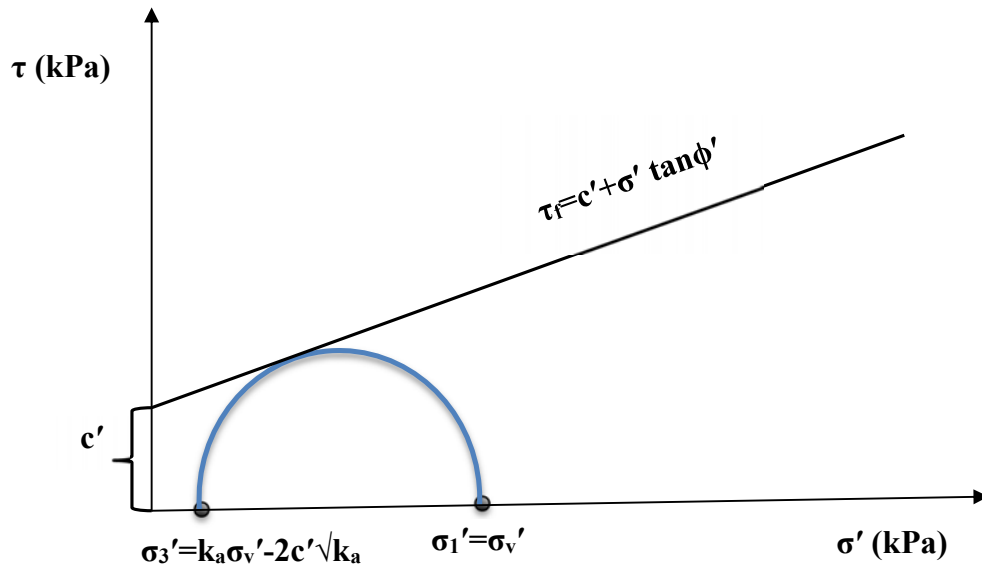


Lateral Earth Pressure II: Cohesive Soils, Coulomb Theory, and Sloping Backfill

1 Rankine Pressures in $c'-\phi'$ Soils

Cohesion reduces the active pressure and increases the passive pressure. In the active state the horizontal stress is the minor principal stress of a Mohr circle tangent to the envelope $\tau = c' + \sigma' \tan \phi'$. The geometry of the circle gives:



Mohr circle in the Rankine active state for a $c'-\phi'$ soil ($c' = 20$ kPa, $\phi' = 25^\circ$, $\sigma'_v = 120$ kPa).

$$\sigma'_a = K_a \sigma'_v - 2c' \sqrt{K_a} \quad (1)$$

$$\sigma'_p = K_p \sigma'_v + 2c' \sqrt{K_p} \quad (2)$$

Tension Crack

Near the surface the computed active pressure is negative (tension). Soil cannot sustain tension against a wall; it separates and a crack opens. The depth at which $\sigma'_a = 0$ is the depth of the tension crack:

$$z_c = \frac{2c'}{\gamma \sqrt{K_a}} - \frac{q}{\gamma} \quad (3)$$

For an undrained clay ($\phi = 0$, $K_a = 1$, $c = c_u$) with no surcharge, $z_c = 2c_u/\gamma$. Two design rules follow:

1. **Ignore the negative part** of the diagram. The active force is the area of the positive triangle below z_c :
2. **Fill the crack with water** when rain can enter it. Add the hydrostatic pressure $\gamma_w z_c$ over the crack depth. Many designers also use a minimum pressure of about $5\gamma z$ to cover cracks and softening.

$$P_a = \frac{1}{2}(H - z_c)(K_a \gamma H - 2c' \sqrt{K_a}) \quad (q=0) \quad (4)$$

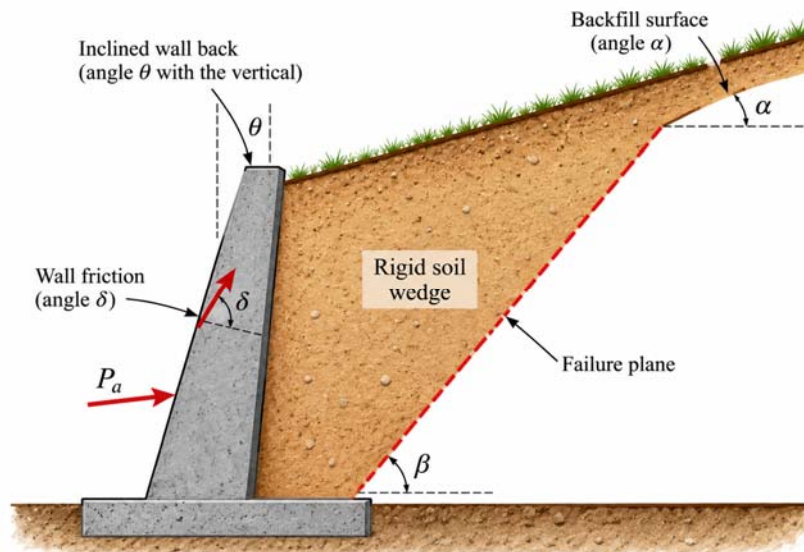
Short-Term and Long-Term Conditions in Clay

Immediately after construction a saturated clay backfill behaves undrained ($\phi_u = 0$, $c = c_u$). With time, pore pressures dissipate and the clay is governed by c' and ϕ' . In clays c' is small, so the long-term active pressure is usually **larger** than the short-term pressure. Walls retaining clay must be checked for both conditions. Clean granular backfill is preferred because it drains, does not crack, and does not swell.

