Chapter Two

(Binding Energy & Nuclear Models)

(2-1) Nuclear Binding Energy

Since an atom contains Z positively charged particles (protons) and N=A-Z neutral particles (neutrons), the total charge of a nucleus is +Ze, where e represents the charge of one electron. Thus, the mass of a neutral atom, M_{atom}, can be expressed in terms of the mass of its nucleus, M_{nuc} and its electrons m_e.

$$M_{atom} = M_{nuc} + Zm_e \qquad M_{nuc} = Zm_p + (A-Z)m_n$$

where m_p is the proton mass, m_e the mass of an electron and m_n the mass of a neutron. For example the mass of the rubidium nucleus, ^{87}Rb, which contains 37 protons and 50 neutrons, can be calculated as:

$$M_{nuc}(^{87}Rb) = 37 \times 1.007277 + 50 \times 1.008665 = 87.7025 amu$$

The atomic mass, indicated on most tables of the elements, is the sum of the nuclear mass and the total mass of the electrons present in a neutral atom. In the case of ^{87}Rb, 37 electrons are present to balance the charge of the 37 protons. The atomic mass of ^{87}Rb is then:

$$M_{atom}(^{87}\mathrm{Rb}) = M_{nuc}(^{87}\mathrm{Rb}) + Zm_e$$
$$= 87.7025 + 37 \times 0.00055 = 87.7228 amu$$

From the periodic table, the measured mass of a ^{87}Rb atom is found to be $M_A^{measured}(^{87}$Rb) = 86.909187 amu. These two masses are not equal and the difference is given by:

$$\Delta m = M_{atom}(^{87}Rb) - M_{atom}^{measured}(^{87}Rb) = 0.813613 amu$$

Expanding the terms in this equation, shows that the difference in mass corresponds to a difference in the mass of the nucleus

$$\Delta m = M_{atom} - M_{atom}^{measured}$$
$$= Zm_p + Zm_e + (A - Z)m_n - M_{nuc}^{measured} - Zm_e$$

which reduces to

$$\Delta m = M_{atom} - M_{atom}^{measured}$$
$$= Zm_p + (A - Z)m_n - M_{nuc}^{measured} = M_{nuc} - M_{nuc}^{measured}$$

Thus, when using atomic mass values given by the periodic table, the mass difference between the measured and calculated is given by

$$\Delta m = M_{nuc} - M_{nuc}^{measured} = Zm_p + Zm_e + (A - Z)m_n - M_{atom}^{measured}$$

Notice also that

$$Zm_p + Zm_e = Zm_H$$

Where m_H is a mass of the hydrogen atom.

From this and other examples it can be concluded that the actual mass of an atomic nucleus is always smaller than the sum of the rest masses of all its nucleons (protons and neutrons). This is because some of the mass of the nucleons is converted into the energy that is needed to form that nucleus and hold it together. This converted mass, Δm, is called the "mass defect" and the corresponding energy is called the "binding energy" and is related to the stability of the nucleus; the greater binding energy leads to the more stable the nucleus. This energy also represents the minimum energy required to separate a nucleus into protons and neutrons. The mass defect and binding energy can be directly related, as shown below:

$$B(A, Z) = \Delta m \times 931.5 \text{MeV} / \text{amu} \qquad \text{or}$$
$$B(A, Z) = 931.5(Zm_p + Zm_e + Nm_n - M_{atom}^{measured})$$

Since the total binding energy of the nucleus depends on the number of nucleons, a more useful measure of the cohesiveness is the average binding energy B_{ave}.

