} \quad (8.52)$$

The common boundary conditions and the corresponding frequency equations for the longitudinal vibration of uniform bars are shown in Fig. 8.7.

EXAMPLE 8.2

Boundary Conditions for a Bar

A uniform bar of cross-sectional area A , length l , and Young's modulus E is connected at both ends by springs, dampers, and masses, as shown in Fig. 8.8(a). State the boundary conditions.

Solution: The free-body diagrams of the masses m_1 and m_2 are shown in Fig. 8.8(b). From this, we find that at the left end ($x = 0$), the force developed in the bar due to positive u and $\partial u / \partial x$ must be equal to the sum of spring, damper, and inertia forces:

$$AE \frac{\partial u}{\partial x}(0, t) = k_1 u(0, t) + c_1 \frac{\partial u}{\partial t}(0, t) + m_1 \frac{\partial^2 u}{\partial t^2}(0, t) \quad (E.1)$$

Similarly at the right end ($x = l$), the force developed in the bar due to positive u and $\partial u / \partial x$ must be equal to the negative sum of spring, damper, and inertia forces:

$$AE \frac{\partial u}{\partial x}(l, t) = -k_2 u(l, t) - c_2 \frac{\partial u}{\partial t}(l, t) - m_2 \frac{\partial^2 u}{\partial t^2}(l, t) \quad (E.2)$$

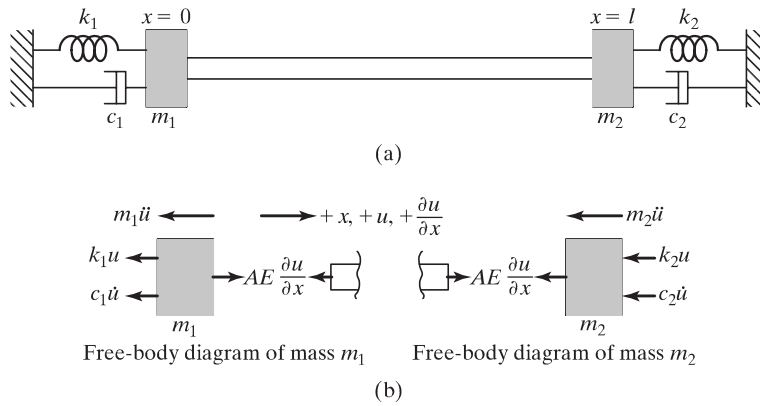


FIGURE 8.8 Bar connected to springs-masses-dampers at ends.

8.3.2 Orthogonality of Normal Functions

The normal functions for the longitudinal vibration of bars satisfy the orthogonality relation

$$\int_0^l U_i(x)U_j(x) dx = 0 \quad (8.53)$$

where $U_i(x)$ and $U_j(x)$ denote the normal functions corresponding to the i th and j th natural frequencies ω_i and ω_j , respectively. When $u(x, t) = U_i(x)T(t)$ and $u(x, t) = U_j(x)T(t)$ are assumed as solutions, Eq. (8.49) gives

$$c^2 \frac{d^2 U_i(x)}{dx^2} + \omega_i^2 U_i(x) = 0 \quad \text{or} \quad c^2 U_i''(x) + \omega_i^2 U_i(x) = 0 \quad (8.54)$$

and

$$c^2 \frac{d^2 U_j(x)}{dx^2} + \omega_j^2 U_j(x) = 0 \quad \text{or} \quad c^2 U_j''(x) + \omega_j^2 U_j(x) = 0 \quad (8.55)$$

where $U_i'' = \frac{d^2 U_i}{dx^2}$ and $U_j'' = \frac{d^2 U_j}{dx^2}$. Multiplication of Eq. (8.54) by U_j and Eq. (8.55) by U_i gives

$$c^2 U_i'' U_j + \omega_i^2 U_i U_j = 0 \quad (8.56)$$

$$c^2 U_j'' U_i + \omega_j^2 U_j U_i = 0 \quad (8.57)$$

Subtraction of Eq. (8.57) from Eq. (8.56) and integration from 0 to l results in

$$\begin{aligned} \int_0^l U_i U_j dx &= -\frac{c^2}{\omega_i^2 - \omega_j^2} \int_0^l (U_i'' U_j - U_j'' U_i) dx \\ &= -\frac{c^2}{\omega_i^2 - \omega_j^2} [U_i' U_j - U_j' U_i] \Big|_0^l \end{aligned} \quad (8.58)$$

