

Lateral Earth Pressure I: At-Rest State and Rankine Theory for Granular Soils

Week 2 | Contact time: 4 hours (2 h theory + 2 h tutorial)

1 Introduction

Vertical or near-vertical faces of soil are supported by retaining walls, sheet pile walls, braced cuts, basement walls, bridge abutments and the walls of culverts and pumping-station sumps. The design of these structures needs the magnitude, distribution and line of action of the **lateral earth pressure**. The pressure depends on:

1. the type and amount of wall movement;
2. the shear strength parameters (c' , ϕ') and unit weight of the soil;
3. the surcharge loads on the backfill;
4. the slope of the backfill surface and the inclination of the wall back;
5. the friction between wall and soil;
6. the drainage conditions and the position of the water table.

In this lecture the wall is rigid, the backfill is granular ($c' = 0$), and the wall is smooth.

2 Wall Movement and the Three Earth Pressure States

Figure 1 shows the three states. If the wall does not move, the soil stays in elastic equilibrium and the pressure is the **at-rest pressure**. If the wall moves away from the soil, the horizontal stress falls until the soil fails along a sliding wedge; the minimum pressure is the **active pressure**. If the wall is pushed into the soil, the horizontal stress rises until a larger wedge is pushed up; the maximum pressure is the **passive pressure**.

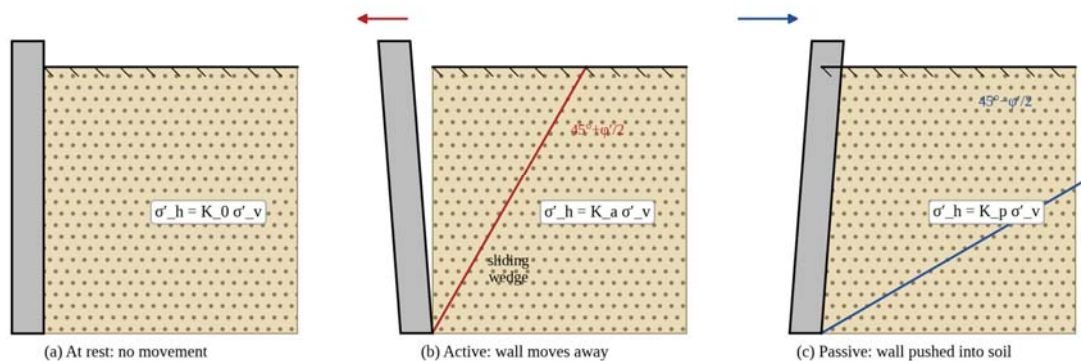


Figure 1: Earth pressure states and the corresponding failure planes.

The movement needed to reach the passive state is much larger than that needed for the active state (**Figure 2 and Table 1**). Therefore, the pressures satisfy active thrust (E_a) < thrust at rest

$(E_0) < \text{passive thrust } (E_p)$, and the passive resistance is usually reduced in design because the movement needed to mobilize it is not acceptable.