2 Coulomb Theory

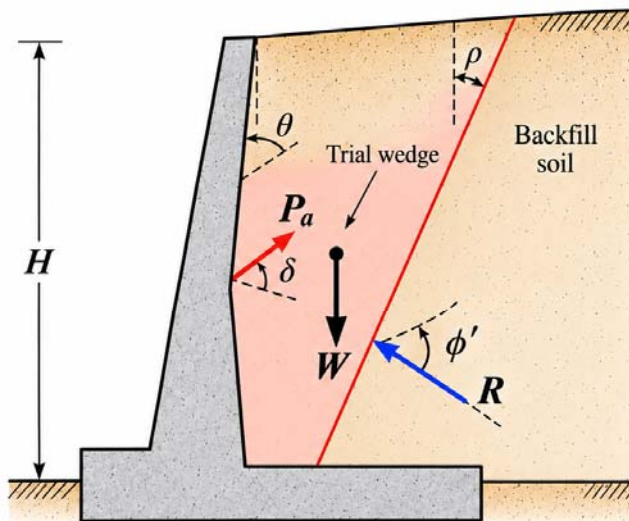
Assumptions

Coulomb (1776) considered the equilibrium of a rigid soil wedge sliding on a plane surface. Unlike Rankine, the theory includes wall friction δ , a battered wall back (angle θ with the vertical) and a sloping backfill (angle α).

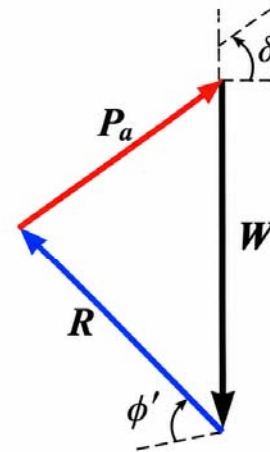


1. The soil is homogeneous, isotropic and cohesionless.
2. The failure surface is a plane passing through the heel of the wall.
3. The wedge is a rigid body in limiting equilibrium; friction is fully mobilised on the plane and on the wall.
4. The wall moves enough to reach the active (or passive) state.

Three forces act on the wedge: its weight W , the reaction R on the failure plane (inclined ϕ' to the normal), and the wall force P (inclined δ to the normal of the wall back). Their directions are known, so the force polygon gives P for each trial plane. The active force is the **maximum** P over all planes; the passive force is the **minimum**.



(a) Trial wedge behind a rough battered wall



(b) Closed force polygon

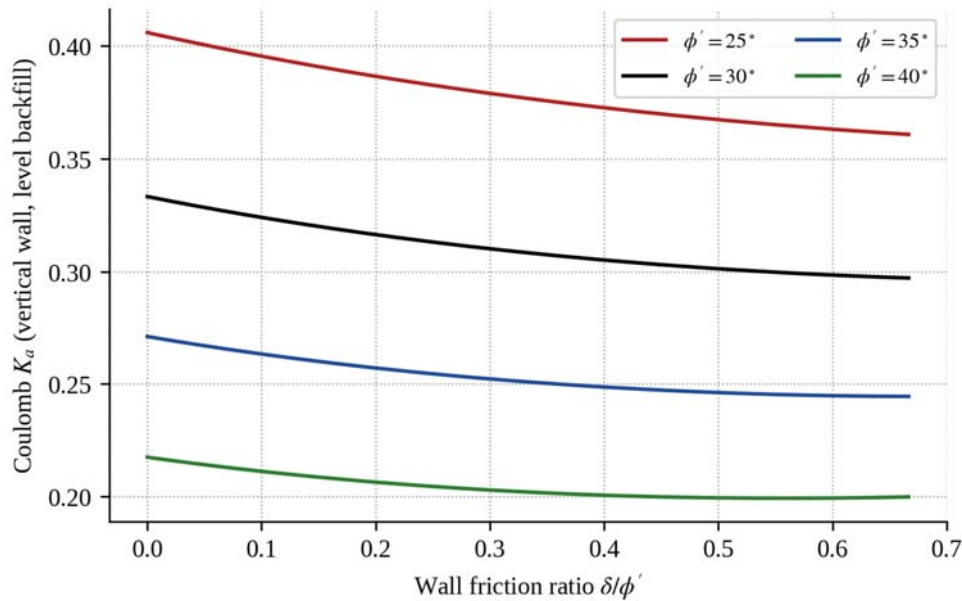
Coulomb active wedge and its force polygon.

