Chapter One

Course Outline

Second Semester

Course Title: Finite Mathematics II

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Stage: The First

Contents

Chapter One

Complex Numbers 1.1:

One of the advantages of dealing with the real numbers instead of the rational numbers is that certain equations which do not have any solutions in the rational numbers have a solution in real numbers. For instance, $x^2 = 2$ is such an equation. However, we also know some equations having no solution in real numbers, for instance $x^2 = -1$, or $x^2 = -2$. We define a new kind of number where such equations have solutions. The new kind of numbers will be called **complex numbers**.

Definition 1.2:

A complex number is an expression of ordered pairs (x, y) of real numbers. The set of all complex numbers are denoted by $\mathbb{C}$ or C. That is;

$$\mathbb{C} = \{z = (x, y) \in R \times R\}.$$

1-Two complex numbers are equal only when there are actually the same. That is; $(x, y) = (u, v)$ precisely when $x = u$ and $y = v$.

2- We define the sum (subtract) and product of two complex numbers:

Sum : $(x, y) + (u, v) = (x + u, y + v)$.
Subtract: $(x, y) - (u, v) = (x - u, y - v)$.
Product : $(x, y).(u, v) = (xu - yv, xv + yu)$.

Now let's consider the arithmetic of the complex numbers with second coordinate 0:
$(x, 0) + (u, 0) = (x + u, 0)$, and $(x, 0).(u, 0) = (xu, 0)$.
We simply use x as an abbreviation for $(x, 0)$ and there is no danger of confusion: $x + u$ is short-hand for $(x, 0) + (u, 0) = (x + u, 0)$ and xu is short-hand for $(x, 0).(u, 0)$.
Next, notice that
$x.(u, v) = (u, v).x = (x, 0).(u, v) = (xu, xv)$. Now then, any complex number $z = (x, y)$ may be written

$$z = (x, y) = (x, 0) + (0, y) = x + y(0, 1).$$

i- When we let $i = (0,1)$, then we have

$i^2 = (0,1).(0,1) = (-1,0)$, and we have agreed that we can safely abbreviate $(-1,0)$ as -1. Thus, $i^2 = -1$ and $i = \sqrt{-1}$ **.**

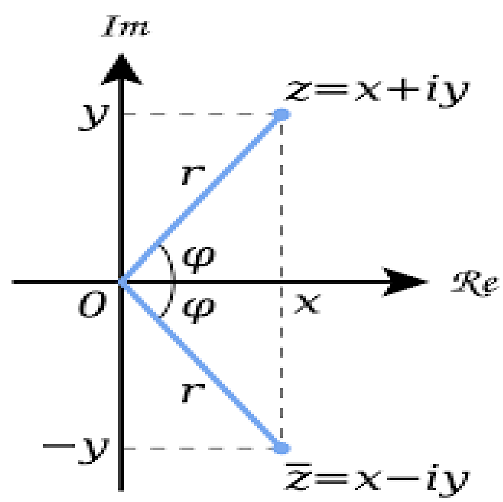
ii- $z = (x,y) = x + y(0,1) = x + iy.$

The real number x is called the **real part** denoted by **R(**z**)** and real number y is called the **imaginary part** of z, and denoted by **Im(**z**).**
Now, suppose $z = (x,y) = x + iy$ and $w = (u,v) = u + iv$. Then we have

$$zw = (x + iy).(u + iv) \\ = xu - yv + i\,(xv + yu).$$

3- Let $z = x + iy$. Define the **conjugate of** the complex number by $\bar{z}$ by

$$\bar{z} = \overline{x + iy} = x - iy.$$

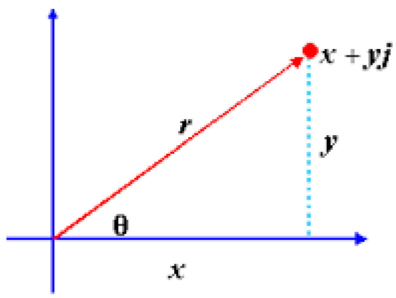


4- **Division:** Let $z = a + ib,\ w = c + id\ \ and\ c^2 + d^2 \neq 0$. Then

$$\frac{z}{w} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$
$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$
$$= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}.$$

5- The **modulus(**absolute value) of a complex number $z = x + iy$ is defined to be the nonnegative real number $r = |z| = \sqrt{x^2 + y^2}$.

If we think of z as a point in the plane (x, y), then $r = |z|$ is the length of the line segment from the origin to z.



6- Distance: the distance between two complex numbers $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ is defined as follows:

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2|.$$

<u>Example 1.3:</u>

1- Let $\alpha = 2 + 3i$ and $\beta = 1 - i$.

$$\begin{aligned} \alpha\beta = (2+3i)(1-i) &= 2(1-i) + 3i(1-i) \\ &= 2 - 2i + 3i - 3i^2 \\ &= 2 + i - 3(-1) \\ &= 2 + 3 + i \\ &= 5 + i. \end{aligned}$$

2- Write $z = \frac{1+3i}{2-5i}$ in the form $a + ib$.

Solution:

$$\frac{1+3i}{2-5i}$$

$$= \frac{1+3i}{2-5i} \cdot \frac{2+5i}{2+5i}$$

$$\frac{2+5i+6i+15i^2}{4+10i-10i-25i^2} = \frac{2+11i+15i^2}{4-25i^2}$$

$$= \frac{2+11i-15}{4+25} = \frac{-13+11i}{29}$$

$$= \frac{-13}{29} + \frac{11i}{29}$$

3- Write $z = \frac{3-5i}{2+9i}$ in the form $a + ib$.

Solution:

$$\begin{aligned}\frac{3-5i}{2+9i} \cdot \frac{2-9i}{2-9i} &= \frac{6-27i-10i+45i^2}{4-18i+18i-81i^2} \\ &= \frac{6-37i-45}{4+81} \\ &= \frac{-39-37i}{85} \\ &= -\frac{39}{85} - \frac{37}{85}i\end{aligned}$$

4- Write $(2+3i).(1-i)$ in the form $a + ib$

Solution:

$$\begin{aligned}(2+3i)(1-i) &= 2 \cdot 1 + 2 \cdot (-i) + 3i \cdot 1 + 3i \cdot (-i) \\ &= 2 - 2i + 3i - 3i^2 \\ &= 2 - 2i + 3i - 3 \cdot (-1) \\ &= 5 + i\end{aligned}$$

Polar Coordinates1.4:

Let $z = (x, y) = x + iy$ be a complex number. We know that any point in the plane can be represented by polar coordinates (r, θ).

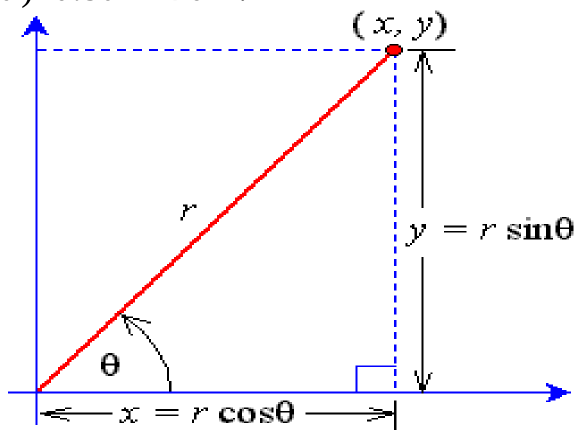
Let

$$r = |z| = \sqrt{x^2 + y^2}$$

If (r, θ) are the polar coordinates of the point (x, y) in the plane, then

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Hence, $z = r(\cos\theta + i\sin\theta) = cis\theta = re^{i\theta}$.



The number θ is called an **argument (angle)** of z and we write

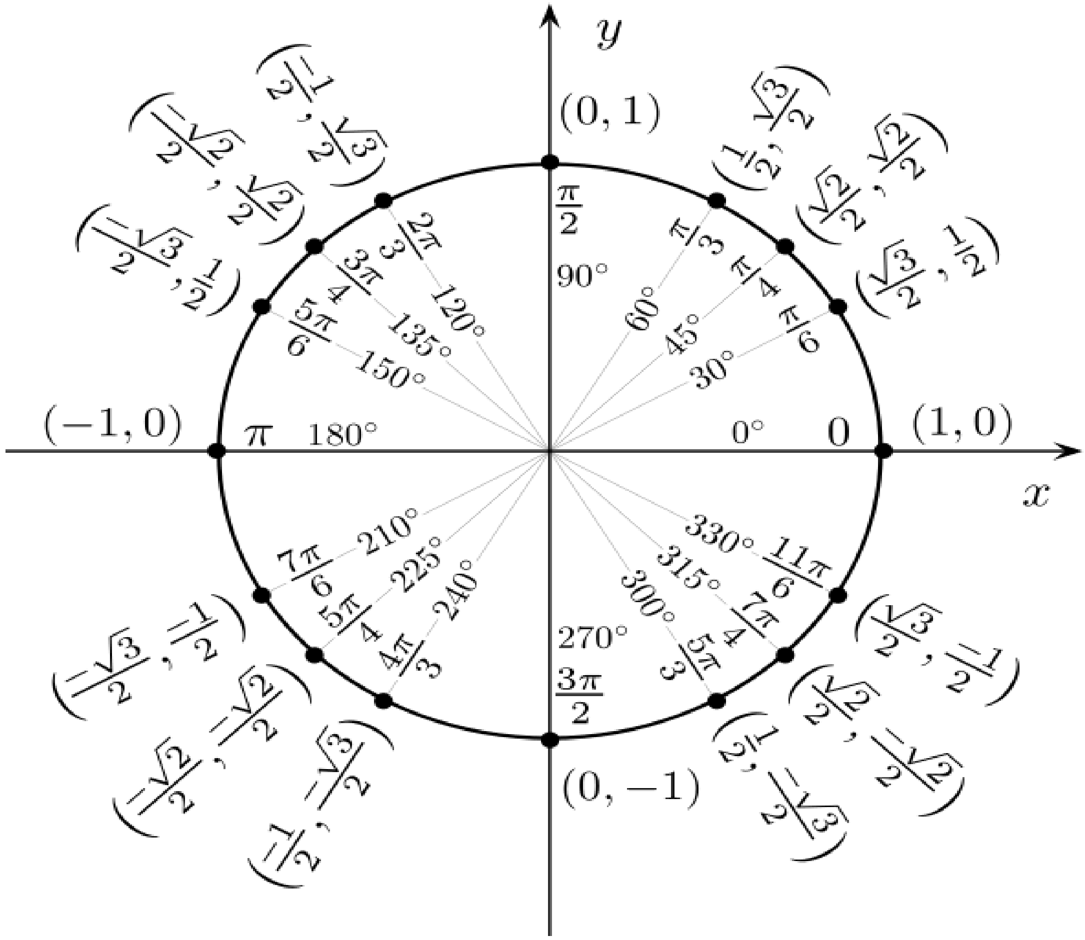
$$\theta = \arg = arctan(\frac{y}{x}) .$$

Thus a complex numbers has an infinite number of arguments, any two of which differ by an integral multiple of 2π; that is,

$$e^{i(\theta+2\pi)} = e^{i\theta}.$$

The **principal argument** of z is the unique argument that lies on the interval $[0,2\pi)$ (in some books $(-\pi, \pi]$).