$$B_{ave}(A, Z) = \frac{B(A, Z)}{A} \quad (\text{MeV} / \text{nucleon})$$

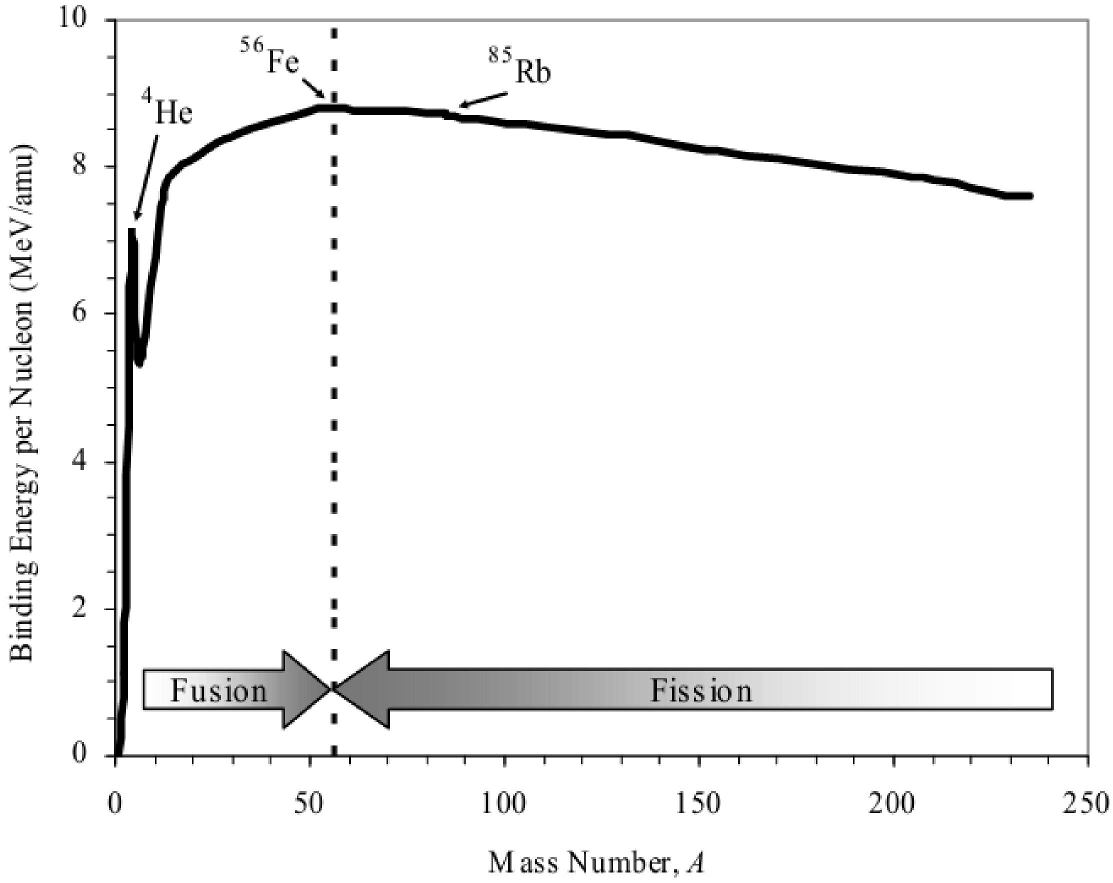

Figure (2-1): Variation of binding energy per nucleon with the atomic mass number

The binding energy per nucleon varies with the atomic mass number A, as shown in figure (2-1). For example, the binding energy per nucleon in a rubidium nucleus is 8.7MeV, while in helium it is 7.3MeV. The curve indicates three characteristic regions:

- Region of stability: A flat region between (A) equal to approximately 35 and 70 where the binding energy per nucleon is nearly constant. This region exhibits a peak near A = 60. These nuclei are near iron and are called the iron peak nuclei representing the most stable elements.
- Region of fission reactions: From the curve it can be seen that the heaviest nuclei are less stable than the nuclei near A = 60, which suggests that energy can be released if heavy nuclei split apart into smaller nuclei having masses nearer the iron peak. This process is called fission (the basic nuclear reaction used in atomic

bombs as uncontrolled reactions and in nuclear power and research reactors as controlled chain reactions). Each fission event generates nuclei commonly referred to as fission fragments with mass numbers ranging from 80 to 160.

- Region of fusion reactions: The curve of binding energy suggests a second way in which energy could be released in nuclear reactions. The lightest elements (like hydrogen and helium) have nuclei that are less stable than heavier elements up to the iron peak. If two light nuclei can form a heavier nucleus a significant energy could be released. This process is called fusion, and represents the basic nuclear reaction in hydrogen (thermonuclear) weapons.