Closed-Form Coefficients

$$K_a = \frac{\cos^2(\phi' - \theta)}{\cos^2 \theta \cos(\delta + \theta) \left[1 + \sqrt{\frac{\sin(\delta + \phi') \sin(\phi' - \alpha)}{\cos(\delta + \theta) \cos(\theta - \alpha)}} \right]^2} \quad (5)$$

$$K_p = \frac{\cos^2(\phi' + \theta)}{\cos^2 \theta \cos(\delta - \theta) \left[1 - \sqrt{\frac{\sin(\delta + \phi') \sin(\phi' + \alpha)}{\cos(\delta - \theta) \cos(\theta - \alpha)}} \right]^2} \quad (6)$$

The force is $P_a = \frac{1}{2}K_a\gamma H^2$ (or $P_p = \frac{1}{2}K_p\gamma H^2$). It acts at $H/3$ above the heel and is inclined at δ to the normal of the wall back. For $\theta = 0$, $\alpha = 0$ and $\delta = 0$ the expressions reduce to Rankine. Typical values of δ : $\frac{1}{2}\phi'$ to $\frac{2}{3}\phi'$ for concrete on sand.



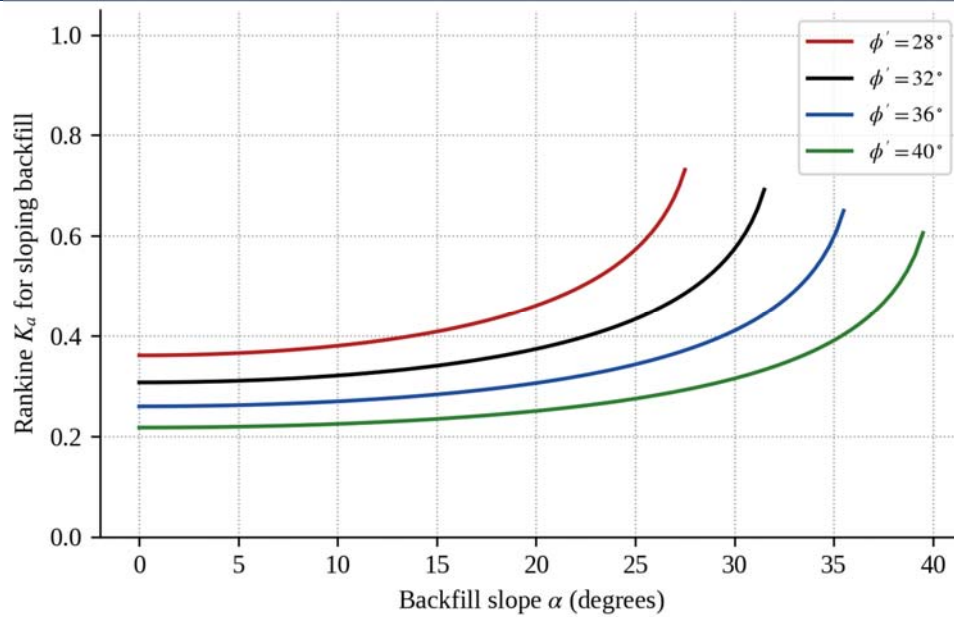
Effect of wall friction on the Coulomb active coefficient.

3 Rankine Pressure for a Sloping Backfill

For a cohesionless backfill sloping at α ($\alpha < \phi'$), Rankine pressure on a vertical plane is parallel to the ground surface (inclined at α). On a vertical plane through the heel:

$$K_a = \cos \alpha \frac{\cos \alpha - \sqrt{\cos^2 \alpha - \cos^2 \phi'}}{\cos \alpha + \sqrt{\cos^2 \alpha - \cos^2 \phi'}}, \quad P_a = \frac{1}{2} K_a \gamma H'^2 \quad (7)$$

H' is the height of the vertical plane through the heel, measured to the sloping surface. P_a has a horizontal component $P_a \cos \alpha$ and a vertical component $P_a \sin \alpha$. The vertical component helps stability against overturning. The following figure shows that K_a rises sharply as α approaches ϕ' .



Rankine active coefficient for a sloping cohesionless backfill.

4 Layered Backfill with Cohesive Layers

When a sand layer overlies a clay layer, K and c change at the interface. Compute σ'_a just above the interface with the sand parameters and just below it with the clay parameters ($K_a\sigma'_v - 2c'\sqrt{K_a}$). A negative value below the interface means that the clay stands unsupported over that depth; the pressure there is taken as zero.