Chapter One



Algebraic Properties of Complex Numbers 1.5:

1- Commutative law for addition:

$$z_1 + z_2 = z_2 + z_1.$$

2- Commutative law for multiplication:

$$z_1 z_2 = z_2 z_1.$$

3- Associative law for addition:

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

4- Associative law for multiplication:

$$z_1(z_2 z_3) = (z_1 z_2) z_3$$

5-Multiplication is distributive with respect to addition:

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

6- The product of two complex numbers is zero if and only if at least one of the factors is zero. That is; if

$$z_1z_2 = 0 \text{ then either } z_1 = 0 \text{ or } z_2 = 0.$$

7- Additive Inverses:

Any complex number z has a unique negative $-z$ such that $z + (-z) = 0$.

If $z = x + yi$, the negative $-z = -x - yi$.

8- Multiplicative Inverses:

Any nonzero complex number $z = x + yi$ has a unique inverse $\frac{1}{z} = z^{-1}$ such that

$zz^{-1} = 1$. Thus, $\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$.

9- Additive Identity.

There is a complex number w such that $z + w = z$ for all complex numbers z. The number w is the ordered pair $(0,0)$.

10- Multiplicative Identity.

There is a complex number w such that $z\,\mathrm{w} = z$ for all complex numbers z. The ordered pair $(1,0) = 1 + 0i$ is the unique complex number with this property.

11- $|z_1z_2| = |z_1|\,|z_2|$.

12- $|z| = \sqrt{z\bar{z}} = |\bar{z}|$.

13- $\overline{z_1 \mp z_2} = \bar{z_1} \mp \bar{z_2}$.

14- $\overline{z_1 . z_1} = \bar{z_1} . \bar{z_2}$ and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$, $z_2 \neq 0$.

15- $\bar{\bar{z}} = z$.

16-Triangle Inequality:

i- $|Z_1 + Z_2| \leqslant |Z_1| + |Z_2|$. (Without prove).

ii- $|Z_1 + Z_2| \geqslant |Z_1| - |Z_2|$.

iii- $|Z_1 - Z_2| \geqslant |Z_1| - |Z_2|$.

iv- $|Z_1 + Z_2 + Z_3| \leqslant |Z1| + |Z_2| + |Z_3|$.

17- If $z_1 = r_1[(\cos(\theta_1) + i(\sin(\theta_1)] = e^{i\theta_1}$ and

$z_2 = r_2[(\cos(\theta_2) + i(\sin(\theta_2)] = e^{i\theta_2}$ then

$z_1 z_2 = r_1 r_2[(\cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2)] = r_1 r_2 e^{i(\theta_1+\theta_2)}$ and

$z_1/z_2 = r_1/r_2\,[(\cos(\theta_1-\theta_2) + i(\sin(\theta_1-\theta_2)] = r_1/r_2\, e^{i(\theta_1-\theta_2)}$; $r_2 \neq 0$.

18- **De Moivre's Formula.** Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. Then

$$z^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta}.$$

Example 1.6:

1- Write $1 - i$, in the polar coordinate.

Solution:

$$1 - i = \sqrt{2}\,(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right))$$
$$= \sqrt{2}\,(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right))$$
$$= \sqrt{2}\,(\cos\left(\frac{399\pi}{4}\right) + i\sin\left(\frac{399\pi}{4}\right))$$

etc., etc., etc. Each of the numbers $\frac{7\pi}{4}$, $-\frac{\pi}{4}$, and $\frac{399\pi}{4}$ is an argument of $1 - i$.

2- Write $z = 1 - i\sqrt{3}$ in the polar coordinate.

Solution:

The point is located in Quadrant IV.

$x = 1$ and $y = -\sqrt{3}$

So, $\sin\theta = \frac{y}{r} = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{x}{r} = \frac{1}{2}$, $0 \le \theta < 2\pi$

Then $\theta = \frac{5\pi}{3}$ and $r = 2$, so the polar form is $z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

3- Write $z = (1+i)^8$ in the form $a + ib$.

Solution:

First we write $w = (1+i)$ in the polar coordinate.

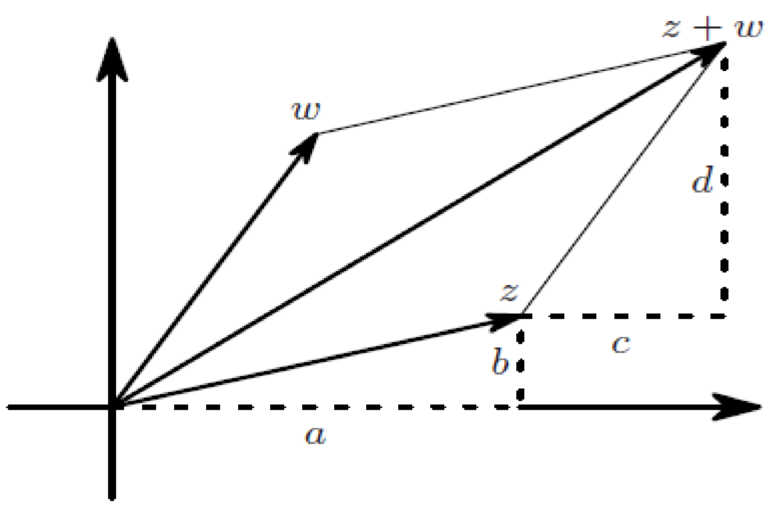
$r = |w| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$. $\theta = \arctan\left(\frac{1}{1}\right) = \arctan(1)$. Since w lies in the first quadrant, therefore, $\theta = \frac{\pi}{4} = 45^{\circ}$. So, by **De Moivre's Formula**

$$z = w^8 = r^8 cis(8\theta) = \sqrt{2}^8 cis\left(\frac{8\pi}{4}\right) = 2^4 cis(2\pi) = 16(1 + 0i) = 16.$$

Geometry of Arithmetic 1.7:

Since we can picture complex numbers as points in the complex plane, we can also try to visualize the arithmetic operations "addition" and "multiplication."

1- To add z and w one forms the parallelogram with the origin, z and w as vertices. The fourth vertex then is $z + w$.

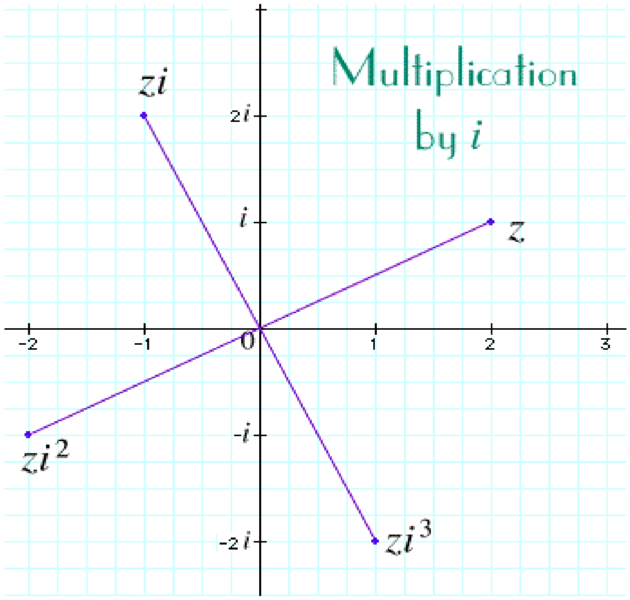


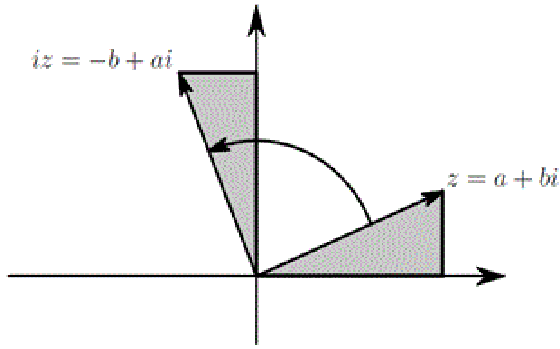
2- To understand multiplication we first look at multiplication with $\boldsymbol{i}$.

If $z = a + bi$, then

$$iz = i(a + bi) = ia + bi2 = ai - b = -b + ai.$$

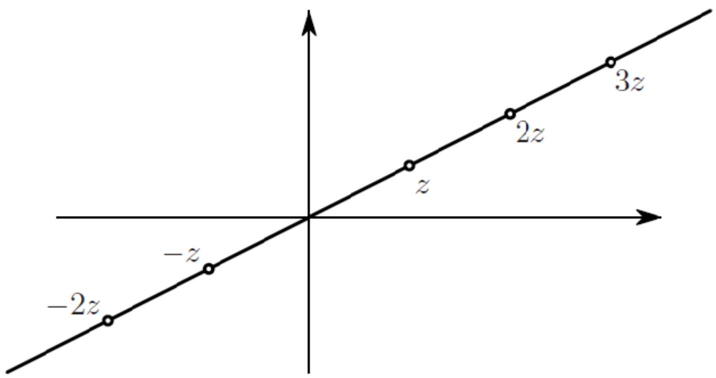
Thus, to form iz from the complex number z one rotates z counterclockwise by 90 degrees.