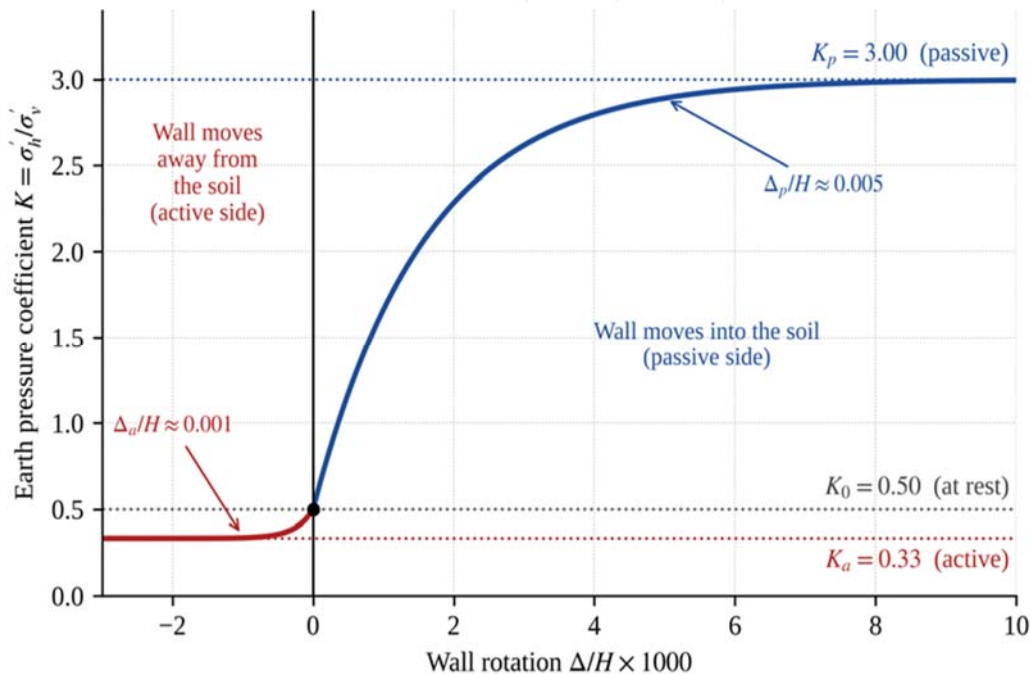


Figure 2: Variation of the earth pressure coefficient with wall movement ($\phi' = 30^\circ$).

Table 1: Typical wall movement needed to reach the active and passive states.

Soil	Active state Δ_a/H	Passive state Δ_p/H
Loose sand	0.001–0.002	0.01
Dense sand	0.0005–0.001	0.005
Soft clay	0.02	0.05
Stiff clay	0.01	0.02

3 Earth Pressure at Rest

Consider a vertical wall of height H that does not move. At depth z the vertical effective stress is $\sigma'_v = q + \gamma z$ and the horizontal effective stress is:

$$\sigma'_h = K_0 \sigma'_v = K_0 (q + \gamma z) \quad (1)$$

For normally consolidated soils, Jaky (1944) gave:

$$K_0 = 1 - \sin \phi' \quad (2)$$

For overconsolidated soils (Mayne and Kulhawy 1982):

$$K_0 = (1 - \sin \phi') OCR^{\sin \phi'} \quad (3)$$

where OCR is the **overconsolidation ratio**: $OCR = \sigma'_c / \sigma'_{v0}$. Here σ'_c is the preconsolidation pressure, the largest effective vertical stress the soil has carried in its history, and σ'_{v0} is the present effective vertical stress at the same depth. $OCR = 1$ for a normally consolidated soil, and Eq. (3) then reduces to Eq. (2). $OCR > 1$ for an overconsolidated soil.

Overconsolidation, compaction of the backfill and preloading increase K_0 . For a compacted backfill behind a rigid basement wall, K_0 can exceed 1.0 near the surface. With a surcharge q and no water table, the diagram is a trapezium. The resultant per meter length of wall is the sum of a rectangle and a triangle:

$$P_0 = qK_0H + \frac{1}{2}\gamma H^2 K_0, \quad \bar{z} = \frac{P_1(H/2) + P_2(H/3)}{P_0} \quad (4)$$

where $\bar{z}$ is measured from the base of the wall. With a water table at depth H_1 below the surface, the effective stress is used below the water table and the hydrostatic pressure $u = \gamma_w(z - H_1)$ is added separately. Water pressure has $K = 1$ in every direction; it is never multiplied by K_0 , K_a or K_p .

4 Rankine Theory of Active and Passive Pressure

Assumptions

1. The soil is homogeneous, isotropic and cohesionless (extended to $c' > 0$ in Lecture 3).
2. The back of the wall is vertical and smooth (no wall friction, $\delta = 0$).
3. The ground surface is horizontal (extended to a sloping surface in Lecture 3).
4. The soil mass is in a state of plastic equilibrium: every point is on the edge of failure.

Derivation from the Mohr Circle

At depth z the vertical stress $\sigma'_v = \gamma z$ is a principal stress because the smooth wall carries no shear. In the active state the horizontal stress decreases until the Mohr circle touches the failure envelope (**Figure 3**). The vertical stress is then the major principal stress σ'_1 and the horizontal stress is the minor one σ'_3 . From the geometry of the circle tangent to $\tau = \sigma' \tan \phi'$:

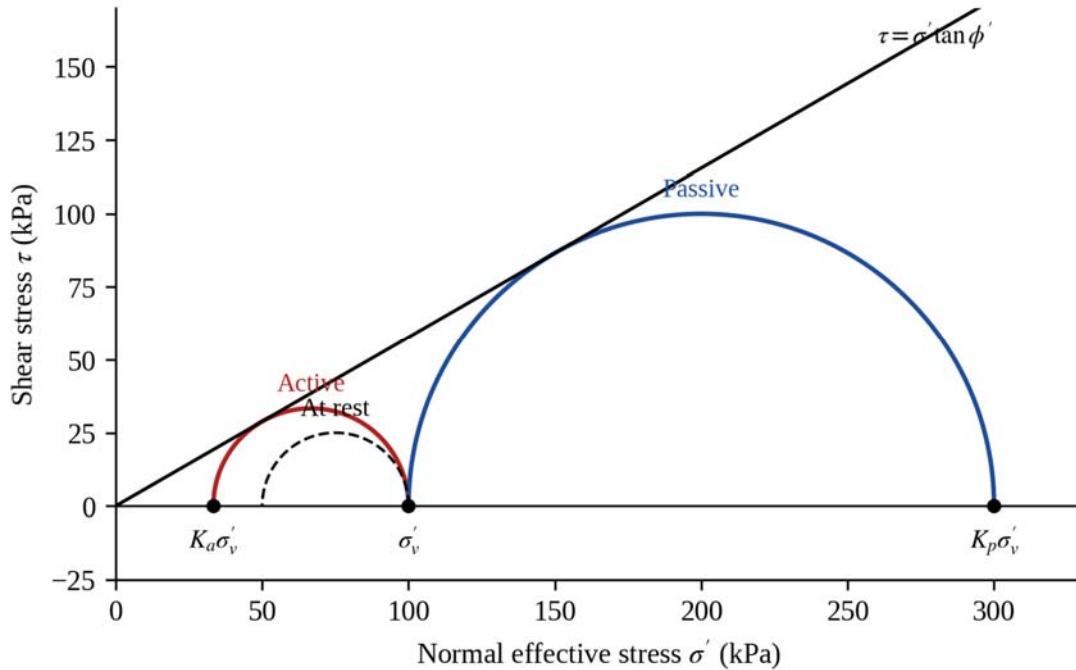


Figure 3: Mohr circles for the at-rest, active and passive states ($\sigma'_v = 100$ kPa, $\phi' = 30^\circ$).

$$\sin \phi' = \frac{(\sigma'_1 - \sigma'_3)/2}{(\sigma'_1 + \sigma'_3)/2} \Rightarrow \frac{\sigma'_3}{\sigma'_1} = \frac{1 - \sin \phi'}{1 + \sin \phi'} \quad (5)$$

Hence the Rankine active and passive coefficients are:

$$K_a = \frac{1 - \sin \phi'}{1 + \sin \phi'} = \tan^2 \left(45^\circ - \frac{\phi'}{2} \right) \quad (6)$$

$$K_p = \frac{1 + \sin \phi'}{1 - \sin \phi'} = \tan^2 \left(45^\circ + \frac{\phi'}{2} \right) = \frac{1}{K_a} \quad (7)$$

The failure planes make $45^\circ + \phi'/2$ with the horizontal in the active state and $45^\circ - \phi'/2$ in the passive state (**Figure 1**). The active wedge is therefore narrower and the passive wedge wider.

Figure 4 plots the three coefficients (also see **Table 2**).