(2-2) Separation energy

Are the analogous of the ionization energies in atomic physics, reflecting the energies of the valence nucleons. The separation energy of any particle is defined as the amount of energy needed to remove a particle from the nucleus. For a given N,Z; S_n , S_p is larger for nuclei with even N or Z than with odd one, this due to the pair effect of nuclear force which increase the binding energy and separation energy.

There are two equations to determine the separation energy using the mass or the binding energy. For neutron as follow:

$S_n=931.5[M(A-1,Z)+m_n-M(A,Z)]$

$S_n=B(A,Z)-B(A-1,Z)$

For proton:

$S_p=931.5[M(A-1,Z-1)+m_H-M(A,Z)]$

$S_p=B(A,Z)-B(A-1,Z-1)$

For alpha particle:

$S_\alpha=931.5[M(A-4,Z-2)+m_\alpha-M(A,Z)]$

$S_\alpha=B(A,Z)-B(A-4,Z-2)-B(\alpha\equiv{}^{4}_{2}He)$

The fact, that each pair of these equations represents two equivalent equations.

Example: calculate the separation energy of neutron for ^{209}Pb by using the two methods, where M($^{209}_{82}Pb$)=209.05398u, M($^{208}_{82}Pb$)=208.04754u.

Sol.:

1- S_n=931.5[M(A-1,Z)+m_n-M(A,Z)]= 931.5[M($^{208}_{82}Pb$)+m_n-M($^{209}_{82}Pb$)]

=931.5[208.04754+1.008665-209.05398]=2.07259MeV

2- $B(A,Z) = 931.5(Zm_p + Zm_e + Nm_n - M_{atom}^{measured})$

B($^{209}_{82}Pb$)=931.5(82x1.007276+82x0.000549+127x1.008665-209.05398]

=1572.48844MeV

B($^{208}_{82}Pb$)=931.5(82x1.007276+82x0.000549+126x1.008665-208.04754]