If a is any real number, then multiplication of $w = c + di$ by a gives

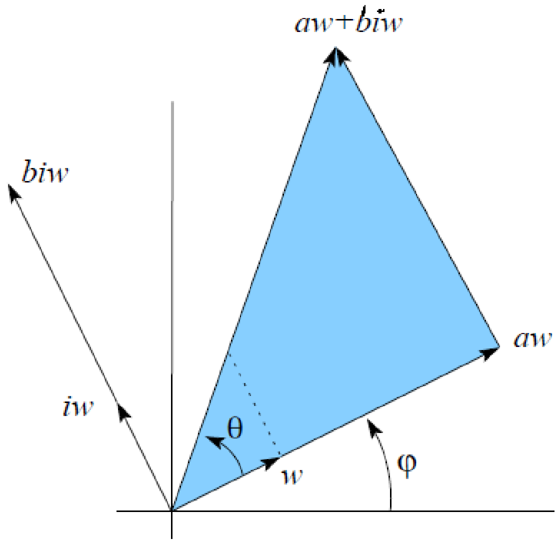
$$aw = ac + adi,$$

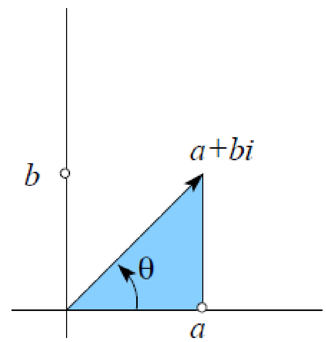


So, aw points in the same direction, but is a times as far away from the origin. If $a < 0$ then aw points in the opposite direction.

Next, to multiply $z = a + bi$ and $w = c + di$ we write the product as

$$zw = (a + bi)w = aw + biw.$$





Complex Roots of a Number 1.8:

For any given complex number $w \neq 0$ there is a method of finding all complex solutions of the equation

$$z^n = w \text{ -------------- } (1)$$

if $n = 2, 3, 4, \ldots$ is a given integer.

To find these solutions you write w in polar form, that is, you find $r > 0$ and θ such that $w = re^{i\theta}$. Then

$$z = r^{1/n} e^{i\theta/n} = \sqrt[n]{w} = w^{1/n}$$

is a solution to (1). But it isn't the only solution, because the angle θ for which $w = re^{i\theta}$ isn't unique, it is only determined up to a multiple of 2π. Thus, if we have found one angle θ for which $w = re^{i\theta}$, then we can also write

$$w = re^{i(\theta+2k\pi)}, k = 0, \mp 1, \mp 2, \ldots$$

The n^{th} roots of w are then

$$z_k = r^{1/n} e^{i(\frac{\theta}{n}+2\pi\frac{k}{n})} = r^{1/n}[\cos\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right) + i\sin(\frac{\theta}{n} + 2\pi\frac{k}{n})].$$

Here k can be any integer, so it looks as if there are infinitely many solutions. However, if you increase k by n, then the exponent above increases by $2\pi i$, and hence z_k does not change. In a formula:

$$z_n = z_0, \; z_{n+1} = z_1, \; z_{n+2} = z_2, \ldots, z_{n+k} = z_k.$$

So if you take $k = 0, 1, 2, \ldots, n-1$ then you have had all the solutions.

Example 1.9:

1- Find all sixth roots of $w = 1$.

Solution: We have to solve $z^6 = 1$. First we write 1 in polar form.

$$1 = 1 \cdot e^{0i} = 1 \cdot e^{2k\pi i}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Then we take the 6^{th} root and find

$$z_k = 1^{1/6} e^{2k\pi i/6} = e^{k\pi i/3}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

The six roots are

$$z_0 = 1 \qquad z_1 = e^{\pi i/3} = \tfrac{1}{2} + \tfrac{i}{2}\sqrt{3} \qquad z_2 = e^{2\pi i/3} = -\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3}$$

$$z_3 = -1 \qquad z_4 = e^{4\pi i/3} = -\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3} \qquad z_5 = e^{5\pi i/3} = \tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}$$

2- Find all cubic roots of $w = -1 + i$.

Solution:

Let $z^3 = w$. First we write $-1 + i$ in polar form.

$r = |w| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}.$

$\theta = \arctan\left(\frac{-1}{1}\right) = \arctan(-1)$. Since w lies in the second quadrant, therefore, $\theta = \frac{3\pi}{4} = 135^\circ$.

$$z_k = r^{1/n} e^{i(\frac{\theta}{n} + 2\pi\frac{k}{n})} = r^{1/n}[\cos\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right) + i\sin(\frac{\theta}{n} + 2\pi\frac{k}{n})].$$

n=3, k =0,1,2.

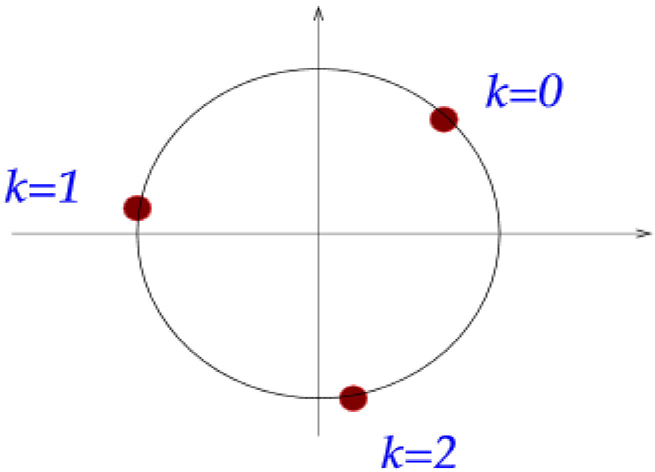
$$k = 0: \qquad z_0 = \sqrt{2}^{1/3} e^{i(\frac{\frac{3\pi}{4}}{3})} = 2^{1/6}[\cos\left(\frac{\pi}{4}\right) + i\sin(\frac{\pi}{4})].$$

$$k = 1: \qquad z_1 = \sqrt{2}^{1/3} e^{i(\frac{\pi}{4} + 2\pi\frac{1}{3})} = 2^{1/6}[\cos\left(\frac{\pi}{4} + 2\pi\frac{1}{3}\right) + i\sin(\frac{\pi}{4} + 2\pi\frac{1}{3})$$

$$= 2^{1/6}[\cos\left(\frac{11\pi}{12}\right) + i\sin(\frac{11\pi}{12})].$$

$$k = 2: \qquad z_2 = \sqrt{2}^{1/3} e^{i(\frac{\pi}{4} + 2\pi\frac{2}{3})} = 2^{1/6}[\cos\left(\frac{\pi}{4} + 2\pi\frac{2}{3}\right) + i\sin(\frac{\pi}{4} + 2\pi\frac{2}{3})$$

$$= 2^{1/6}[\cos\left(\frac{19\pi}{12}\right) + i\sin(\frac{19\pi}{12})].$$



3- Find the solution of the equation $z^5 + 32 = 0$.

Solution:

First we write $w = -32$ in polar form.

$r = |w| = \sqrt{(-32)^2 + (0)^2} = 32.$

$\theta = \arctan\left(\frac{0}{-32}\right) = \arctan(0)$. Since w lies in the second quadrant, therefore,

$\theta = \pi.$

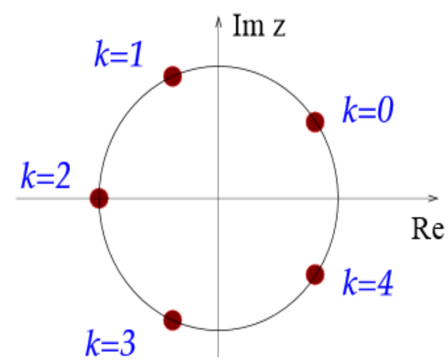
$$z_k = r^{1/n} e^{i\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right)} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right) + i\sin\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right)\right].$$

n=5, k =0,1,2,3,4.

$$(-32)^{1/5} = 32^{1/5}\left[\cos\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right)\right] \quad , \quad k = 0,1,2,3,4$$

that is,

$$k = 0: \quad 2\left(\cos\tfrac{\pi}{5} + i\sin\tfrac{\pi}{5}\right)$$
$$k = 1: \quad 2\left(\cos\tfrac{3\pi}{5} + i\sin\tfrac{3\pi}{5}\right)$$
$$k = 2: \quad -2$$
$$k = 3: \quad 2\left(\cos\tfrac{7\pi}{5} + i\sin\tfrac{7\pi}{5}\right)$$
$$k = 4: \quad 2\left(\cos\tfrac{9\pi}{5} + i\sin\tfrac{9\pi}{5}\right)$$



Chapter One

Exercise1.10:

1- Write in polar form $re^{i\theta}$, the following:
(1) i (2) -2 (3) $\sqrt{3}+3i$ (4) $-3i$ (5) $1-i\sqrt{3}$ (6) $1+i\sqrt{3}$ (7) $4i$.

2- Put the following complex numbers in the form $x+iy$.
(1) $e^{3i\pi}$ (2) $3e^{i\frac{\pi}{4}}$ (3) $\pi e^{-i\frac{\pi}{3}}$ (4) $e^{-5i\frac{\pi}{4}}$ (5) $e^{i100\pi}$.

3- Put $(-1+i)^{100}$ in the form $x+iy$.

4- Let $z_1=\sqrt{3}+3i$, $z_2=1+i\sqrt{2}$, $z_3=4i$. Find the following in the form $x+iy$.

$$\frac{z_3^2(z_1^2+z_2^2)^2}{z_1\ z_2}$$

5- Prove that $\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3=1$.

6- Prove that $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos(\frac{n\pi}{4})$.

7- Prove by polar form that $i\left(1-i\sqrt{3}\right)\left(\sqrt{3}+i\right)=2+i2\sqrt{3}$.

8- Find the modulus of the following:
(1) $\left|\frac{-2+3i}{3-2i}\right|$
(2) $\left|\frac{1-4i}{4+3i}\right|$

9- Find and draw all real complex solutions of the following:
(1) $z^2+7z+10=0$. (2) $z^3+8=0$.
(3) $z^5-16z=0$. (4) $z^5-32=0$.
(5) $z^4+2z^2-3=0$. (6) $3z^6=z^3+2$.
(7) $z^3-125=0$. (8) $z^2+6z+10=0$.

10- Compute the following:
(1) $\left|(1-i\sqrt{3})^2-(4-i\sqrt{3})\right|$.
(2) $\left|(3-i)^2(5-i\sqrt{2})\right|$.

Chapter Two

Polynomial Functions

Definition 2.1: (Polynomial Function)

If n is a nonnegative integer, a function in one variable that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0, \ a_n \neq 0 \quad \text{--- (1)}$$

is called a **polynomial function of degree n.** The numbers $a_n, \ a_{n-1}, \dots, a_1, a_0$ are called the **coefficients** of $P(x)$.