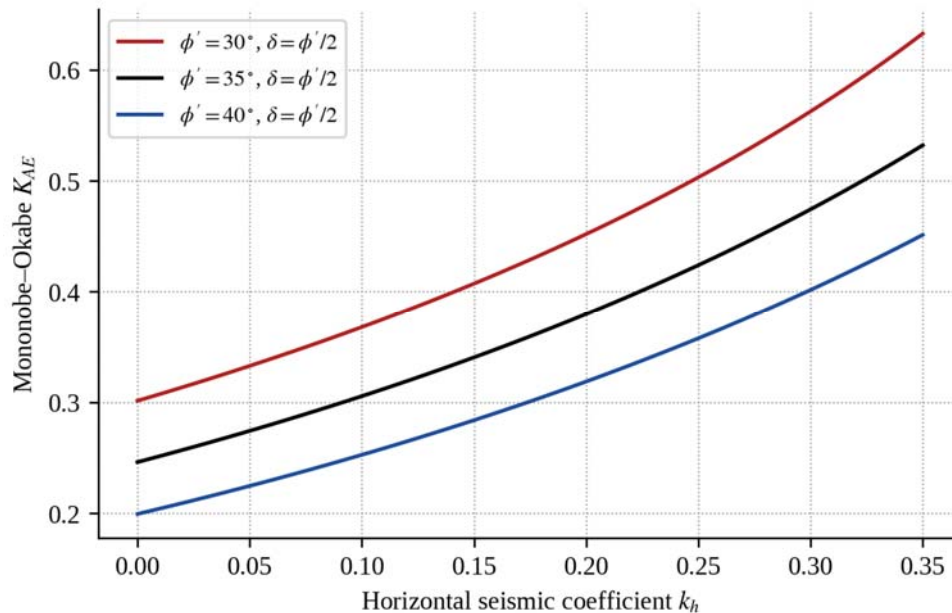
5 Earthquake Loading: Mononobe–Okabe Method

Parts of northern and eastern Iraq lie in seismically active zones. During an earthquake the active wedge is accelerated, and the thrust on the wall increases. The Mononobe–Okabe (M–O) method extends Coulomb theory by adding the inertia forces $k_h W$ and $k_v W$ to the wedge, where k_h and k_v are the horizontal and vertical seismic coefficients. The total (static + dynamic) active force is:

$$P_{AE} = \frac{1}{2} \gamma H^2 (1 - k_v) K_{AE}, \quad \psi = \tan^{-1} \frac{k_h}{1 - k_v} \quad (8)$$

$$K_{AE} = \frac{\cos^2(\phi' - \theta - \psi)}{\cos \psi \cos^2 \theta \cos(\delta + \theta + \psi) \left[1 + \sqrt{\frac{\sin(\phi' + \delta) \sin(\phi' - \alpha - \psi)}{\cos(\delta + \theta + \psi) \cos(\theta - \alpha)}} \right]^2} \quad (9)$$

The equation has no real solution when $\phi' - \alpha - \psi < 0$: the backfill itself is unstable. The dynamic increment $\Delta P_{AE} = P_{AE} - P_a$ is commonly assumed to act at about $0.6H$ above the base. The following figure shows the rapid increase of K_{AE} with k_h .



Mononobe-Okabe active coefficient for a vertical wall with level backfill ($k_v = 0$).

6 Selecting a Method

Choice of earth pressure method.

Situation	Recommended method
Vertical smooth wall, level granular backfill	Rankine (K_a , K_p)
Cantilever wall, pressure on vertical plane through heel	Rankine (sloping backfill if $\alpha \neq 0$)
Gravity wall with rough battered back	Coulomb K_a with $\delta \approx \frac{2}{3}\phi'$
Passive resistance in front of wall	Rankine K_p , reduced by a factor (often 50%)
Clay backfill	Rankine with c' ; check undrained and drained; tension crack
Irregular backfill, line loads	Trial wedge (graphical or numerical)

Tutorial: Worked Examples**Example 1**

A smooth wall 6.7 m high retains a clayey sand with $c' = 19.2$ kPa, $\phi' = 28^\circ$, $\gamma = 18.1$ kN/m³. A surcharge $q = 24$ kPa acts on the backfill (this is the SI version of the 22 ft wall with $q = 500$ psf and $c = 400$ psf of the earlier course notes). Draw the active pressure diagram, find the tension-crack depth and the active force.