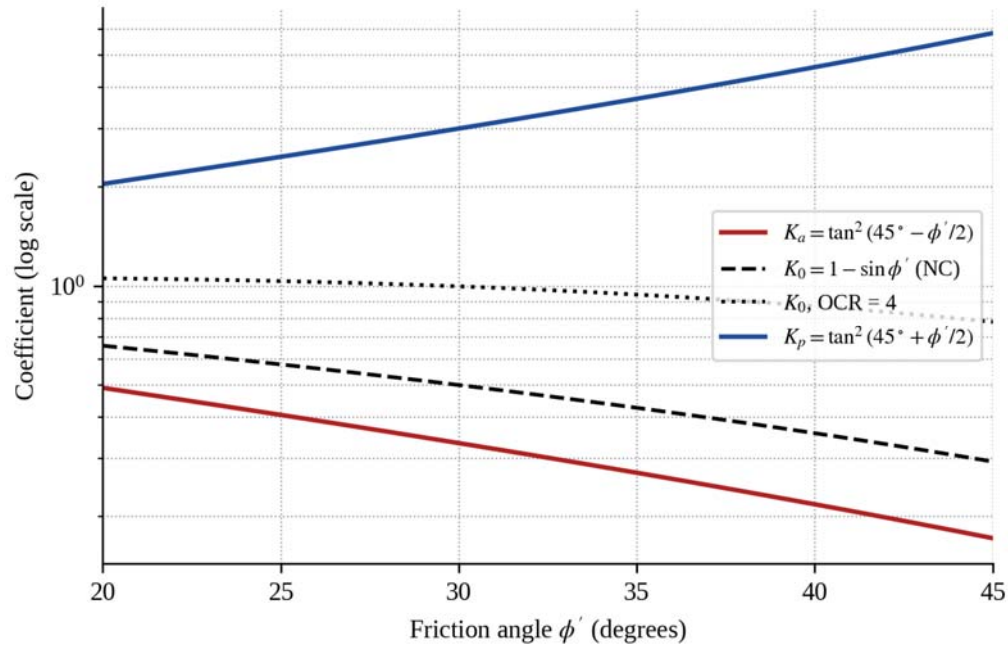


Figure 4: Earth pressure coefficients versus friction angle.

Table 2: Values of the Rankine and at-rest coefficients.

$\phi' (^{\circ})$	25	28	30	32	34	36	38	40
K_a	0.406	0.361	0.333	0.307	0.283	0.260	0.238	0.217
K_0 (NC)	0.577	0.531	0.500	0.470	0.441	0.412	0.384	0.357
K_p	2.46	2.77	3.00	3.25	3.54	3.85	4.20	4.60

5 Pressure Diagrams for Granular Backfill

Dry Backfill

$$\sigma'_a = K_a \gamma z, \quad P_a = \frac{1}{2} K_a \gamma H^2 \text{ acting at } H/3 \text{ above the base} \quad (8)$$

$$\sigma'_p = K_p \gamma z, \quad P_p = \frac{1}{2} K_p \gamma H^2 \text{ acting at } H/3 \text{ above the base} \quad (9)$$

Uniform Surcharge

A uniform surcharge q on the backfill adds a constant pressure $K_a q$ over the full height. Its resultant $q K_a H$ acts at $H/2$. Traffic and stored material behind walls are often represented by $q = 10\text{--}20$ kPa.

Water Table in the Backfill

Below the water table use the effective (submerged) unit weight $\gamma' = \gamma_{\text{sat}} - \gamma_w$ for the soil and add the full water pressure:

$$\sigma_a = K_a [q + \gamma H_1 + \gamma' (z - H_1)] + \gamma_w (z - H_1) \quad (z > H_1) \quad (10)$$

Water pressure often exceeds the soil pressure below the water table. This is why drainage of the backfill is one of the most effective ways to reduce the load on a wall.

Layered Backfill

At the interface between two layers the vertical stress is continuous but K changes. The diagram therefore has a jump at the interface: compute the pressure just above with K of the upper layer and just below with K of the lower layer. The resultant is found by splitting the diagram into rectangles and triangles and taking moments about the base.

6 Lateral Pressure from a Strip Surcharge

Loads that do not cover the whole backfill (a road, a crane track, a footing near the wall) produce a lateral pressure that depends on their position. For a strip load of intensity q and width b placed at a distance a from a rigid (non-yielding) wall, the theory of elasticity, doubled to account for the rigid wall, gives:

$$\sigma_h = \frac{2q}{\pi} (\beta - \sin \beta \cos 2\alpha), \quad \theta_1 = \tan^{-1} \frac{a}{z}, \quad \theta_2 = \tan^{-1} \frac{a+b}{z}, \quad \beta = \theta_2 - \theta_1, \quad \alpha = \frac{\theta_1 + \theta_2}{2} \quad (11)$$

The angles are in radians. The pressure is zero at the top, peaks at a depth comparable to a , and then decays. The resultant is found by numerical integration (trapezoidal rule) or from charts.

Figure 5 shows the distribution for three load positions.