The right-hand side of Eq. (8.58) can be proved to be zero for any combination of boundary conditions. For example, if the bar is fixed at $x = 0$ and free at $x = l$,

$$u(0, t) = 0, \quad t \geq 0 \quad \text{or} \quad U(0) = 0 \quad (8.59)$$

$$\frac{\partial u}{\partial x}(l, t) = 0, \quad t \geq 0 \quad \text{or} \quad U'(l) = 0 \quad (8.60)$$

Thus $(U_i' U_j - U_j' U_i)|_{x=l} = 0$ due to U' being zero (Eq. (8.60)) and $(U_i' U_j - U_j' U_i)|_{x=0} = 0$ due to U being zero (Eq. (8.59)). Equation (8.58) thus reduces to Eq. (8.53), which is also known as the *orthogonality principle for the normal functions*.

EXAMPLE 8.3 Free Vibrations of a Fixed-Free Bar

Find the natural frequencies and the free-vibration solution of a bar fixed at one end and free at the other.

Solution: Let the bar be fixed at $x = 0$ and free at $x = l$, so that the boundary conditions can be expressed as

$$u(0, t) = 0, \quad t \geq 0 \quad (\text{E.1})$$

$$\frac{\partial u}{\partial x}(l, t) = 0, \quad t \geq 0 \quad (\text{E.2})$$

The use of Eq. (E.1) in Eq. (8.51) gives $\tilde{A} = 0$, while the use of Eq. (E.2) gives the frequency equation

$$\tilde{B} \frac{\omega}{c} \cos \frac{\omega l}{c} = 0 \quad \text{or} \quad \cos \frac{\omega l}{c} = 0 \quad (\text{E.3})$$

The eigenvalues or natural frequencies are given by

$$\frac{\omega_n l}{c} = (2n + 1) \frac{\pi}{2}, \quad n = 0, 1, 2, \dots$$

or

$$\omega_n = \frac{(2n + 1)\pi c}{2l}, \quad n = 0, 1, 2, \dots \quad (\text{E.4})$$

Thus the total (free-vibration) solution of Eq. (8.49) can be written, using the mode superposition method, as

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= \sum_{n=0}^{\infty} \sin \frac{(2n + 1)\pi x}{2l} \left[C_n \cos \frac{(2n + 1)\pi c t}{2l} + D_n \sin \frac{(2n + 1)\pi c t}{2l} \right] \end{aligned} \quad (\text{E.5})$$

where the values of the constants C_n and D_n can be determined from the initial conditions, as in Eqs. (8.33) and (8.34):

$$C_n = \frac{2}{l} \int_0^l u_0(x) \sin \frac{(2n + 1)\pi x}{2l} dx \quad (\text{E.6})$$

$$D_n = \frac{4}{(2n + 1)\pi c} \int_0^l \dot{u}_0(x) \sin \frac{(2n + 1)\pi x}{2l} dx \quad (\text{E.7})$$

■

EXAMPLE 8.4**Natural Frequencies of a Bar Carrying a Mass**

Find the natural frequencies of a bar with one end fixed and a mass attached at the other end, as in Fig. 8.9.

Solution: The equation governing the axial vibration of the bar is given by Eq. (8.49) and the solution by Eq. (8.51). The boundary condition at the fixed end ($x = 0$)

$$u(0, t) = 0 \quad (\text{E.1})$$

leads to $\dot{u} = 0$ in Eq. (8.51). At the end $x = l$, the tensile force in the bar must be equal to the inertia force of the vibrating mass M , and so

$$AE \frac{\partial u}{\partial x}(l, t) = -M \frac{\partial^2 u}{\partial t^2}(l, t) \quad (\text{E.2})$$

With the help of Eq. (8.51), this equation can be expressed as

$$AE \frac{\omega}{c} \cos \frac{\omega l}{c} (C \cos \omega t + D \sin \omega t) = M \omega^2 \sin \frac{\omega l}{c} (C \cos \omega t + D \sin \omega t)$$

That is,

$$\frac{AE\omega}{c} \cos \frac{\omega l}{c} = M \omega^2 \sin \frac{\omega l}{c}$$

or

$$\alpha \tan \alpha = \beta \quad (\text{E.3})$$

where

$$\alpha = \frac{\omega l}{c} \quad (\text{E.4})$$

and

$$\beta = \frac{AE l}{c^2 M} = \frac{A \rho l}{M} = \frac{m}{M} \quad (\text{E.5})$$

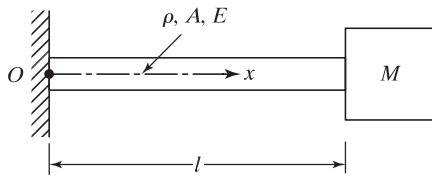


FIGURE 8.9 Bar carrying an end mass.