Solution:

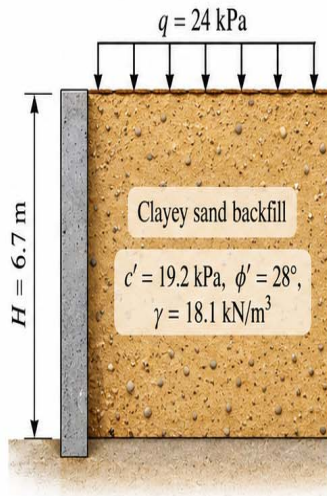
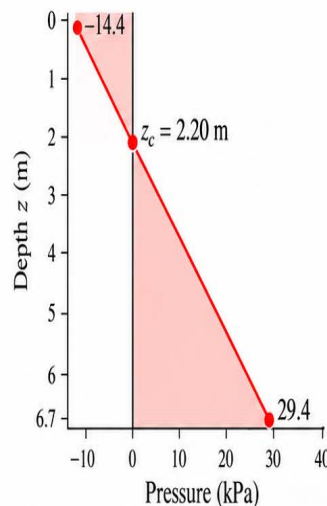
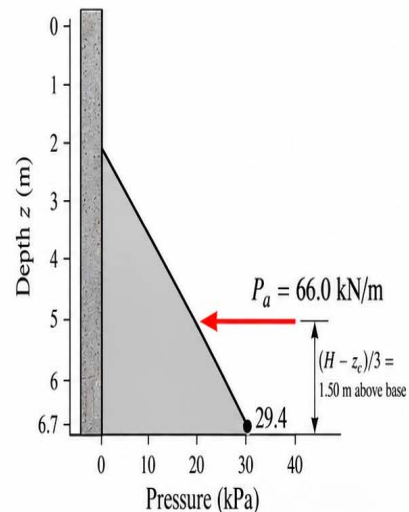
$$K_a = \tan^2(45 - 14) = 0.361, \quad \sqrt{K_a} = 0.601$$

$$z = 0: \sigma'_a = 24(0.361) - 2(19.2)(0.601) = -14.41 \text{ kPa (tension)}$$

$$z = H: \sigma'_a = (24 + 18.1 \times 6.7)(0.361) - 23.07 = 29.37 \text{ kPa}$$

$$z_c = \frac{2(19.2)}{18.1(0.601)} - \frac{24}{18.1} = 2.20 \text{ m}$$

$$P_a = \frac{1}{2}(29.37)(6.7 - 2.20) = 66.0 \text{ kN/m, at } (H - z_c)/3 = 1.50 \text{ m above the base}$$

1. Retaining wall and backfill**2. Computed σ'_a** **3. Design (tension ignored)****Active pressure with a tension crack.**

A conservative alternative ignores cohesion completely ($c' = 0$): $P_a = qK_aH + \frac{1}{2}K_a\gamma H^2 = 204.7$ kN/m. The difference (210%) shows how strongly the result depends on c' . Because cohesion of a backfill is unreliable, many designers use the $c' = 0$ value.

Example 2

A wall 5 m high retains a saturated clay backfill ($\gamma = 18.5 \text{ kN/m}^3$). Undrained: $c_u = 30 \text{ kPa}$, $\phi_u = 0$. Drained: $c' = 5 \text{ kPa}$, $\phi' = 24^\circ$. Compare the short-term and long-term active forces (tension ignored). Then add water in the crack for the short-term case.

Solution:

Short term: $K_a = 1$, $z_c = 2(30)/18.5 = 3.24 \text{ m}$, $\sigma_a(H) = 18.5(5) - 60 = 32.5 \text{ kPa}$

$$P_{a,u} = \frac{1}{2}(32.5)(5 - 3.24) = 28.5 \text{ kN/m}$$

Water in crack: $P_w = \frac{1}{2}(9.81)(3.24)^2 = 51.6 \text{ kN/m} \Rightarrow P_{total} = 80.1 \text{ kN/m}$

Long term: $K_a = 0.422$, $z_c = 0.83 \text{ m}$, $\sigma'_a(H) = 32.52 \text{ kPa}$, $P_{a,d} = 67.8 \text{ kN/m}$

The long-term force (67.8 kN/m) is 2.4 times the short-term force without water. The drained condition governs, as expected for clay backfills.

Example 3

A rough vertical wall 5 m high retains a level sand ($\gamma = 18 \text{ kN/m}^3$, $\phi' = 32^\circ$, $\delta = 20^\circ$). (a) Find the Coulomb active force. (b) Compare with Rankine.

Solution:

Step 1: Calculate the Coulomb active earth pressure coefficient

For a vertical retaining wall supporting horizontal cohesionless backfill, Coulomb's active earth pressure coefficient is:

$$K_a = \frac{\cos^2 \phi'}{\cos \delta \left[1 + \sqrt{\frac{\sin(\phi' + \delta) \sin \phi'}{\cos \delta}} \right]^2}$$

Substituting:

$$K_a = \frac{\cos^2 32}{\cos 20 \left[1 + \sqrt{\frac{\sin(32 + 20) \sin 32}{\cos 20}} \right]^2} = 0.2755$$

Step 2: Calculate the total active earth pressure

$$P_a = \frac{1}{2} K_a \gamma H^2 = \frac{1}{2} (0.2755)(18)(5)^2 = 62 \frac{kN}{m}$$

Step 3: Calculate horizontal and vertical components, Because the retaining wall is rough, the resultant active pressure is inclined at ($\delta=20^\circ$) below the horizontal.