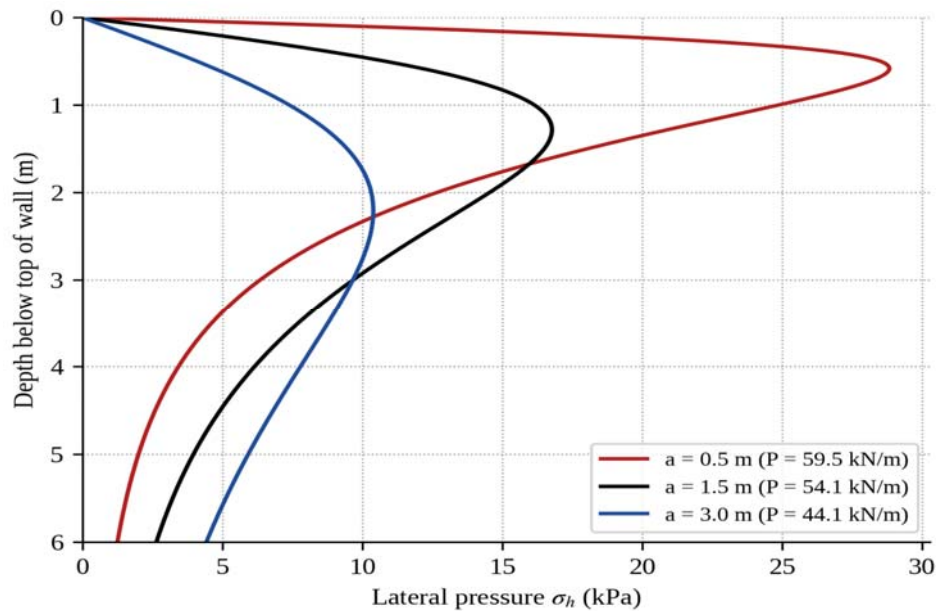


Figure 5: Lateral pressure on a 6 m rigid wall from a 2 m wide strip load $q = 50$ kPa placed at three distances a .

The closer the load, the larger and shallower the peak. A load placed farther than about H from the wall has a small effect on the wall but may still affect its overall stability.

7 Summary of the Procedure

1. Decide the state (at rest, active or passive) from the expected wall movement.
2. Compute K for each layer (K_0 , K_a or K_p).
3. Compute σ'_v at the top and bottom of each layer and at the water table, using γ above and γ' below the water table. Add q .
4. Multiply by K to get σ'_h . Compute values just above and just below each interface.
5. Add the hydrostatic pressure u below the water table.
6. Split the diagram into rectangles and triangles. Compute each force and its arm about the base.
7. Sum the forces for P ; divide the sum of moments by P for $\bar{z}$.

Limits of the Rankine theory

1. A real wall back is rough. Wall friction rotates the resultant and reduces the active force slightly; Rankine gives a conservative active force for a vertical wall.
2. For passive pressure on a rough wall, the Rankine plane wedge overestimates the safety only slightly, but Coulomb plane wedges overestimate K_p when $\delta > \phi'/2$.
3. The theory gives limit pressures. It does not give the pressure at intermediate movements.

Tutorial: Worked Examples

Example 1

A rigid basement wall 5 m high retains a normally consolidated sand ($\phi' = 30^\circ$). The water table is 2 m below the surface. $\gamma = 17 \text{ kN/m}^3$ above and $\gamma_{\text{sat}} = 19.5 \text{ kN/m}^3$ below the water table. A surcharge $q = 10 \text{ kPa}$ acts on the surface. Draw the at-rest pressure diagram and find the resultant force per meter and its location.

Solution

$$K_0 = 1 - \sin 30^\circ = 0.500, \quad \gamma' = 19.5 - 9.81 = 9.69 \text{ kN/m}^3$$

$$z = 0: \sigma'_h = 0.500(10) = 5.00 \text{ kPa}$$

$$z = 2 \text{ m}: \sigma'_h = 0.500(10 + 17 \times 2) = 22.00 \text{ kPa}$$

$$z = 5 \text{ m}: \sigma'_h = 0.500(10 + 34 + 9.69 \times 3) = 36.53 \text{ kPa}, \quad u = 9.81 \times 3 = 29.43 \text{ kPa}$$

