3.1 The Rank of a Matrix.

In this section we obtain another effective method for finding a basis for a vector space V spanned by a given set of vectors $S=\{A_1, A_2, \ldots, A_n\}$.

Def. (3.1.1): An $m \times n$ matrix is said to be in reduced row echelon form when it satisfies the following properties:

(a) All rows consisting entirely of zeros, if any, are at the bottom of the matrix.

(b) The first nonzero entry in each row that does not consist entirely of zeros is a 1, called the leading entry of its row.

(c) If rows i and $i+1$ are two successive rows that do not consist entirely of zeros, then the leading entry of row $i+1$ is to the right of the leading entry of row i.

(d) If a column contains a leading entry of some row, then all other entries in that column are zero.

Ex. The matrices $A=\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

are in reduced row echelon form.

But the matrices $A=\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are not in reduced row echelon form, since they fail to satisfy (a) and (d) respectively.

We shall now turn to the discussion of how to transform a given matrix to a matrix in reduced row echelon form.

Def. (3.1.2): An elementary row operation on an $m \times n$ matrix $A = [a_{ij}]$ is any one of the following operations:

(a) Interchange rows r and s of A. That is, replace $a_{r1}, a_{r2}, \dots, a_{rn}$ by $a_{s1}, a_{s2}, \dots, a_{sn}$ and $a_{s1}, \dots, a_{sn}$ by $a_{r1}, \dots, a_{rn}$.

(b) Multiply row r of A by $c \neq 0$. That is, replace $a_{r1}, \dots, a_{rn}$ by $ca_{r1}, \dots, ca_{rn}$.

(c) Add d times row r of A to rows s of A, $r \neq s$. That is replace $a_{s1}, \dots, a_{sn}$ by $a_{s1} + da_{r1}, \dots, a_{sn} + da_{rn}$.

Remark: If $[A : B]$ is the augmented matrix of a linear system ($AX = B$), then we observe that: المصفوفة الموسعة

The elementry row operations are equivalent, respectively, to interchanging two equations, multiplying an equation by a nonzero constant, and adding a multiple of one equation to another equation.

Ex. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix} \xrightarrow{\text{interchanging } r_1 \leftrightarrow r_3} B = \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$A \xrightarrow{(1/3)\, r_3} C = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

$$A \xrightarrow{-2r_2 + r_3} D = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Def. (3.1.3): An $m \times n$ matrix A is said to be row equivalent to an $m \times n$ matrix B if B can be obtained by applying a finite sequence of elementary row operations to A.

Ex. The matrix

$$A=\begin{bmatrix}1&2&4&3\\2&1&3&2\\1&-1&2&3\end{bmatrix} \text{ is row equivalent to } D=\begin{bmatrix}2&4&8&6\\1&-1&2&3\\4&-1&7&8\end{bmatrix}.$$

because

$$A=\begin{bmatrix}1&2&4&3\\2&1&3&2\\1&-1&2&3\end{bmatrix}\xrightarrow{2r_3+r_2}\begin{bmatrix}1&2&4&3\\4&-1&7&8\\1&-1&2&3\end{bmatrix}\xrightarrow[r_2 \leftrightarrow r_3]{\text{interchanging}}\begin{bmatrix}1&2&4&3\\1&-1&2&3\\4&-1&7&8\end{bmatrix}.$$

$$\xrightarrow{2r_1} D=\begin{bmatrix}2&4&8&6\\1&-1&2&3\\4&-1&7&8\end{bmatrix}.$$

Remark: It is easy to show that, if A, B and C are matrices, the

(1) A is row equivalent to A

(2) If A is row equivalent to B, then B is row equivalent to A

(3) If A is 〃 B and B 〃 C,
then A is 〃 C.

In view of (2), both statements, "A is row equivalent to B" and "B is row equivalent to A" can be replaced by "A and B are row equivalent"

Theorem (3.1.1): Every nonzero $m \times n$ matrix is row equivalent to a unique matrix in reduced row echelon form

We shall illustrate the proof of the theorem by given the steps that must be carried out on a specific matrix A to obtain a matrix in reduced row echelon form that is row equivalent to A.

Ex. Let

$$A=\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

The procedure for transforming a matrix to reduced row echelon form is as follows.