Horizontal component: $P_{aH} = P_a \cos \delta = 62 * \cos 20^\circ = 58.23 \frac{kN}{m}$

Vertical component: $P_{aV} = P_a \sin \delta = 62 * \sin 20^\circ = 21.2 \frac{kN}{m}$ (downward)

Step 4: Determine the point of application

The active earth pressure distribution is triangular. Therefore, the resultant acts at:

$$y = \frac{H}{3} = \frac{5}{3} = 1.67 \text{ m above the base}$$

(b) Comparison With Rankine's Theory

Step 1: Calculate Rankine's coefficient

For horizontal cohesionless backfill:

$$K_{a,R} = \tan^2 \left(45^\circ - \frac{\phi'}{2} \right) = \tan^2 \left(45^\circ - \frac{32}{2} \right) = 0.3073$$

Step 2: Calculate Rankine's active thrust

$$P_{a,R} = \frac{1}{2} K_{a,R} \gamma H^2 = \frac{1}{2} 0.3073 * 18 * 5^2 = 69.13 \frac{kN}{m}$$

Because Rankine's theory assumes a smooth vertical wall in this case, the force acts horizontally.

Example 4

The vertical plane through the heel of a cantilever wall has $H' = 7.16$ m (stem 6.0 m + base 0.7 m + backfill rise $2.6 \tan 10^\circ = 0.46$ m). The backfill slopes at $\alpha = 10^\circ$; $\gamma = 18$ kN/m³, $\phi' = 30^\circ$. Find P_a and its components.

Solution:

Step 1: Calculate the effective height

The effective height is measured along the imaginary vertical plane passing through the heel of the wall.

$$H' = 6.0 + 0.7 + 2.6 \tan 10^\circ = 7.16 \text{ m}$$

Step 2: Calculate Rankine's active earth pressure coefficient

For a vertical plane retaining a sloping cohesionless backfill:

$$K_a = \cos \alpha \left[\frac{\cos \alpha - \sqrt{\cos^2 \alpha - \cos^2 \phi'}}{\cos \alpha + \sqrt{\cos^2 \alpha - \cos^2 \phi'}} \right]$$

Substituting ($\alpha=10^\circ$) and ($\phi'=30^\circ$):

$$K_a = \cos 10^\circ \left[\frac{\cos 10^\circ - \sqrt{\cos^2 10^\circ - \cos^2 30^\circ}}{\cos 10^\circ + \sqrt{\cos^2 10^\circ - \cos^2 30^\circ}} \right] = 0.3495 \approx 0.35$$

Step 3: Calculate the total active earth pressure

The total active thrust per unit length of the retaining wall is:

$$P_a = \frac{1}{2} K_a \gamma H'^2 = \frac{1}{2} 0.3495 * 18 * 7.16^2 = 161.27 \frac{kN}{m}$$

Step 4: Calculate horizontal and vertical components

According to Rankine's theory for a sloping backfill, the active earth pressure resultant acts parallel to the backfill surface, inclined at ($\alpha=10^\circ$) below the horizontal.

$$\text{Horizontal component: } P_{aH} = P_a \cos \alpha = 161.27 * \cos 10^\circ = 158.82 \frac{kN}{m}$$

$$\text{Vertical component: } P_{aV} = P_a \sin \alpha = 161.27 * \sin 10^\circ = 28 \frac{kN}{m}$$

The vertical component acts downward on the imaginary vertical plane through the heel.

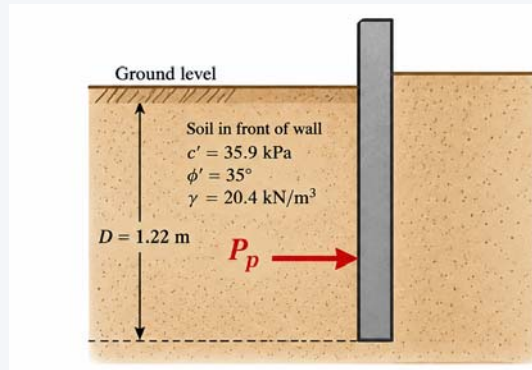
Step 5: Location of the resultant force

Because the lateral earth pressure distribution is triangular, the resultant acts at one-third of the effective height above the bottom of the imaginary plane.

$$y = \frac{H'}{3} = \frac{7.16}{3} = 2.39 \text{ m}$$

Example 5

The soil in front of a wall has $c' = 35.9 \text{ kPa}$, $\phi' = 35^\circ$, $\gamma = 20.4 \text{ kN/m}^3$. The embedment is 1.22 m. Draw the Rankine passive diagram and compute P_p .