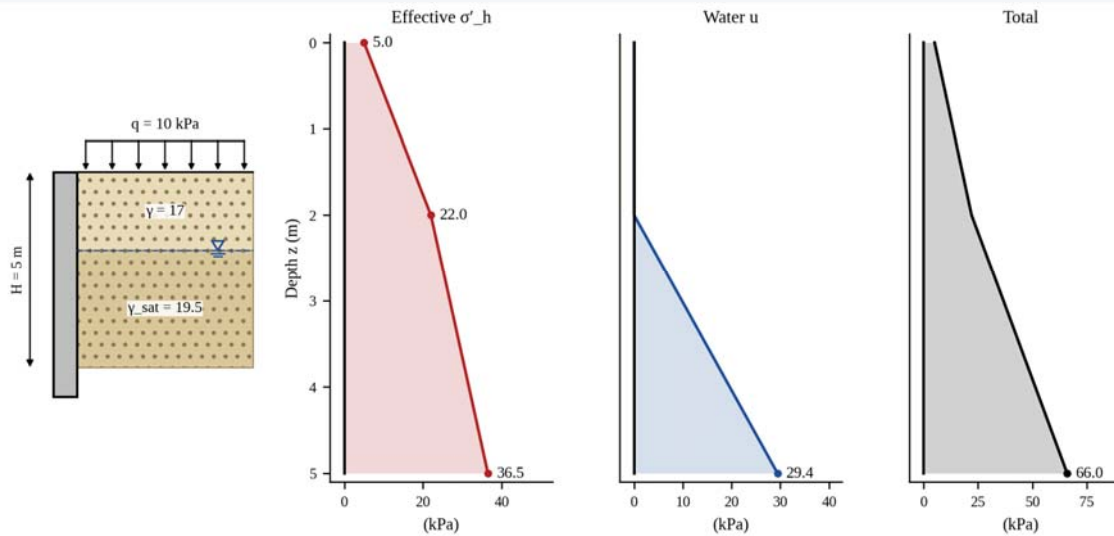


Figure 6: Example 1: at-rest pressure diagrams.

Part	Force (kN/m)	Arm above base (m)	Moment (kN·m/m)
Rectangle 0–2 m	10.00	4.00	40.00
Triangle 0–2 m	17.00	3.67	62.33
Rectangle 2–5 m	66.00	1.50	99.00
Triangle 2–5 m (soil)	21.80	1.00	21.80
Triangle 2–5 m (water)	44.14	1.00	44.14
Sum	158.95		267.28

$$P_0 = 158.9 \text{ kN/m}, \quad \bar{z} = 267.3/158.9 = 1.68 \text{ m above the base}$$

Example 2

A smooth vertical wall 6 m high retains a dry sand with $\gamma = 17 \text{ kN/m}^3$ and $\phi' = 36^\circ$. Determine (a) the Rankine active force and its location, and (b) the Rankine passive force and its location, and the ratio P_p/P_a .

Solution

$$K_a = \tan^2(45 - 18) = 0.260, \quad K_p = \tan^2(45 + 18) = 3.852$$

$$\sigma'_a(z=6) = 0.260 \times 17 \times 6 = 26.5 \text{ kPa}, \quad P_a = \frac{1}{2}(26.5)(6) = 79.4 \text{ kN/m}$$

$$\sigma'_p(z=6) = 3.852 \times 17 \times 6 = 392.9 \text{ kPa}, \quad P_p = \frac{1}{2}(392.9)(6) = 1178.7 \text{ kN/m}$$

Both resultants act at $H/3 = 2.0 \text{ m}$ above the base. The ratio $P_p/P_a = K_p/K_a = 14.8$. This large ratio is the reason why a small embedment in front of a wall gives significant resistance, provided the wall can move enough to mobilize it.

Example 3

A smooth wall 7 m high retains two sand layers. Layer 1 (0–3 m): $\phi' = 30^\circ$, $\gamma = 16.5 \text{ kN/m}^3$. Layer 2 (3–7 m): $\phi' = 34^\circ$, $\gamma_{\text{sat}} = 19.0 \text{ kN/m}^3$. The water table is at the interface (3 m). A surcharge of 15 kPa acts on the surface. Find the Rankine active force, including water, and its location.