Remark 2.2:

(i) We will assume that the coefficients of a polynomial function are complex numbers, or real numbers, or rational numbers, or integers, depending on our interest. Similarly, the domain of a polynomial function can be the set of complex numbers, the set of real numbers, or an appropriate subset of either, depending on the situation.

(ii) If $n = 0$ and $a_0 \neq 0$, then (1) consists only the number a_0 and its degree is 0.

(iii) The polynomial consisting of just the number 0 does not have degree, and it is called **Zero polynomial(function)**, $P(x) = 0$.

(iv) The coefficient a_n is called the leading coefficient.

(v) The coefficient a_0 is called the constant term.

(vi) Each $a_i x^i$ $i = 0,1,\dots,n$ is called a term of the polynomial.

Examples 2.3:

(i) $f(x) = ax + b, \quad a \neq 0$ **(Linear function)**.

(1) The degree is 1.

(2) The coefficients are $a_1 = a, \ a_0 = b$.

(3) The leading coefficient is a.

(4) The constant term is b.

(5) The terms are ax, b.

(ii) $f(x) = ax^2 + bx + c, \ a \neq 0$ **(Quadratic function)**.

(1) The degree is 2.

(2) The coefficients are $a_2 = a, \ a_1 = b, \ a_0 = c$.

(3) The leading coefficient is a.

(4) The constant term is c.

(5) The terms are ax^2, bx, c.

(iii) $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$ **(Cubic function)**.
(1) The degree is 3.
(2) The coefficients are $a_3 = a,\ a_2 = b,\ a_1 = c,\ a_0 = d$.
(3) The leading coefficient is a.
(4) The constant term is d.
(5) The terms are ax^3, bx^2cx, d.

Properties of Polynomials 2.4:

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$.
and

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0 = \sum_{j=0}^{m} b_j x^j$$

be two polynomials. Then

(i) $p(x) = q(x) \Leftrightarrow n = m \quad and \ a_n = b_n, a_{n-1} = b_{n-1}, \ldots a_0 = b_0$.
(ii) If $m \leq n$ then $p(x) + q(x)$ is a polynomial of degree $\leq n$.

$p(x) = x^2 - 2x + 1 \qquad q(x) = 5x - x^2 + 1$
$p(x) + q(x) = (1-1)x^2 + (-2+5)x + (1+1) = 3x + 2$ (of degree 1).

(iii) $p(x) \cdot q(x)$ is a polynomial of degree $n + m$.

$p(x) = x^2 - 2x + 1 \qquad q(x) = 5x - x^2 + 1$

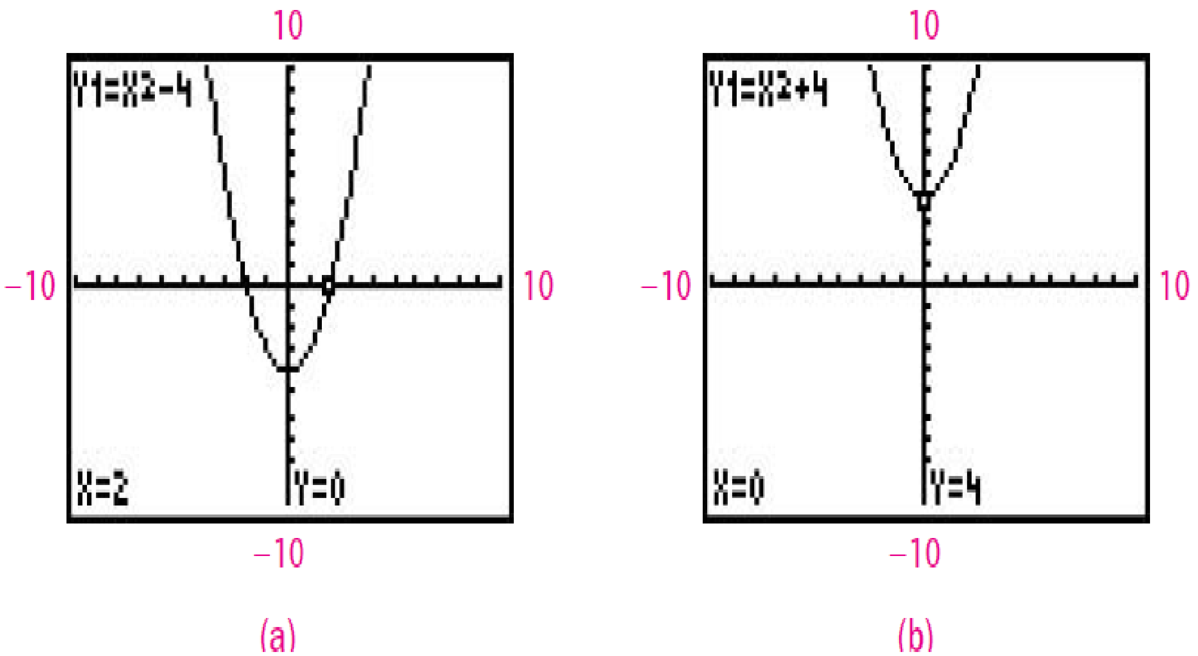
$$p(x) \cdot q(x) = x^2(5x - x^2 + 1) - 2x(5x - x^2 + 1) + (5x - x^2 + 1)$$
$$= -x^4 + 7x^3 - 10x^2 + 3x + 1$$ (of degree 4).

(iv) If $p(x) \neq 0$ and $p(x) \cdot q(x) = 0$ then $q(x) = 0$ and if $q(x) \neq 0$ and $f(x) \cdot q(x) = p(x) \cdot q(x)$ then $f(x) = p(x)$.

Definition 2.5: **(Zeros or Roots)**
A number r is said to be a **zero** or **root** of a function $P(x)$ if $P(r) = 0$.

The zeros of $P(x)$ are the solutions of the equation $P(x) = 0$. So if the coefficients of a polynomial $P(x)$ are real numbers, then the real zeros of $P(x)$ are just the x intercepts of the graph of $P(x)$. For example, the real zeros of the polynomial $P(x) = x^2 - 4$ are 2 and -2, the x- intercepts of the graph of $P(x)$ [Fig. (a)]. However, a polynomial may have zeros that are not x - intercepts. $Q(x) = x^2 + 4$, for example, has zeros $2i$ and $-2i$, but its graph has no

x- intercepts [Fig. (b)].

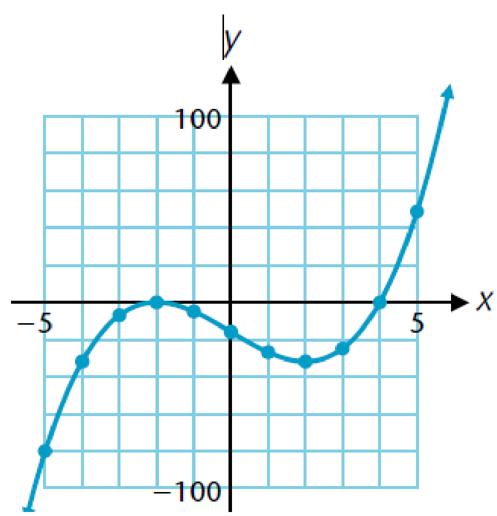


Example 2.6:

Graph the polynomial $P(x) = x^3 - 12x - 16, \ -5 \le x \le 5$. List the real zeros points.

Solution: First we construct a table of values by calculating $P(x)$ for each integer $x, \ -5 \le x \le 5$.

x	$P(x)$	x	$P(x)$
-5	-81	1	-27
-4	-32	2	-32
-3	-7	3	-25
-2	0	4	0
-1	-5	5	49
0	-16		



Then we plot the points in the table and join them with a smooth curve. The zeros are -2 and 4.

Caution 2.7: Finding the real zeros points of a polynomial is usually more difficult than suggested by the example. In example above, how did we know that the real zeros were between -5 and 5 rather than between, say, 95 and 105? Could there be another real zero just to the left or right of -2?

To answer such questions we must view polynomials from an algebraic perspective. Polynomials can be factored. So next we will study the division and factorization of polynomials.

Polynomial Division 2.8:

We can find quotients of polynomials by a long-division process similar to the one used in arithmetic. Example below will illustrate the process.

Example 2.9: (Polynomial Long Division)

Divide $P(x) = 3x^3 - 5 + 2x^4 - x$ by $(2 + x)$.

Solution: First, rewrite the dividend $P(x)$ in descending powers of x, inserting 0 as the coefficient for any missing terms of degree less than 4:

$$P(x) = 2x^4 + 3x^3 + 0x^2 - x - 5.$$

Similarly, rewrite the divisor $(2 + x)$.in the form $(x + 2)$. Then divide the first term x of the divisor into the first term $2x^4$ of the dividend. Multiply the result, $2x^3$, by the divisor, obtaining $2x^4 + 4x^3$. Line up like terms, subtract as in arithmetic, and bring down $0x^2$. Repeat the process until the degree of the remainder is less than the degree of the divisor.

$$
\begin{array}{rll}
 & 2x^3 - x^2 + 2x - 5 & \text{Quotient} \\
\text{Divisor} \quad x + 2 & \overline{)2x^4 + 3x^3 + 0x^2 - x - 5} & \text{Dividend} \\
 & \underline{2x^4 + 4x^3} & \text{Subtract} \\
 & -x^3 + 0x^2 & \\
 & \underline{-x^3 - 2x^2} & \text{Subtract} \\
 & 2x^2 - x & \\
 & \underline{2x^2 + 4x} & \text{Subtract} \\
 & -5x - 5 & \\
 & \underline{-5x - 10} & \text{Subtract} \\
 & 5 & \text{Remainder}
\end{array}
$$

Therefore,

$$\frac{2x^4 + 3x^3 - x - 5}{x + 2} = 2x^3 - x^2 + 2x - 5 + \frac{5}{x + 2}$$

CHECK You can always check division using multiplication:

$$
\begin{aligned}
(x + 2)\left[2x^3 - x^2 + 2x - 5 + \frac{5}{x + 2}\right] & \\
&= (x + 2)(2x^3 - x^2 + 2x - 5) + 5 \quad \text{Multiply and collect like terms} \\
&= 2x^4 + 3x^3 - x - 5
\end{aligned}
$$

The procedure illustrated in example above is called the **division algorithm.** The concluding equation of example may be multiplied by the divisor $(x + 2)$ to give the following form:

$$\underbrace{2x^4 + 3x^3 - x - 5}_{\text{Dividend}} = \underbrace{(x + 2)(2x^3 - x^2 + 2x - 5)}_{\text{Divisor} \cdot \text{Quotient}} + \underbrace{5}_{\text{Remainder}}$$

This last equation is an *identity:* it is true for all replacements of x by real or complex numbers including $x = -2$. Theorem below, which we state without proof, gives the general result of applying the division algorithm when the divisor has the form $(x - r)$.