**Solution:**

Step 1: Passive earth pressure coefficient

$$K_p = \tan^2 \left(45^\circ + \frac{\phi'}{2} \right) = \tan^2(62.5^\circ) = 3.69$$

Step 2: Pressure at the top of the embedded depth

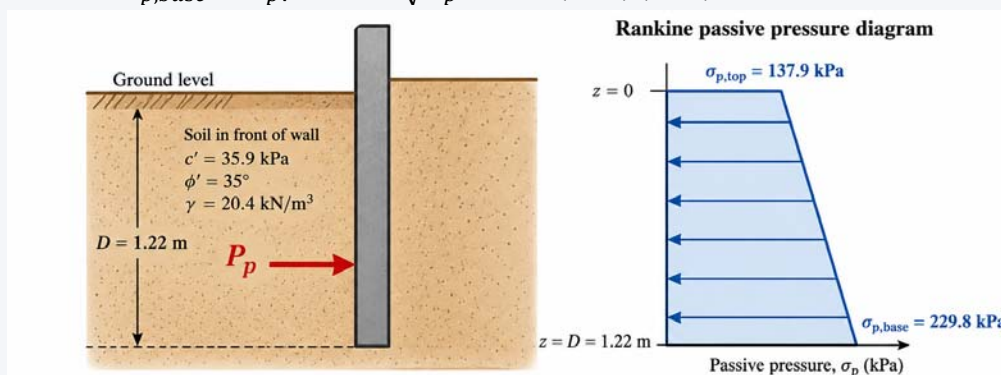
At ($z=0$),

$$\sigma_{p,top} = 2c' \sqrt{K_p} = 2(35.9)(\sqrt{3.69}) = 137.9 \text{ kPa}$$

Step 3: Pressure at the base of embedment

At ($z=D=1.22 \text{ m}$),

$$\sigma_{p,base} = K_p \gamma D + 2c' \sqrt{K_p} = 3.69(20.4)(1.22) + 137.9 = 229.8 \text{ kPa}$$



So, the Rankine passive pressure diagram is a trapezoid, starting at 137.9 kPa at the ground level and increasing linearly to 229.8 kPa at the bottom of the embedment.

Step 4: Resultant passive force (P_p)

Area of trapezoid:

$$P_p = \frac{\sigma_{p,top} + \sigma_{p,base}}{2} D = \frac{137.9 + 229.8}{2} (1.22) = 224.3 \frac{kN}{m}$$

Step 5: Point of application

Split the trapezoid into:

1. rectangle: (137.9 * 1.22)
2. triangle: ((229.8-137.9)/2) * 1.22 = 91.9 kPa over 1.22 m

The resultant acts at: 0.661 m below ground level or 0.559 m above the bottom of embedment.

Example 6

A wall 7 m high retains 3 m of sand ($\gamma = 17 \text{ kN/m}^3$, $\phi' = 32^\circ$) over 4 m of saturated clay ($\gamma_{sat} = 19 \text{ kN/m}^3$, $c' = 15 \text{ kPa}$, $\phi' = 22^\circ$, long-term). The water table is at the sand-clay interface.

Draw the active pressure diagram and compute the total thrust.

Solution

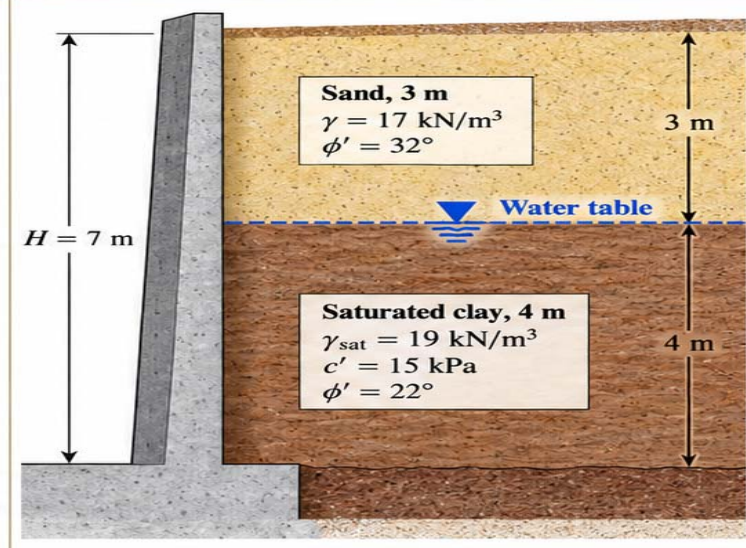
$$K_{a1} = 0.307, K_{a2} = 0.455, \sigma'_v(3) = 51.0 \text{ kPa}, \sigma'_v(7) = 87.8 \text{ kPa}$$

$$z = 3^- : \sigma'_a = 0.307(51.0) = 15.67; \quad z = 3^+ : \sigma'_a = 0.455(51.0) - 2(15)\sqrt{0.455} = 2.97 \text{ kPa}$$