=1570.41585MeV

S_n=B(A,Z)-B(A-1,Z)=1572.48844-1570.41585=2.07259MeV

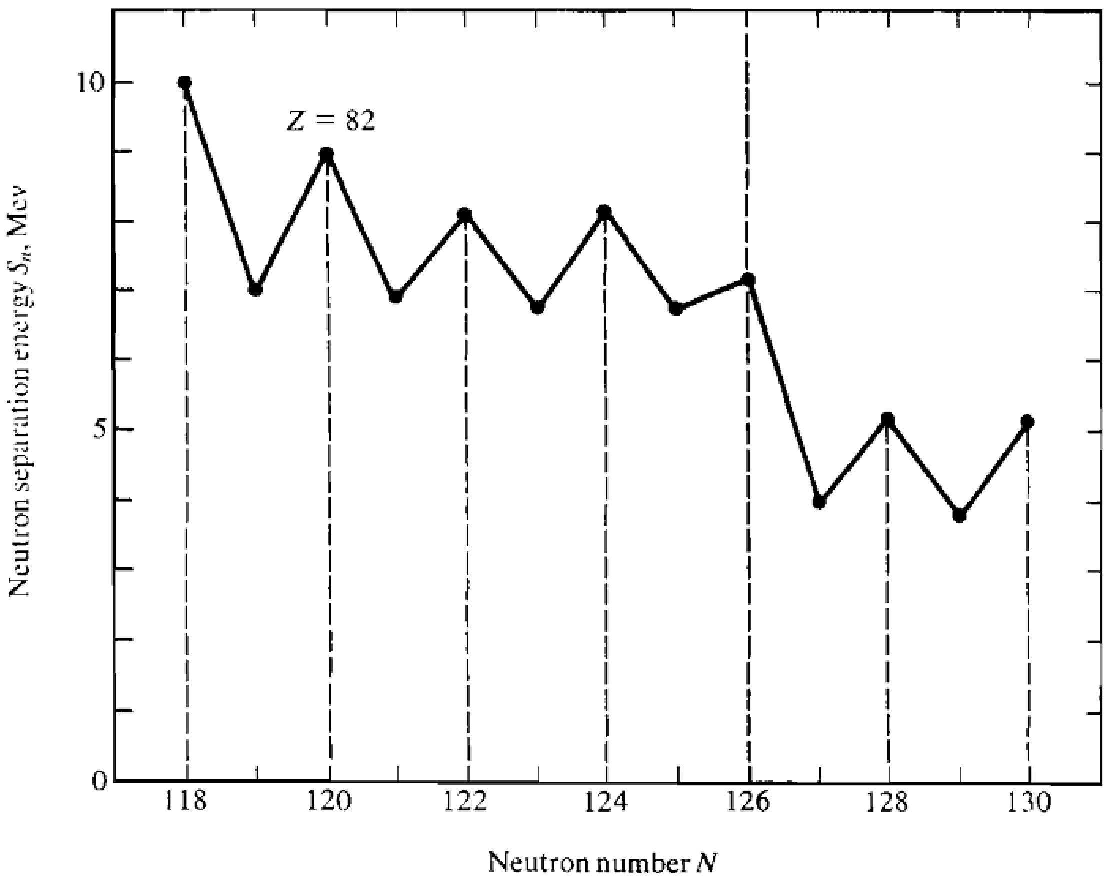

Figure (2-2): Neutron separation energy of lead isotopes as a function of neutron number.

(2-3) Nuclear Spins and Dipole Moments

Both the proton and the neutron have spin angular momentum of $\frac{1}{2}\hbar$. Furthermore, just as electrons in an atom can have orbital angular momentum, so also can nucleons inside a nucleus. We know from quantum mechanics that orbital angular momentum can take on only integral values. The total angular momentum of the constituents-namely, the vector sum of the orbital and intrinsic spin angular momenta-defines the spin of the nucleus.

Thus, the nuclei with even atomic number have integral nuclear spin whereas nuclei with odd atomic number have half-integral nuclear spin. However, the nuclei with an even number of protons and an even number of neutrons (even-even nuclei) have zero nuclear spin. These facts lend credence to the hypothesis that spins of nucleons inside a nucleus are very strongly paired so as to cancel their overall effect.

To explain the fine structure of the spectral lines, suppose that each of the electron, proton and neutron has spin momentum result from rotation on its axis and therefore they has a magnetic moment due to this rotation, the interaction of magnetic moment of the electron with the magnetic moment of the nucleus leads to increase or decrease the tension between them and then will increase or decrease energy of electron, i.e. split the energy levels of the electron and thus will be divided every line of spectral lines into several lines. Every charged particle has a magnetic dipole moment associated with its spin, given by:

$$\vec{\mu}_s = g_s \frac{e}{2mc} \vec{s}$$

where e, m and s are the charge, mass and the intrinsic spin of the charged particle. The constant g is known as the Lande factor (Gyromagnetic ratio), which for a point particle, such as the electron, is expected to have the value g = 2.

When $g \neq 2$ the particle is said to possess an anomalous magnetic moment, which is usually ascribed to the particle having a substructure like proton and neutron:

$$g_s \approx \begin{cases} 5.5857 & \text{proton} \\ -3.8261 & \text{neutron} \end{cases}$$

Relate to angular momentum

$$\vec{\mu}_\ell = g_\ell \frac{e}{2mc} \vec{\ell}$$

$g_\ell = 1$ for proton

$g_\ell = 0$ for neutron

For the electron (with $s = \frac{1}{2}\hbar$) , the dipole moment $\mu_e \approx \mu_B$, where μ_B is the Bohr magneton, defined as:

$$\mu_B = \frac{e\hbar}{2m_e c} = 5.79 \times 10^{-11}\ \text{MeV/T}$$

Where a magnetic field of 1 tesla (T) corresponds to 10^4 gauss (G), the magnetic dipole moment for nucleons is measured in terms of the nuclear magneton, defined using the proton mass:

$$\mu_N = \frac{e\hbar}{2m_p c} = 3.15 \times 10^{-14}\ \text{MeV/T}$$

From the ratio of m_p/m_e , we deduce that the Bohr magneton is about 2000 times larger than the nuclear magneton due to $m_p = 1837 m_e \approx 2000 m_e$, i.e.

atomic moment >> nuclear moment.