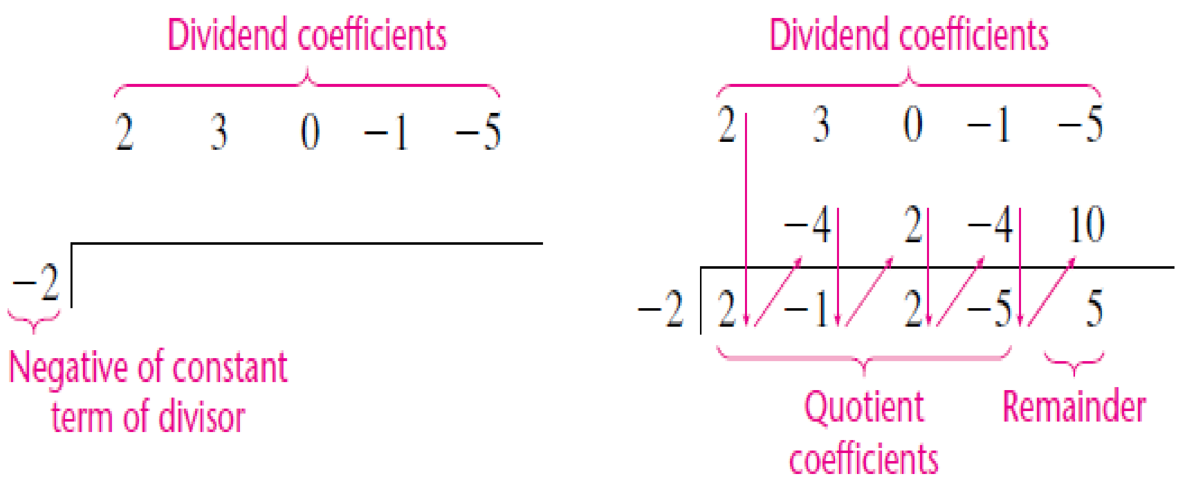
<u>Theorem 2.10:</u> (Division Algorithm)
For each polynomial $P(x)$ of degree greater than 0 and each number r, there exists a unique polynomial $Q(x)$ of degree less than $P(x)$ and a unique number R such that

$$P(x) = (x - r)Q(x) + R.$$

The polynomial $P(x)$ is called the **dividend**, $Q(x)$ is the **quotient,** $(x - r)$ is the **divisor,** and R is the **remainder.** Note that R may be 0.

<u>Synthetic Division 2.11:</u>

Divide $P(x) = 3x^3 - 5 + 2x^4 - x$ by $(2 + x)$.
synthetic division for the long division of $P(x)$. First write the coefficients of the dividend and the *negative* of the constant term of the divisor in the format shown below at the left. Bring down the 2 as indicated next on the right, multiply by -2, and record the product -4. Add 3 and -4, bringing down their sum -1. Repeat the process until the coefficients of the quotient and the remainder are obtained.



KEY STEPS IN THE SYNTHETIC DIVISION PROCESS

To divide the polynomial $P(x)$ by $x - r$:

Step 1. Arrange the coefficients of $P(x)$ in order of descending powers of x. Write 0 as the coefficient for each missing power.

Step 2. After writing the divisor in the form $x - r$, use r to generate the second and third rows of numbers as follows. Bring down the first coefficient of the dividend and multiply it by r; then add the product to the second coefficient of the dividend. Multiply this sum by r, and add the product to the third coefficient of the dividend. Repeat the process until a product is added to the constant term of $P(x)$.

Step 3. The last number to the right in the third row of numbers is the remainder. The other numbers in the third row are the coefficients of the quotient, which is of degree less than $P(x)$.

Example 2.12: Use synthetic division to divide

$$P(x) = 4x^5 - 30x^3 - 50x - 2 \text{ by } (x + 3).$$

Find the quotient and remainder. Write the conclusion in the form

$$P(x) = (x - r)Q(x) + R$$

of Division Algorithm Theorem.

Solution: Because $x + 3 = x - (-3)$, we have $r = -3$ and

	4	0	−30	0	−50	−2
		−12	36	−18	54	−12
−3	4	−12	6	−18	4	−14

The quotient is $4x^4 - 12x^3 + 6x^2 - 18x + 4$ with a remainder of -14. So

$$4x^5 - 30x^3 - 50x - 2 = (x + 3)(4x^4 - 12x^3 + 6x^2 - 18x + 4) - 14$$

Major Problems 2.13:

(i) Given that $P(x)$ is a polynomial and r is a number, find $P(r)$.

(ii) Given that $P(x)$ is a polynomial and M is a number, find the solution set of the polynomial equation $P(x) = M$.

<u>Theorem 2.14:</u> (Remainder Theorem)

If R is the remainder after dividing the polynomial $P(x)$ by $(x - r)$, then

$$P(r) = R.$$

<u>Example 2.15:</u> If $P(x) = 4x^4 + 10x^3 + 19x + 5$, find $P(-3)$ by

(i) Using the remainder theorem and synthetic division.

(ii) Evaluating $P(-3)$ directly.

<u>Solution:</u>

(i) Use synthetic division to divide $P(x)$ by $x - (-3)$.

	4	10	0	19	5	
		-12	6	-18	-3	
-3	4	-2	6	1	2	$= R = P(-3)$

(ii) $P(-3) = 4(-3)^4 + 10(-3)^3 + 19(-3) + 5 = 2.$

(iii)

<u>Theorem 2.16:</u> (Factor Theorem)

If r is a zero of the polynomial $P(x)$, then $(x - r)$ is a factor of $P(x)$. Conversely, if $(x - r)$ is a factor of $P(x)$, then r is a zero of $P(x)$.

Proof:

The remainder theorem shows that the division algorithm equation,

$$P(x) = (x - r)Q(x) + R$$

can be written in the form where R is replaced by $P(r)$:

$$P(x) = (x - r)Q(x) + P(r)$$

Therefore, x- r is a factor of $P(x)$ if and only if $P(r) = 0$, that is, if and only if r is a zero of the polynomial $P(x)$.

<u>Example 2.17:</u>

Use the factor theorem to show that $x + 1$ is a factor of $P(x) = x^{25} + 1$ but is not a factor of $Q(x) = x^{25} - 1$.

SOLUTION:

Because

$$P(-1) = (-1)^{25} + 1 = -1 + 1 = 0$$

$x - (-1) = x + 1$ is a factor of $x^{25} + 1$. On the other hand,

$$Q(-1) = (-1)^{25} - 1 = -1 - 1 = -2$$

and $x + 1$ is not a factor of $x^{25} - 1$.

Example 2.18:
Find the remainder when $f(x) = 2x^4 - 3x^3 + 7x$ is divided by $(x + 2)$.
Solution:

(i) Remainder Theorem. Since $(x - r) = (x + 2)$, it follows that $r = -2$. Thus,

$$R = f(-2) = 32 + 24 - 14 = 42.$$

(ii) Long Division.

$$\begin{array}{r|l} & 2x^3 - 7x^2 + 14x - 21 \\ \hline (x+2) & 2x^4 - 3x^3 + 0x^2 + 7x + 0 \\ & \mp 2x^4 \mp 4x^3 \\ \hline & -7x^3 + 0x^2 + 7x + 0 \\ & \pm 7x^3 \pm 14x^2 \\ \hline & 14x^2 + 7x + 0 \\ & \mp 14x^2 \mp 28x \\ \hline & -21x + 0 \\ & \pm 21x \pm 42 \\ \hline & +42 \end{array}$$

This division shows that not only that the remainder is 42 but also that quotient $q(x)$ is $q(x) = 2x^3 - 7x^2 + 14x - 21$.

(iii) Synthetic substitution.

$$\begin{array}{r|rrrrr} & 2 & -3 & +0 & +7 & +0 \\ & & -4 & +14 & -28 & +42 \\ \hline -2 & 2 & -7 & +14 & -21 & +42 = R = f(-2) \end{array}$$

Now compare the other number in line three with the coefficient in $q(x)$; they are identical.

<u>Example 2.19:</u> Find the quotient and remainder when $f(x) = 2x^3 - 3x^2 + 4$ is divided by $(x - 2)$.

<u>Solution:</u>

Here, $x - a = x - 2$, and hence $a = 2$. Using synthetic division, we obtain

$$\begin{array}{r|rrrl} & 2 & -3 & +0 & +4 \\ & & +4 & +2 & +4 \\ \hline 2 & 2 & +1 & +2 & +8 = R = f(2) \end{array}$$

Thus, $q(x) = 2x^2 + x + 2$ and $R = 8$.

We can check this result by long division.

<u>Example 2.20:</u> Determine whether $(x - 2)$ is a factor of $f(x) = x^6 - 64$.

<u>Solution:</u>

(i) Factor theorem. $a = 2,\ f(2) = 2^6 - 64 = 64 - 64 = 0$.
Thus, $x - 2$ is a factor of $f(x)$.
(ii) Synthetic division.

$$\begin{array}{r|rrrrrrl} & 1 & +0 & +0 & +0 & +0 & +0 & -64 \\ & & +2 & +4 & +8 & +16 & +32 & +64 \\ \hline 2 & 1 & +2 & +4 & +8 & +16 & +32 & +0 = R = f(2) \end{array}$$

Hence, $x - 2$ is a factor of $f(x) = x^6 - 64$ and the second factor is

$x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$. That is,

$f(x) = x^6 - 64 = (x - 2)(x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32)$.

Example 2.21: Determine whether $(x-1)$ is a factor of

$$f(x) = x^3 + 7x^2 - 3x - 4.$$

Solution: Synthetic division gives

$$\begin{array}{r|rrrl} & 1 & +7 & -3 & -4 \\ & & +1 & +8 & +5 \\ \hline 1 & 1 & +8 & +5 & +1 = R = f(1) \end{array}$$

Since $f(1) = 1 \neq 0$, then $x-1$ is not a factor of the given polynomial.