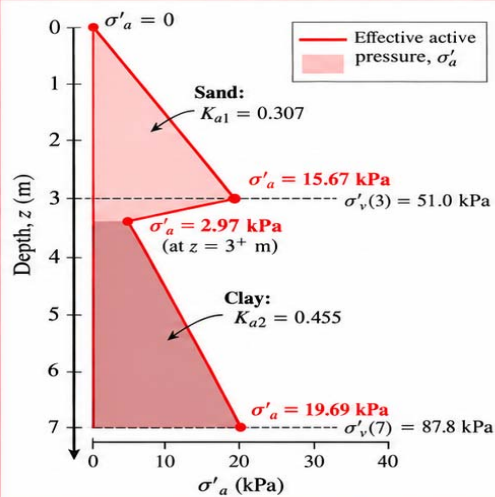
$$z = 7 : \sigma'_a = 0.455(87.8) - 20.24 = 19.69 \text{ kPa}, \quad u = 39.24 \text{ kPa}$$

The clay pressure is positive over the whole layer.

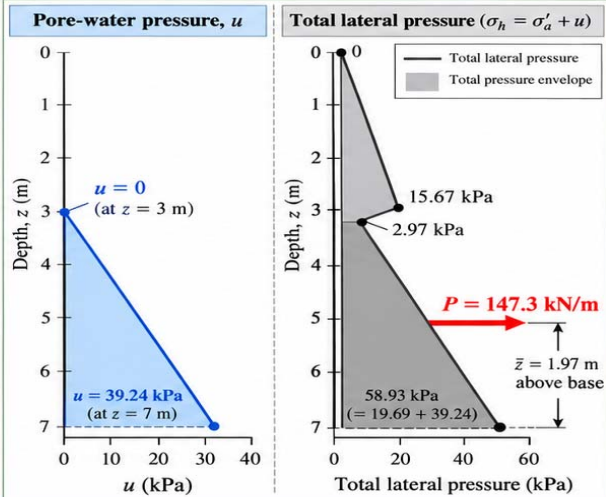
1. Retaining wall and soil profile



2. Effective active pressure diagram, σ'_a



3. Pore-water pressure and total lateral pressure



Active pressure for sand over clay.

$$P = 23.5 \text{ (sand)} + 45.3 \text{ (clay)} + 78.5 \text{ (water)} = 147.3 \text{ kN/m}, \quad \bar{z} = 1.97 \text{ m}$$

Key Terms (English – Arabic)

English term	المصطلح العربي
Cohesion	التماسك
Angle of internal friction	زاوية الاحتكاك الداخلي
Tension crack	شق الشد
Undrained condition	الحالة غير المصرفة
Drained (long-term) condition	الحالة المصرفة (طويلة الأمد)
Coulomb wedge	إسفين كولوم
Trial wedge method	طريقة الإسفين التجريبي
Sloping backfill	ردم مائل
Battered wall	جدار مائل الظهر
Seismic coefficient	المعامل الزلزالي
Force polygon	مضلع القوى

References for Further Reading:

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- Seed, H.B. and Whitman, R.V. (1970). Design of earth retaining structures for dynamic loads. *Proceedings, ASCE Specialty Conference on Lateral Stresses in the Ground and Design of Earth Retaining Structures*, Cornell University, Ithaca, NY, 103–147.

Homework No. 3:

- Q1. A smooth wall 6 m high retains a clay with $c' = 12$ kPa, $\phi' = 22^\circ$, $\gamma = 18.5$ kN/m³. Draw the active diagram, find z_c and P_a . Repeat with the tension crack full of water.
- Q2. Explain, with reference to effective stress, why the long-term active pressure in a clay backfill is usually larger than the short-term pressure.
- Q3. A wall 7 m high retains sand ($\gamma = 17.5$ kN/m³, $\phi' = 34^\circ$). The back is inclined 10° from the vertical (leaning into the backfill), $\delta = 2\phi'/3$, and the backfill slopes at 8° . Compute the Coulomb P_a , its horizontal and vertical components.
- Q4. Solve Q3 by the trial-wedge method using at least six trial planes. Tabulate W and P for each plane and compare the maximum with the closed-form result.
- Q5. Compute the Rankine P_a for a vertical plane 6.5 m high behind a cantilever wall with a backfill slope of 12° , $\gamma = 18$ kN/m³, $\phi' = 32^\circ$. Give P_h and P_v .
- Q6. The soil in front of a wall (embedment 1.5 m) has $c' = 20$ kPa, $\phi' = 25^\circ$, $\gamma = 19$ kN/m³. Find P_p and its line of action.
- Q7. Why is the Coulomb passive coefficient unsafe for large wall friction? What alternative do you recommend?
- Q8. A cantilever wall 6 m high retains level sand ($\gamma = 18$ kN/m³, $\phi' = 34^\circ$, $\delta = 17^\circ$). Compute the static Coulomb force and the Mononobe–Okabe force for $k_h = 0.1$ and 0.2 ($k_v = 0$). Plot P_{AE}/P_a against k_h .