Solution

$$K_{a1} = 0.333, \quad K_{a2} = 0.283, \quad \gamma'_2 = 9.19 \text{ kN/m}^3$$

$$z=0: \sigma'_a = 0.333(15) = 5.00 \text{ kPa}$$

$$z=3^-: \sigma'_a = 0.333(15 + 49.5) = 21.50; \quad z=3^+: \sigma'_a = 0.283(64.5) = 18.24 \text{ kPa}$$

$$z=7: \sigma'_a = 0.283(64.5 + 9.19 \times 4) = 28.63 \text{ kPa}, \quad u = 9.81 \times 4 = 39.24 \text{ kPa}$$

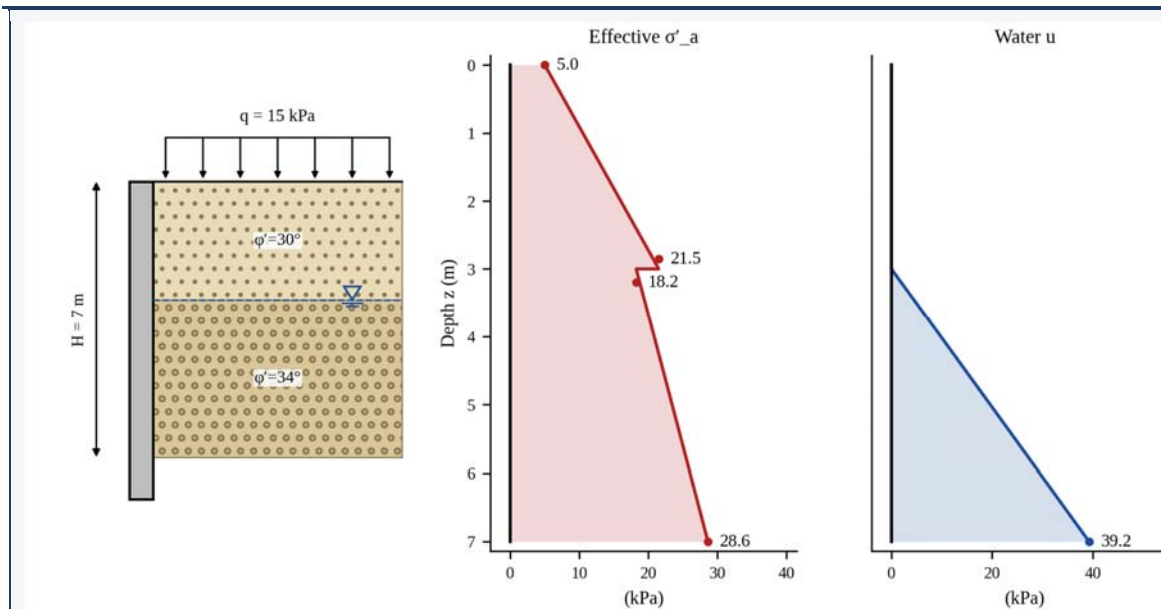


Figure 7: Example 3: active pressure for layered backfill with water table and surcharge.

Part	Force (kN/m)	Arm above base (m)	Moment (kN·m/m)
Rect. layer 1	15.00	5.50	82.50
Tri. layer 1	24.75	5.00	123.75
Rect. layer 2	72.94	2.00	145.88
Tri. layer 2	20.79	1.33	27.71
Water	78.48	1.33	104.64
Sum	211.96		484.48

$$P_a = 212.0 \text{ kN/m}, \quad \bar{z} = 484.5/212.0 = 2.29 \text{ m above the base}$$

The water force (78.5 kN/m) is 37 % of the total. If the backfill were fully drained (soil weight 19.0 kN/m^3 with no pore pressure), the active force would be 155.7 kN/m, a reduction of 27 %.

Example 4

A 2.5 m wide access road carrying $q = 40 \text{ kPa}$ runs parallel to a rigid wall 5 m high. The near edge of the road is 1.0 m from the wall. Compute the lateral pressure at $z = 1, 2, 3$ and 5 m, the resultant force and its height above the base.

Solution

z (m)	θ_1 (rad)	θ_2 (rad)	β (rad)	α (rad)	σ_h (kPa)
1	0.785	1.292	0.507	1.039	18.92
2	0.464	1.052	0.588	0.758	14.19
3	0.322	0.862	0.540	0.592	8.82
5	0.197	0.611	0.413	0.404	3.46

Numerical integration of σ_h over the height with 500 intervals gives:

$$P = \int_0^H \sigma_h dz = 52.6 \text{ kN/m}, \quad \bar{z} = H - \frac{\int \sigma_h z dz}{P} = 3.01 \text{ m above the base}$$

This force is added to the active or at-rest force of the backfill. Its arm (3.01 m) is larger than $H/3$, so it has a relatively large overturning effect.