TABLE 8.1

	Values of the Mass Ratio β				
	0.01	0.1	1.0	10.0	100.0
Value of $\alpha_1 \left(\omega_1 = \frac{\alpha_1 c}{l} \right)$	0.1000	0.3113	0.8602	1.4291	1.5549
Value of $\alpha_2 \left(\omega_2 = \frac{\alpha_2 c}{l} \right)$	3.1448	3.1736	3.4267	4.3063	4.6658

where m is the mass of the bar. Equation (E.3) is the frequency equation (in the form of a transcendental equation) whose solution gives the natural frequencies of the system. The first two natural frequencies are given in Table 8.1 for different values of the parameter β .

Note: If the mass of the bar is negligible compared to the mass attached, $m \simeq 0$,

$$c = \left(\frac{E}{\rho} \right)^{1/2} = \left(\frac{EA}{m} \right)^{1/2} \rightarrow \infty \quad \text{and} \quad \alpha = \frac{\omega l}{c} \rightarrow 0$$

In this case

$$\tan \frac{\omega l}{c} \simeq \frac{\omega l}{c}$$

and the frequency equation (E.3) can be taken as

$$\left(\frac{\omega l}{c} \right)^2 = \beta$$

This gives the approximate value of the fundamental frequency

$$\omega_1 = \frac{c}{l} \beta^{1/2} = \frac{c}{l} \left(\frac{\rho A l}{M} \right)^{1/2} = \left(\frac{EA}{lM} \right)^{1/2} = \left(\frac{g}{\delta_s} \right)^{1/2}$$

where

$$\delta_s = \frac{Mg l}{EA}$$

represents the static elongation of the bar under the action of the load Mg .

■

EXAMPLE 8.5

Vibrations of a Bar Subjected to Initial Force

A bar of uniform cross-sectional area A , density ρ , modulus of elasticity E , and length l is fixed at one end and free at the other end. It is subjected to an axial force F_0 at its free end, as shown in Fig. 8.10(a). Study the resulting vibrations if the force F_0 is suddenly removed.

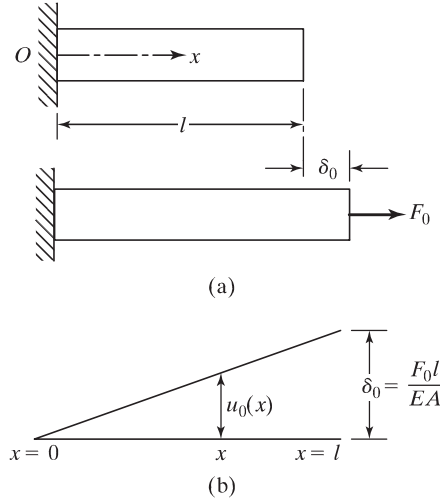


FIGURE 8.10 Bar subjected to an axial force at end.

Solution: The tensile strain induced in the bar due to F_0 is

$$\varepsilon = \frac{F_0}{EA}$$

Thus the displacement of the bar just before the force F_0 is removed (initial displacement) is given by (see Fig. (8.10b))

$$u_0 = u(x, 0) = \varepsilon x = \frac{F_0 x}{EA}, \quad 0 \leq x \leq l \quad (\text{E.1})$$

Since the initial velocity is zero, we have

$$\dot{u}_0 = \frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 \leq x \leq l \quad (\text{E.2})$$

The general solution of a bar fixed at one end and free at the other end is given by Eq. (E.5) of Example 8.3:

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= \sum_{n=0}^{\infty} \sin \frac{(2n+1)\pi x}{2l} \left[C_n \cos \frac{(2n+1)\pi ct}{2l} + D_n \sin \frac{(2n+1)\pi ct}{2l} \right] \end{aligned} \quad (\text{E.3})$$

where C_n and D_n are given by Eqs. (E.6) and (E.7) of Example 8.3. Since $\dot{u}_0 = 0$, we obtain $D_n = 0$. By using the initial displacement of Eq. (E.1) in Eq. (E.6) of Example 8.3, we obtain

$$C_n = \frac{2}{l} \int_0^l \frac{F_0 x}{EA} \cdot \sin \frac{(2n+1)\pi x}{2l} dx = \frac{8F_0 l}{EA\pi^2} \frac{(-1)^n}{(2n+1)^2} \quad (\text{E.4})$$