Procedure	Example
Step 1. Find the first (counting from left to right) column in A not all of whose entries are zero. This column is called the pivotal column. العمود المحوري	$A=\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$ pivotal column of A
Step 2. Identify the first (counting from top to bottom) nonzero entry in the pivotal column. This element is called the pivot.	$A=\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$ pivot محور
Step 3. Interchange, if necessary, the first row with the row where the pivot occurs so that the pivot is now in the first row. Call the new matrix A_1.	$A_1=\begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$ A_1 is by interchange $r_1 \leftrightarrow r_3$ of A
Step 4. Divide the first row of A_1 by the pivot. Thus the entry in the first row and pivotal column is now a 1. see A_2	$A_2=\begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$ A_2 is by $r_1/2$ of A_1.
Step 5. Add multiples of the first row of A_2 to all other rows to make all entries in the pivotal column, except the entry where the pivot was located, equal to zero. Thus all entries in the pivotal column and rows $2, 3, \dots, m$ are zero. see A_3	$A_3=\begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$ $-2r_1 + r_4$ of $A_2 \Rightarrow A_3$

Step 6. Identify B as the $(m-1)\times n$ submatrix of A_3 obtained by deleting the first row of A_3; do not erase the first row of A_3. Repeat step 1 through 5 on B

$$\begin{array}{c} \begin{array}{ccccc} 1 & 1 & -5/2 & 1 & 2 \end{array} \\ B=\begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \end{array}$$

pivotal column of B pivot

Interchange r_1 & r_2 of $B \Rightarrow B_1$

$$\begin{array}{c} \begin{array}{ccccc} 1 & 1 & -5/2 & 1 & 2 \end{array} \\ B_1=\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \end{array}$$

$r_1/2$ of $B_1 \Rightarrow B_2$

$$\begin{array}{c} (*) \rightarrow \begin{array}{ccccc} 1 & 1 & -5/2 & 1 & 2 \end{array} \\ B_2=\begin{bmatrix} 0 & 1 & 3/2 & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \end{array}$$

$2r_1 + r_3$ of $B_2 \Rightarrow B_3$

$$\begin{array}{c} \begin{array}{ccccc} 1 & 0 & -4 & 3 & 3/2 \end{array} \\ B_3=\begin{bmatrix} 0 & 1 & 3/2 & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \end{array}$$

$-r_1$ (of B_3) + (*)

Step 7. Add multiples of the first row of B_3 to all the row of A_3 above so that all entries in the pivotal column, except for the pivot, become zero.

$$\begin{array}{c} \begin{array}{ccccc} 1 & 0 & -4 & 3 & 3/2 \end{array} \\ B_4=\begin{bmatrix} 0 & 1 & 3/2 & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \end{array}$$

let $(*_1) \rightarrow$ 1 0 −4 3 3/2
$(*_2) \rightarrow$ 0 1 3/2 −2 1/2

Step 8. Identify C as the $(m-2)\times n$ submatrix of B_4 obtained by deleting the first row of B_4; do not erase the first row of B_4. Repeat Steps 1–7 on C.

$$C=\begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

pivotal column of C. pivot

Thus $r_1/2$ of $C \Rightarrow C_1 \xrightarrow[\text{of } C_1]{-2r_1+r_2} C_2 \xrightarrow[\& \ -\frac{3}{2}r_1(\text{of } C_2)+(*_2)]{4r_1(\text{of } C_2)+(*_1)}$ The final matrix

i.e., the final matrix

$$\begin{bmatrix} 1 & 0 & 0 & 9 & 19/2 \\ 0 & 1 & 0 & -17/4 & -5/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in reduced row echelon form

Theorem (3.1.2): Let $AX=B$ and $CX=D$ be two linear systems each of m equations in n unknowns. If the augmented matrices $[A \vdots B]$ and $[C \vdots D]$ of these systems are row equivalent, then both linear systems have exactly the same solutions.

Corollary (3.1.3): If A and C are row equivalent $m \times n$ matrices, then the linear systems $AX=0$ and $CX=0$ have exactly the same solutions.

The Gauss-Jordan reduction procedure for solving the linear system $AX=B$ is as follows.

Step 1. Form the augmented matrix $[A \vdots B]$.

Step 2. Transform the augmented matrix to reduced row echelon form by using elementary row operations.

Step 3. The linear system that corresponds to the matrix in reduced row echelon form that has been obtained in Step 2 has exactly the same solutions as the given linear system.

Remark: The rows consisting entirely of zeros can be ignored.

Ex. Solve the linear system
$$\begin{aligned} x+2y+3z&=9 \\ 2x-y+z&=8 \\ 3x \quad\quad -z&=3 \end{aligned}$$
by Gauss-Jordan reduction.

Solution: The augmented matrix of this linear system is

Step 1.
$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$