Example 5

The toe of a wall is embedded 3 m in sand ($\phi' = 30^\circ$, $\gamma = 18 \text{ kN/m}^3$, $\gamma_{\text{sat}} = 20 \text{ kN/m}^3$). The water table is 1 m below the ground surface in front of the wall. Compute the Rankine passive force including the water pressure on the front face.

Solution

$$K_p = 3.0; \quad z = 1: \sigma'_p = 3(18)(1) = 54.0; \quad z = 3: \sigma'_p = 54.0 + 3(10.19)(2) = 115.1 \text{ kPa}$$

$$P_p = \frac{1}{2}(54.0)(1) + 54.0(2) + \frac{1}{2}(61.1)(2) + \frac{1}{2}(19.62)(2) = 215.8 \text{ kN/m}$$

The water pressure on the front face (19.6 kN/m) is balanced by an equal water pressure on the back face only when the water levels on both sides are the same. When they differ, the net water pressure must be included.

Example 6

For the wall of Example 2, compare the at-rest force with the active force. The wall is a basement wall connected to floor slabs. Which value should be used in design?

Solution

$$K_0 = 1 - \sin 36^\circ = 0.412, \quad P_0 = \frac{1}{2}(0.412)(17)(6^2) = 126.1 \text{ kN/m} \quad (P_a = 79.4 \text{ kN/m})$$

$P_0/P_a = 1.59$. The floor slabs prevent the wall from moving, so the active state never develops. The basement wall must be designed for the at-rest force. A free-standing cantilever wall can rotate by more than $\Delta/H \approx 0.001$ and is designed for the active force.

Key Terms (English – Arabic)

English term	المصطلح العربي
Lateral earth pressure	ضغط التربة الجانبي
At-rest pressure	الضغط في حالة السكون
Active pressure	الضغط الفعال (الإيجابي)
Passive pressure	الضغط السلبي (المقاوم)
Backfill	الردم / التربة المسنودة
Surcharge	الحمل الإضافي
Plastic equilibrium	التوازن اللدن
Failure plane	مستوى الفشل
Wall friction	احتكاك الجدار
Overconsolidation ratio	نسبة الانضمام المسبق
Hydrostatic pressure	الضغط الهيدروستاتيكي
Resultant force	القوة المحصلة

References for Further Reading

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- Mayne, P.W. and Kulhawy, F.H. (1982). K_0 –OCR relationships in soil. *Journal of the Geotechnical Engineering Division, ASCE*, 108(GT6), 851–872.
- Bowles, J.E. (1996). *Foundation Analysis and Design*, 5th ed. McGraw-Hill, New York.

Homework No. 2

Answer all questions. Show every step, draw the pressure or force diagrams, and state units. Take $\gamma_w = 9.81 \text{ kN/m}^3$ unless stated otherwise.

- Q1. Explain with sketches why the passive state needs a larger wall movement than the active state. Give typical values of Δ/H for dense sand.
- Q2. A rigid wall 4.5 m high retains an overconsolidated sand ($\phi' = 32^\circ$, $\text{OCR} = 3$, $\gamma = 18 \text{ kN/m}^3$). Compute K_0 , draw the at-rest diagram and find P_0 and its location. Repeat for $\text{OCR} = 1$.
- Q3. A smooth wall 8 m high retains sand with $\phi' = 33^\circ$ and $\gamma = 17.8 \text{ kN/m}^3$. The water table rises to 3 m below the surface; $\gamma_{\text{sat}} = 20 \text{ kN/m}^3$. Draw the active pressure diagrams (effective, water, total) and find P_a and its location.
- Q4. Repeat Q3 with a surcharge of 20 kPa. By what percentage does the overturning moment about the base increase?
- Q5. A wall 6 m high retains 2.5 m of loose sand ($\phi' = 28^\circ$, $\gamma = 16 \text{ kN/m}^3$) over 3.5 m of dense sand ($\phi' = 38^\circ$, $\gamma = 18.5 \text{ kN/m}^3$). No water table. Draw the active diagram showing the jump at the interface. Find P_a and $\bar{z}$.
- Q6. Derive $K_p = \tan^2(45^\circ + \phi'/2)$ from the Mohr circle. Show on the sketch the orientation of the failure planes.
- Q7. The front of a wall is embedded 1.5 m in a sand with $\phi' = 30^\circ$, $\gamma = 18 \text{ kN/m}^3$. Find the Rankine passive force. If only 50% of it can be relied upon, what is the design value?