Thus the solution becomes

$$u(x, t) = \frac{8F_0 l}{EA\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{2l} \cos \frac{(2n+1)\pi ct}{2l} \quad (\text{E.5})$$

Equations (E.3) and (E.5) indicate that the motion of a typical point at $x = x_0$ on the bar is composed of the amplitudes

$$C_n \sin \frac{(2n+1)\pi x_0}{2l}$$

corresponding to the circular frequencies

$$\frac{(2n+1)\pi c}{2l}$$

■

8.4 Torsional Vibration of a Shaft or Rod

Figure 8.11 represents a nonuniform shaft subjected to an external torque $f(x, t)$ per unit length. If $\theta(x, t)$ denotes the angle of twist of the cross section, the relation between the torsional deflection and the twisting moment $M_t(x, t)$ is given by [8.8]

$$M_t(x, t) = GJ(x) \frac{\partial \theta}{\partial x}(x, t) \quad (8.61)$$

where G is the shear modulus and $GJ(x)$ is the torsional stiffness, with $J(x)$ denoting the polar moment of inertia of the cross section in the case of a circular section. If the mass polar moment of inertia of the shaft per unit length is I_0 , the inertia torque acting on an element of length dx becomes

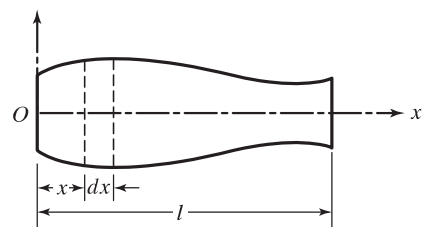
$$I_0 dx \frac{\partial^2 \theta}{\partial t^2}$$

If an external torque $f(x, t)$ acts on the shaft per unit length, the application of Newton's second law yields the equation of motion:

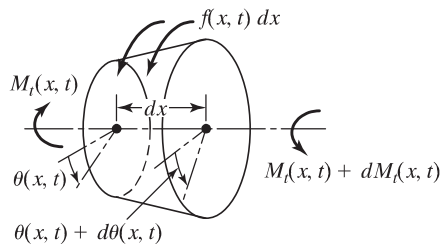
$$(M_t + dM_t) + f dx - M_t = I_0 dx \frac{\partial^2 \theta}{\partial t^2} \quad (8.62)$$

By expressing dM_t as

$$\frac{\partial M_t}{\partial x} dx$$



(a)



(b)

FIGURE 8.11 Torsional vibration of a shaft.

and using Eq. (8.61), the forced torsional vibration equation for a nonuniform shaft can be obtained:

$$\frac{\partial}{\partial x} \left[GJ(x) \frac{\partial \theta}{\partial x}(x, t) \right] + f(x, t) = I_0(x) \frac{\partial^2 \theta}{\partial t^2}(x, t) \quad (8.63)$$

For a uniform shaft, Eq. (8.63) takes the form

$$GJ \frac{\partial^2 \theta}{\partial x^2}(x, t) + f(x, t) = I_0 \frac{\partial^2 \theta}{\partial t^2}(x, t) \quad (8.64)$$

which, in the case of free vibration, reduces to

$$c^2 \frac{\partial^2 \theta}{\partial x^2}(x, t) = \frac{\partial^2 \theta}{\partial t^2}(x, t) \quad (8.65)$$

where

$$c = \sqrt{\frac{GJ}{I_0}} \quad (8.66)$$

Notice that Eqs. (8.63) to (8.66) are similar to the equations derived in the cases of transverse vibration of a string and longitudinal vibration of a bar. If the shaft has a uniform cross section, $I_0 = \rho J$. Hence Eq. (8.66) becomes