Theorem 2.22: (Zeros of Polynomials)
A polynomial of degree *n* with real coefficients has at most *n* real zeros.

Example 2.23:

(i) $f(x) = x^2 - 1 = (x+1)(x-1)$ has two real roots 1 and -1.
(ii) $f(x) = x^2 + 1$ has no real roots.
(iii) $f(x) = x^3 + x^2 + x + 1 = (x+1)(x^2+1)$ has one real roots -1.

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,\ a_n \neq 0$

(i) a_i are complex coefficients $\longrightarrow$ Complex zeros(Imaginary zeros).

(ii) a_i are real coefficients $\longrightarrow \begin{cases} \text{Pure real zeros} \\ \text{Pure complex zeros} \\ \text{Real zeros and complex zeros} \end{cases}$

Real Zeros 2.24:

The real zeros of a polynomial $P(x)$ with real coefficients are just the x- intercepts of the graph of $P(x)$. So an obvious strategy for finding the real zeros consists of two steps:
(1) Graph $P(x)$.
(2) Approximate each x- intercept.

Chapter Two

We emphasize the approximation of real zeros in this section; the problem of finding zeros exactly, when possible, is considered later.

<u>Theorem 2.25:</u> (Descartes Rule of Signs)
Given a polynomial $P(x)$ with real coefficients:
(1) **<u>Positive Zero:</u>** The number of positive zeros of $P(x)$ is never greater than the number of variations in sign $P(x)$ and if less than always by an even number.
(2) **<u>Negative Zero:</u>** The number of negative zeros of $P(x)$ is never greater than the number of variations in sign $P(-x)$ and if less than always by an even number.
<u>Example 2.26</u>: Construct a table showing the possible combination of positive, negative and imaginary zero of the following polynomials:
(1) $P(x) = 3x^4 - 2x^3 + 3x - 5$.
(2) $Q(x) = 2x^6 + x^4 - x + 3$.
Solution:
(1) $P(-x) = 3x^4 + 2x^3 - 3x - 5$.

$+$	$-$	I
3	1	0
1	1	2

(2) $Q(-x) = 2x^6 + x^4 + x + 3$

$+$	$-$	I
2	0	4
0	0	6

<u>Theorem 2.27:</u> (Upper and Lower Bound Theorem)

Let $P(x)$ be a polynomial of degree $n > 0$ with real coefficients, $a_n > 0$:

1. Upper bound: A number $r > 0$ is an upper bound for the real zeros of $P(x)$ if, when $P(x)$ is divided by $x - r$ by synthetic division, all numbers in the quotient row, including the remainder, are nonnegative.
2. Lower bound: A number $r < 0$ is a lower bound for the real zeros of $P(x)$ if, when $P(x)$ is divided by $x - r$ by synthetic division, all numbers in the quotient row, including the remainder, alternate in sign.

[*Note:* In the lower bound test, if 0 appears in one or more places in the quotient row, including the remainder, the sign in front of it can be considered either positive or negative, but not both. For example, the numbers 1, 0, 1 can be considered to alternate in sign, whereas 1, 0, -1 cannot.]

<u>Example 2.28:</u>

Let $P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90$. Find the smallest positive integer and the largest negative integer that, by Upper and Lower Bound Theorem, are upper and lower bounds, respectively, for the real zeros of $P(x)$.

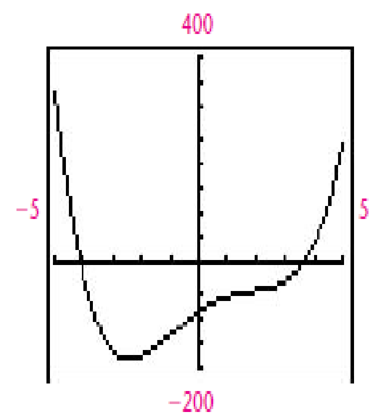
<u>Solution:</u>

We perform synthetic division for $r = 1, 2, 3, \ldots$ until the quotient row turns nonnegative;

then repeat this process for $r = -1, -2, -3, \ldots$ until the quotient row alternates in sign.

We organize these results in the *synthetic division table* shown below. In a synthetic division table we dispense with writing the product of $\boldsymbol{r}$ with each coefficient in the quotient and simply list the results in the table.

		1	−2	−10	40	−90	
	1	1	−1	−11	29	−61	
	2	1	0	−10	20	−50	
	3	1	1	−7	19	−33	
	4	1	2	−2	32	38	
UB	5	1	3	5	65	235	← This quotient row is nonnegative; 5 is an upper bound (UB).
	−1	1	−3	−7	47	−137	
	−2	1	−4	−2	44	−178	
	−3	1	−5	5	25	−165	
	−4	1	−6	14	−16	−26	
LB	−5	1	−7	25	−85	335	← This quotient row alternates in sign; −5 is a lower bound (LB).



The graph of $P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90$ for $-5 \le x \le 5$ is shown in the above Figure. Upper and Lower Bound Theorem guarantees that all the real zeros of $P(x)$ are between -5 and 5. We can be certain that the graph does not change direction and cross the x- axis somewhere outside the viewing window in the Figure.

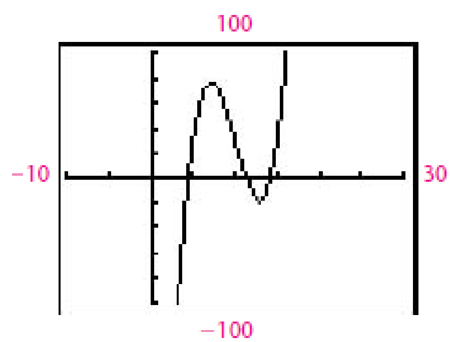
Example 2.29:

Let $P(x) = x^3 - 30x^2 + 275x - 720$. Find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that, by Upper and Lower Bound Theorem, are upper and lower bounds, respectively, for the real zeros of $P(x)$.

Solution: We construct a synthetic division table to search for bounds for the zeros of $P(x)$. The size of the coefficients in $P(x)$ indicates that we can speed up this search by choosing larger increments between test values.

		1	−30	275	−720
	10	1	−20	75	30
	20	1	−10	75	780
UB	30	1	0	275	7,530
LB	−10	1	−40	675	−7,470

Therefore, all real zeros of $P(x) = x^3 - 30x^2 + 275x - 720$ must lie between −10 and 30, as confirmed by in the figure below.



Theorem 2.30: **(Location Theorem)**

If $P(x)$ is a polynomial with real coefficients and if $P(a)$ and $P(b)$ are of opposite sign, then there is at least one real zero between a and b.

Example 2.31: Show that there is at least one real zero of

$$P(x) = x^4 - 2x^3 - 6x^2 + 6x + 9$$

between 1 and 2.

Solution: It is enough to show that $P(1)$ and $P(2)$ have opposite signs.

$$\begin{array}{c|ccccl} & 1 & -2 & -6 & 6 & 9 \\ & & 1 & -1 & -7 & -1 \\ \hline 1 & 1 & -1 & -7 & -1 & 8 = P(1) \end{array} \qquad \begin{array}{c|ccccl} & 1 & -2 & -6 & 6 & 9 \\ & & 2 & 0 & -12 & -12 \\ \hline 2 & 1 & 0 & -6 & -6 & -3 = P(2) \end{array}$$

Complex Zeros and Rational Zeros of Polynomials 2.32:

(i) The Fundamental Theorem of Algebra.
(ii) Factors of Polynomials with Real Coefficient.
(iii) Rational Zeros of Polynomials with Real Coefficient.

Theorem 2.33: (Fundamental Theorem of Algebra)
Every polynomial of degree $n > 0$ with complex coefficients has a complex zero.

Theorem 2.34: (n Linear Factors Theorem)
Every polynomial of degree $n > 0$ with complex coefficients can be factored as a product of n linear factors.

Proof:
If $P(x)$ is a polynomial of degree $n > 0$ with complex coefficients, then by Fundamental Theorem of Algebra it has a zero r_1. So $(x - r_1)$ is a factor of $P(x)$ by Factor Theorem, and

$$P(x) = (x - r_1)Q(x), \quad deg\, Q(x) = n - 1.$$

Now, if deg $Q(x) > 0$, then, applying the Fundamental Theorem to $Q(x), Q(x)$ has a root r_2 and therefore a factor $(x - r_2)$. (It is possible that r_2 is equal to r_1.) By continuing this reasoning we obtain a proof of the theorem.

Definition 2.35: The number of linear factors that have zero r is said to be the **multiplicity** of r.

Example 2.36: The polynomial
$P(x) = (x - 5)^3(x + 1)^2(x - 6i)(x + 2 + 3i)$ has degree 7 and is written as a product of seven linear factors. $P(x)$ has just four zeros,
namely $5, -1, 6i,$ and $-2 - 3i$.
Because the factor $(x - 5)$ appears to the power 3, we say that the zero 5 has multiplicity 3.

-1 has multiplicity 2,
$6i$ has multiplicity 1
$(-2-3i)$ has multiplicity 1.
A zero of multiplicity 2 is called a **double zero**, and a zero of multiplicity 3 is called **triple zero**.

Note that the sum of the multiplicities is always equal to the degree of the polynomial: for $P(x)$ $\quad 3+2+1+1=7.$

<u>Factors of Polynomials with Real Coefficients 2.37:</u>

<u>Theorem 2.38:</u> (Imaginary Zeros of Polynomials with Real Coefficients)
Imaginary zeros of polynomials with real coefficients, if they exist, occur in conjugate pairs.
Proof: The prove is given for quadratic polynomial (polynomial of degree 2). By same way for degree >2.

If $p+qi$ is a zero of $P(x)=ax^2+bx+c$, where a, b, c, p, and q are real numbers, then

$$P(p+qi)=0$$

$$a(p+qi)^2+b(p+qi)+c=0 \qquad \text{Take the conjugate of both sides.}$$

$$\overline{a(p+qi)^2+b(p+qi)+c}=\overline{0} \qquad \overline{z+w}=\overline{z}+\overline{w},\ \overline{zw}=\overline{z}\,\overline{w}$$

$$\overline{a}\,(\overline{p+qi})^2+\overline{b}\,(\overline{p+qi})+\overline{c}=\overline{0} \qquad \overline{z}=z \text{ if } z \text{ is real, } \overline{p+qi}=p-qi$$

$$a(p-qi)^2+b(p-qi)+c=0$$

$$P(p-qi)=0$$

Therefore, $p-qi$ is also a zero of $P(x)$. This method of proof can be applied to any polynomial $P(x)$ of degree $n>0$ with real coefficients.

<u>Theorem 2.39:</u> (Linear and Quadratic Factors Theorem)
If $P(x)$ is a polynomial of degree $n>0$ with real coefficients, then $P(x)$ can be factored as a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros).

Proof:
If a polynomial $P(x)$ of degree $n > 0$ has real coefficients and a linear factor of the form $x - (p + qi)$ where $q \neq 0$, then, by **Imaginary Zeros of Polynomials with Real Coefficients Theorem**, $P(x)$ also has the linear factor $x - (p - qi)$.
But

$$[x - (p + qi)][x - (p - qi)] = x^2 - 2px + p^2 - q^2$$

which is a quadratic factor of $P(x)$ with real coefficients and imaginary zeros.

Example 2.40:
Factor $P(x) = x^3 + x^2 + 4x + 4$ in two ways:
(i) As a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros)
(ii) As a product of linear factors with complex coefficients

Solution:
(i) Note that $P(-1) = 0$, so -1 is a zero of $P(x)$. Therefore, $(x + 1)$ is a factor of $P(x)$. Using synthetic division, the quotient is $x^2 + 4$, which has imaginary roots. Therefore,
$P(x) = (x+1)(x^2 + 4)$

An alternative solution is to factor by grouping:

$$\begin{aligned} x^3 + x^2 + 4x + 4 &= x^2(x + 1) + 4(x + 1) \\ &= (x^2 + 4)(x + 1). \end{aligned}$$

(ii) Because $x^2 + 4$ has roots $2i$ and $-2i$,
$P(x) = (x + 1)(x - 2i)(x + 2i)$.

Theorem 2.41: (Real Zeros and Polynomials of Odd Degree)
Every polynomial of odd degree with real coefficients has at least one real zero.

Example 2.42: Let $P(x)$ be a third-degree polynomial with real coefficients. Are the following statements is false?
(1) $P(x)$ has at least one real zero.
(2) $P(x)$ has three zero.
(3) $P(x)$ can have two real zero and one imaginary zero.

Solution:
(1) True
(2) True.
(3) False.

Example 2.43: What are the possible combinations of real and imaginary zeros for the polynomial $P(x) = 3x^5 - 2x^4 + x^2 - \sqrt{2}x - 5$.

Solution:

Real	Imaginary
1	4
3	2
5	0

Rational Zeros 2.44:

First note that a polynomial with rational coefficients can always be written as a constant times a polynomial with integer coefficients. For example,

$$P(x) = \frac{1}{2}x^3 - \frac{2}{3}x^2 + \frac{7}{4}x \quad + 5$$
$$P(x) = \frac{1}{12}(6x^3 - 8x^2 + 21x \quad + 60).$$

Because the zeros of $P(x)$ are the zeros of $6x^3 - 8x^2 + 21x \quad + 60$, it is sufficient, for the purpose of finding rational zeros of polynomials with rational coefficients, to study just the polynomials with integer coefficients.

We introduce the rational zero theorem by examining the following quadratic polynomial whose zeros can be found easily by factoring:

$P(x) = 6x^2 - 13x - 5 = (2x - 5)(3x + 1)$

Zeros of $P(x)$: $\frac{5}{2}$ $\quad and \quad$ $\frac{-1}{3}$.

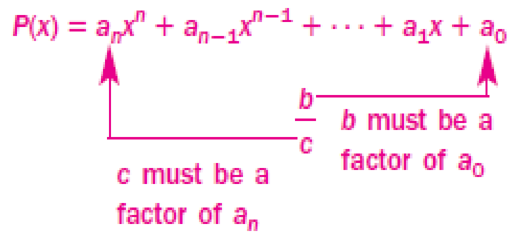
Notice that the numerators 5 and -1 of the zeros are both integer factors of -5, the constant term in $P(x)$. The denominators 2 and 3 of the zeros are both integer factors of 6, the leading coefficient in $P(x)$.

<u>Theorem 2.45:</u> (Rational Zero Theorem)

If the rational number b/c, in lowest terms, is a zero of the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad a_n \neq 0$$

with integer coefficients, then b must be an integer factor of a_0 and c must be an integer factor of a_n.



<u>Remark 2.46:</u> Rational Zero Theorem enables us to construct a finite list of possible rational zeros of $P(x)$. Each number in the list must then be tested to determine whether or not it is actually a zero.

<u>Example 2.47:</u>

Find all the rational zeros for $P(x) = 2x^3 + 9x^2 + 7x - 6$.

If b/c in lowest terms is a rational zero of $P(x)$, then b must be a factor of -6 and c must be a factor of 2.

Possible values of b are the integer factors of -6: $\pm1, \pm2, \pm3, \pm6$ (1)

Possible values of c are the integer factors of 2: $\pm1, \pm2$ (2)

Writing all possible fractions b/c where b is from (2) and c is from (3), we have

Possible rational zeros for $P(x)$: $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}$ (3)

[Note that all fractions are in lowest terms and duplicates like $\pm6/\pm2 = \pm3$ are not repeated.] If $P(x)$ has any rational zeros, they must be in list (4). We can test each number r in this list simply by evaluating $P(r)$.

Here, $-3, -2$ *and* $\frac{1}{2}$ are rational zeros of $P(x)$. Because a third-degree polynomial can have at most three zeros, we have found all the rational zeros. There is no need to test the remaining candidates in list (3).

Remark 2.48:

As we saw in the solution of the above example, rational zeros can be located by simply evaluating the polynomial. However, if we want to find multiple zeros, imaginary zeros, or exact values of irrational zeros, we need to consider *reduced polynomials.*

If r is a zero of a polynomial $P(x)$, then we can write

$$P(x) = (x - r)Q(x)$$

where $Q(x)$ is a polynomial of degree one less than the degree of $P(x)$. The quotient polynomial $Q(x)$ is called a **reduced polynomial** for $P(x)$. In above example, after determining that -3 is a zero of $P(x)$, we can write

$$\begin{array}{r|rrrr} & 2 & 9 & 7 & -6 \\ & & -6 & -9 & 6 \\ \hline -3 & 2 & 3 & -2 & 0 \end{array}$$

$$\begin{aligned} P(x) &= 2x^3 + 9x^2 + 7x - 6 \\ &= (x + 3)(2x^2 + 3x - 2) \\ &= (x + 3)Q(x) \end{aligned}$$

Because the reduced polynomial $Q(x) = 2x^2 + 3x - 2$ is a quadratic, we can find its zeros by factoring or the quadratic formula. We get

$$P(x) = (x + 3)(2x^2 + 3x - 2) = (x + 3)(x + 2)(2x - 1)$$

and we see that the zeros of $P(x)$ are -3, -2, and $\frac{1}{2}$, as before.

Example 2.49:

Find all zeros exactly for $P(x) = 2x^3 - 7x^2 + 4x + 3$.

First, list the possible rational zeros:

$$\pm 1, \pm 3, \pm \tfrac{1}{2}, \pm \tfrac{3}{2}$$

$$P(-\tfrac{1}{2}) = -1 \qquad \text{and} \qquad P(\tfrac{3}{2}) = 0$$

So $\frac{3}{2}$ is a zero, but $-\frac{1}{2}$ is not. Using synthetic division (details omitted), we can write

$$P(x) = (x - \tfrac{3}{2})(2x^2 - 4x - 2)$$

Because the reduced polynomial is quadratic, we can use the quadratic formula to find the exact values of the remaining zeros:

$$2x^2 - 4x - 2 = 0 \qquad \text{Divide both sides by 2.}$$

$$x^2 - 2x - 1 = 0 \qquad \text{Use the quadratic formula.}$$

$$x = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

So the exact zeros of $P(x)$ are $\frac{3}{2}$ and $1 \pm \sqrt{2}$.

Example 2.50:

Find all zeros exactly for $P(x) = x^4 - 6x^3 + 14x^2 - 14x + 5$.

Solution:

The possible rational zeros are ± 1 and ± 5. $P(1) = 0$,

Using synthetic division (details omitted), we find that

$$P(x) = (x - 1)(x^3 - 5x^2 + 9x - 5)$$

The possible rational zeros of the reduced polynomial

$$Q(x) = x^3 - 5x^2 + 9x - 5$$

are ± 1 and ± 5. By substituting 1 in $Q(x)$, we see that 1 is a rational zero. After a division, we have a quadratic reduced polynomial:

$$Q(x) = (x - 1)Q_1(x) = (x - 1)(x^2 - 4x + 5)$$

We use the quadratic formula to find the zeros of $Q_1(x)$:

$$x^2 - 4x + 5 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

So, the exact zeros of $P(x)$ are 1 (multiplicity 2), $(2 + i)$, and $(2 - i)$.

Remark 2.51:

We were successful in finding all the zeros of the polynomials in the above examples because we could find sufficient rational zeros to reduce the original polynomial to a quadratic. This is not always possible. For example, the polynomial

$$P(x) = x^3 + 6x - 2$$

has no rational zeros, but does have an irrational zero at $x \approx 0.32748$. The other two zeros are imaginary. The techniques we have developed will not find the exact value of these roots.

The following examples summarize the strategy for finding rational zeros.

Example 2.52:

(1) Find all rational zeros for $P(x) = 2x^3 - x^2 - 8x + 4$.

Solution:

Step 1. List the possible rational zeros: $\pm 1,\ \pm 2,\ \pm 4,\ \pm\frac{1}{2}$.

Step 2. List possible combinations of zeros:

$$P(x) = 2x^3 - x^2 - 8x + 4$$
$$P(-x) = -2x^3 - x^2 + 8x + 4$$

+	−	I
2	1	0
0	1	2

Step 3. Construct a synthetic division table start with $r = 0$.

	2	−1	−8	4	
0	2	−1	−8	4	$P(x)$ change sign
1	2	1	−7	−3	
$\frac{1}{2}$	2	0	−8	0	$\frac{1}{2}$ is zero of $P(x)$

$$P(x) = (x - r)Q(x) = \left(x - \frac{1}{2}\right)(2x^2 - 8)$$

Using quadratic formula to find the zeros of $(2x^2 - 8)$

$$x = \frac{\pm\sqrt{64}}{4} = \pm 2$$

Thus, the rational zeros of $P(x)$ are $\pm 2,\ \frac{1}{2}$.