$$c = \sqrt{\frac{G}{\rho}} \quad (8.67)$$

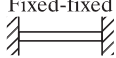
End Conditions of Shaft	Boundary Conditions	Frequency Equation	Mode Shape (Normal Function)	Natural Frequencies
 Fixed-free	$\theta(0, t) = 0$ $\frac{\partial \theta}{\partial x}(l, t) = 0$	$\cos \frac{\omega l}{c} = 0$	$\theta(x) = C_n \sin \frac{(2n+1)\pi x}{2l}$	$\omega_n = \frac{(2n+1)\pi c}{2l};$ $n = 0, 1, 2, \dots$
 Free-free	$\frac{\partial \theta}{\partial x}(0, t) = 0$ $\frac{\partial \theta}{\partial x}(l, t) = 0$	$\sin \frac{\omega l}{c} = 0$	$\theta(x) = C_n \cos \frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l};$ $n = 0, 1, 2, \dots$
 Fixed-fixed	$\theta(0, t) = 0$ $\theta(l, t) = 0$	$\sin \frac{\omega l}{c} = 0$	$\theta(x) = C_n \cos \frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l};$ $n = 1, 2, 3, \dots$

FIGURE 8.12 Boundary conditions for uniform shafts (rods) subjected to torsional vibration.

If the shaft is given an angular displacement $\theta_0(x)$ and an angular velocity $\dot{\theta}_0(x)$ at $t = 0$, the initial conditions can be stated as

$$\begin{aligned}\theta(x, t = 0) &= \theta_0(x) \\ \frac{\partial \theta}{\partial t}(x, t = 0) &= \dot{\theta}_0(x)\end{aligned}\quad (8.68)$$

The general solution of Eq. (8.65) can be expressed as

$$\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \quad (8.69)$$

The common boundary conditions for the torsional vibration of uniform shafts are indicated in Fig. 8.12 along with the corresponding frequency equations and the normal functions.

EXAMPLE 8.6 Natural Frequencies of a Milling Cutter

Find the natural frequencies of the plane milling cutter shown in Fig. 8.13 when the free end of the shank is fixed. Assume the torsional rigidity of the shank as GJ and the mass moment of inertia of the cutter as I_0 .

Solution: The general solution is given by Eq. (8.69). From this equation, by using the fixed boundary condition $\theta(0, t) = 0$, we obtain $A = 0$. The boundary condition at $x = l$ can be stated as

$$GJ \frac{\partial \theta}{\partial x}(l, t) = -I_0 \frac{\partial^2 \theta}{\partial t^2}(l, t) \quad (E.1)$$

That is,

$$BGJ \frac{\omega}{c} \cos \frac{\omega l}{c} = BI_0 \omega^2 \sin \frac{\omega l}{c}$$

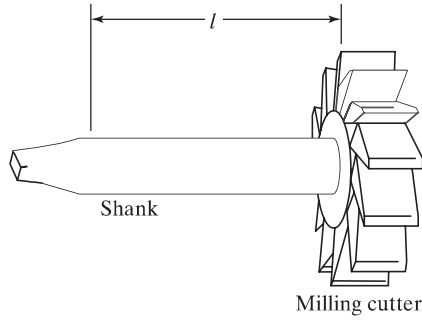


FIGURE 8.13 Plane milling cutter.

or

$$\frac{\omega l}{c} \tan \frac{\omega l}{c} = \frac{Jpl}{I_0} = \frac{J_{\text{rod}}}{I_0} \quad (\text{E.2})$$

where $J_{\text{rod}} = Jpl$. Equation (E.2) can be expressed as

$$\alpha \tan \alpha = \beta \quad \text{where } \alpha = \frac{\omega l}{c} \quad \text{and} \quad \beta = \frac{J_{\text{rod}}}{I_0} \quad (\text{E.3})$$

The solution of Eq. (E.3), and thus the natural frequencies of the system, can be obtained as in the case of Example 8.4.

■

8.5 Lateral Vibration of Beams

8.5.1 Equation of Motion

Consider the free-body diagram of an element of a beam shown in Fig. 8.14, where $M(x, t)$ is the bending moment, $V(x, t)$ is the shear force, and $f(x, t)$ is the external force per unit length of the beam. Since the inertia force acting on the element of the beam is

$$\rho A(x) dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

the force equation of motion in the z direction gives

$$-(V + dV) + f(x, t) dx + V = \rho A(x) dx \frac{\partial^2 w}{\partial t^2}(x, t) \quad (8.70)$$

where ρ is the mass density and $A(x)$ is the cross-sectional area of the beam. The moment equation of motion about the y -axis passing through point O in Fig. 8.14 leads to

$$(M + dM) - (V + dV) dx + f(x, t) dx \frac{dx}{2} - M = 0 \quad (8.71)$$