(2) Find all zeros for $P(x) = 2x^3 - 5x^2 - 8x + 6$.

Solution:

Step 1. List the possible rational zeros: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$.

Step 2. List possible combinations of zeros:

$$P(x) = 2x^3 - 5x^2 - 8x + 6$$

$$P(x) = -2x^3 - 5x^2 + 8x + 6$$

$+$	$-$	I
2	1	0
0	1	2

Step 3. Construct a synthetic division table.

	2	−5	−8	6	
0	2	−5	−8	6	$P(x)$ change sign
1	2	−3	−11	−5	
$\frac{1}{2}$	2	−4	−10	1	$1/2$ is not zero, so between 0 and 1. There is irrational zero(between $1/2$ and 1)
2	2	−1	−10	−14	
3	2	1	−5	−9	$P(x)$ change sign
4	2	3	4	22	There is irrational zero between 3 and 4
−1	2	−7	−1	7	$P(x)$ change sign
−2	2	−9	10	−14	
$-\frac{3}{2}$	2	−8	4	0	

From the synthetic division table, we know that $P(x)$ has three real zeros, one negative and two positive $-\frac{3}{2}$ is the negative zero, and the two positive zeros must be irrational.

$P(x) = (x - r)Q(x) = \left(x + \frac{3}{2}\right)(2x^2 - 8x + 4)$

Using quadratic formula to find the zeros of $(2x^2 - 8x + 4)$

$x = \frac{8 \pm \sqrt{64-32}}{4} = 2 \pm \sqrt{2}$. The zeros of $P(x)$ are $-\frac{3}{2}, 2 \pm \sqrt{2}$.

$2 - \sqrt{2} \approx 0.59$ -------------------------> $0 < 0.59 < 1$

$2 + \sqrt{2} \approx 3.41$ -------------------------> $3 < 3.41 < 4$.

(3) Find all rational zeros for $P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$.

Solution:

Step 1. List the possible rational zeros: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$.

Step 2. List possible combinations of zeros:

$$P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$$
$$P(-x) = x^4 + 7x^3 + 17x^2 + 17x + 6$$

+	−	I
4	0	0
2	0	2
0	0	4

Step 3. Construct a synthetic division table.
From the sign table we don't need to check the negative rational zero.

	1	−7	17	−17	6
1	1	−6	11	−6	0

$P(x) = (x - r)Q(x) = (x - 1)(x^3 - 6x^2 + 11x - 6)$
Using the way for $Q(x) = (x^3 - 6x^2 + 11x - 6)$.

Step 1. List the possible rational zeros: $1,\ 2,\ 3,\ 6.$
Step 2. List possible combinations of zeros:

+	−	I
3	0	0
1	0	2

Step 3. Construct a synthetic division table.

	1	−6	11	−6
1	1	−5	6	$0 = Q(1)$

$Q(x) = (x - 1)(x^2 - 5x + 6).$
$(x^2 - 5x + 6) = (x - 3)(x - 2)$. The zeros of $Q(x)$ are
1 (of multiplicity 2) 2 and 3.

Chapter Two

Relationships Between the Roots and Coefficients 2.54:

Let $f(x)$ be a polynomial of degree n with roots $r_1, r_2, \ldots, r_n$ and coefficients $a_1, a_2, \ldots, a_n$. Then we can written $f(x)$ as follows:

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n, \; a_0 \neq 0 \quad \text{--- (1)}$$
$$P(x) = a_0 (x - \propto_1)(x - \propto_2) \ldots (x - \propto_n).$$

If n=2, we have a polynomial of degree 2 defined as follows:

$$P(x) = a_0 x^2 + a_1 x + a_2 = a_0 (x - \propto_1)(x - \propto_2)$$
$$= a_0 (x^2 - x \propto_1 - x \propto_2 + \propto_1 \propto_2)$$
$$= a_0 \left(x^2 - (\propto_1 + \propto_2) x + \propto_1 \propto_2\right)$$

$$\Rightarrow P(x) = a_0 x^2 + a_1 x + a_2 = a_0 x^2 - a_0 (\propto_1 + \propto_2) x + a_0 \propto_1 \propto_2$$

$$a_1 = -a_0 (\propto_1 + \propto_2) \Rightarrow -a_1/a_0 = \propto_1 + \propto_2.$$
$$a_2 = a_0 (\propto_1 \cdot \propto_2) \Rightarrow +a_2/a_0 = \propto_1 \cdot \propto_2.$$

If n=3, we have a polynomial of degree 3 defined as follows:

$$P(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$$
$$= a_0 (x - \propto_1)(x - \propto_2)(x - \propto_3)$$
$$= a_0 \left(x^2 - (\propto_1 + \propto_2 + \propto_3) x^2 + (\propto_1 \propto_2 + \propto_1 \propto_3 + \propto_2 \propto_3) x - \propto_1 \propto_2 \propto_3\right).$$

And by equal coefficients powers of x we have:

$$-a_1/a_0 = \propto_1 + \propto_2 + \propto_3.$$

$$a_2/a_0 = \propto_1 \propto_2 + \propto_1 \propto_3 + \propto_2 \propto_3.$$
$$-a_3/a_0 = \propto_1 \cdot \propto_2 \cdot \propto_3.$$

Thus, in general for any n roots of a polynomial of degree n we get:

$$\sum \propto_i = -a_1/a_0$$
$$\sum \propto_i \propto_j = +a_2/a_0 \quad i \neq j$$
$$\sum \propto_i \propto_j \propto_k = -a_3/a_0 \quad i \neq j \neq k$$
$$\sum \propto_i \propto_j \propto_k \propto_l = +a_4/a_0 \quad i \neq j \neq k \neq l$$
$$\vdots$$
$$\propto_1 \propto_2 \propto_3 \ldots \propto_n = (-1)^n a_n/a_0 .$$

Example 2.55:

(1) Find the roots of an equation

$$P(x) = 6x^3 - 18x^2 + 24x - 12 = 0$$

when the result of multiplication of two roots of them is equal 2.

Solution:

$$a_0 = 6, \quad a_1 = -18, \quad a_2 = 24, \quad a_3 = -12.$$

$$\propto_1 + \propto_2 + \propto_3 = -a_1/a_0 = -(-18)/6 = 3 \ldots\ldots\ldots (1)$$
$$\propto_1\propto_2 + \propto_1\propto_3 + \propto_2\propto_3 = a_2/a_0 = 24/6 = 4 \ldots\ldots\ldots (2)$$
$$\propto_1 \cdot \propto_2 \cdot \propto_3 = -a_3/a_0 = -(-12)/6 = 2 \ldots\ldots\ldots\ldots (3)$$

From (3) $\propto_1 \cdot \propto_2 \cdot \propto_3 = 2$, but $\propto_1 \cdot \propto_2 = 2$. So, $\propto_3 = 1$.

From (1) we have

$$\propto_1 + \propto_2 + \propto_3 = 3 \Rightarrow \propto_1 + \propto_2 = 3 - 1 = 2 \Rightarrow \propto_1 = 2 - \propto_2 .$$

So, by (2) we have

$$(2 - \propto_2) \propto_2 + (2 - \propto_2).1 + \propto_2 = 4 \Rightarrow \propto_2^2 - 2\propto_2 + 2 = 0.$$
$$\propto_2 = \frac{+2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Therefore, the roots are 1,1+i, 1-i.

(2) Find the roots of an equation

$$P(x) = x^3 - 6x^2 + 11x - 6 = 0.$$

Solution:

$$a_0 = 1, \quad a_1 = -6, \quad a_2 = 11, \quad a_3 = -6.$$

Thus, if $\propto_1, \propto_2, \propto_3$ are roots of $P(x)$, then

$$\propto_1 \cdot \propto_2 \cdot \propto_3 = (-1)^n a_3/a_0 = (-1)^3(-6)/1 = 6$$

Thus, the possible integer roots of $P(x)$ are $\pm 1, \pm 2, \pm 3$ and ± 6.

$P(1)=0$ is a root of $P(x)$;
$P(-1)\neq 0$ is not a root of $P(x)$;
$P(2)=0$ is a root of $P(x)$;
$P(-2)\neq 0$ is not a root of $P(x)$;
$P(3)=0$ is a root of $P(x)$;
$P(-3)\neq 0$ is not a root of $P(x)$;
$P(6)\neq 0$ is not a root of $P(x)$;
$P(-6)\neq 0$ is not a root of $P(x)$.
Therefore, $x=1, x=2, x=3$ are roots of $P(x)$.

(3) If $\propto_1, \propto_2, \propto_3, \propto_4$ are roots of the following equation

$$P(x)= x^4-4x^3+3x^2+x-1=0.$$

Find $(\propto_1{}^2+\propto_2{}^2+\propto_3{}^2+\propto_4{}^2)^3$.

Solution:

$$a_0=1,\quad a_1=-4,\quad a_2=3,\quad a_3=1,\quad a_4=-1.$$

So, $\propto_1+\propto_2+\propto_3+\propto_4=-a_1/a_0=-(-4)/1 \;=4$

and

$$\propto_1.\propto_2+\propto_1.\propto_3+\propto_1.\propto_4+\propto_2.\propto_3+\propto_2.\propto_4+\propto_3.\propto_4$$
$$=a_2/a_0=3/1 \;=3.$$

But

$$(\propto_1+\propto_2+\propto_3+\propto_4)^2=((\propto_1+\propto_2)+(\propto_3+\propto_4))^2$$
$$=(\propto_1+\propto_2)^2+2(\propto_1+\propto_2)(\propto_3+\propto_4)+(\propto_3+\propto_4)^2$$
$$=\propto_1{}^2+\propto_2{}^2+\propto_3{}^2+\propto_4{}^2+2(\propto_1.\propto_2+\propto_1.\propto_3+\propto_1.\propto_4+\propto_2.\propto_3+\propto_2.\propto_4+\propto_3.\propto_4$$

$$\Rightarrow 16=\propto_1{}^2+\propto_2{}^2+\propto_3{}^2+\propto_4{}^2+(2.3)$$
$$\Rightarrow 10=\propto_1{}^2+\propto_2{}^2+\propto_3{}^2+\propto_4{}^2\Rightarrow (10)^3=(\propto_1{}^2+\propto_2{}^2+\propto_3{}^2+\propto_4{}^2)^3$$
$$=1000.$$