The magnetic moments of the proton and the neutron are:

$$\mu_p \approx 2.79\mu_N \quad , \quad \mu_n \approx -1.91\mu_N$$

Thus, the electrons cannot be present inside nuclei because it would then be particularly hard to explain the small values of nuclear moments, since even one electron would produce a moment a thousand times that observed for nuclei.

(2-4) Nuclear Forces

Protons and neutrons are bound inside nuclei, despite the Coulomb repulsion among protons. Therefore there must be different and much stronger force acting among nucleons to bind them together. This force is called nuclear force, nuclear binding force, or in more modern settings, the strong interaction. There are notable properties of the nuclear binding force.

1. It is much stronger than the electromagnetic force (the force is charge independent, i.e. $F_{pp}=F_{nn}=F_{pn}$, we can see that from the equality of energy level, binding energy and total angular momentum of mirror nuclei). As shown in the empirical mass formula [see section (2-5-1)], the coefficient of the Coulomb term is more than an order of magnitude smaller than the other terms in the binding energy.

2. It is short-ranged, acts only up to 1–2 fm.

3. It has the saturation property, giving nearly constant B/A= $B_{ave} \approx 8.5$ MeV.

This is in stark contrast to the electromagnetic force.

4. The force depends on spin and states of the nucleon.

i.e. the nuclear force between two nucleons of the same type (p and p) or (n and n) could be the biggest whenever the total angular momentum for the first has the maximum value and equal with opposite direction to the other, i.e. the angular momentum for both is equal to zero.

For example, let $\vec{s}_1$ and $\vec{s}_2$ are the spin to the two protons, $\vec{\ell}_1$ and $\vec{\ell}_2$ the orbital momentum, therefore the total angular momentum for the first is equal $\vec{j}_1 = \vec{\ell}_1 + \vec{s}_1$, and for the second proton $\vec{j}_2 = \vec{\ell}_2 + \vec{s}_2$

For the maximum nuclear force between the two nucleons, must be $\vec{j}_{1max} = -\vec{j}_{2max}$

Put the two protons in s-state, then:

$\vec{\ell}_1 = 0, \quad \vec{s}_1 = 1/2 \Rightarrow \vec{j}_1 = 1/2$ and $\vec{\ell}_2 = 0, \quad \vec{s}_2 = 1/2 \Rightarrow \vec{j}_2 = 1/2$

For maximum F_{pp}, $\vec{j}_1 = -\vec{j}_2 \Rightarrow \vec{J} = \vec{j}_1 + \vec{j}_2 = 1/2 - 1/2 = 0$

for two protons in p-state, then:

$\vec{\ell}_1 = 1, \quad \vec{s}_1 = 1/2 \Rightarrow \vec{j}_{1min} = 1/2$ (if $\vec{\ell}_1 \# \vec{s}_1$) and $\vec{j}_{1max} = 3/2$ (if $\vec{\ell}_1 // \vec{s}_1$)

$\vec{\ell}_2 = 1, \quad \vec{s}_2 = 1/2 \Rightarrow \vec{j}_{2min} = 1/2$ (if $\vec{\ell}_2 \# \vec{s}_2$) and $\vec{j}_{2max} = 3/2$ (if $\vec{\ell}_2 // \vec{s}_2$)

For maximum F_{pp}, $\vec{j}_{1max} = -\vec{j}_{2max} \Rightarrow \vec{J} = \vec{j}_{1max} + \vec{j}_{2max} = 3/2 - 3/2 = 0$

This phenomenon is called the pairing effect.

5. It is exchange forces. Like of the photon exchange between the electric charges, there are medium mass particles (mesons) were exchange between nucleons.

6. Even though the nuclear force is attractive to bind nucleons, there is a repulsive core when they approach too closely, around 0.5fm. They basically cannot go closer. i.e. 2fm>r>0.5fm leads to attractive nuclear force, while r<0.5fm repulsive force.

(2-5) Nuclear Models

The aim of nuclear models is to understand how certain combinations of N neutrons and Z protons form bound states and to understand the masses, spins and parities of those states. The great majorities of nuclear species contain excess neutrons or protons and are therefore β-unstable. Many heavy nuclei decay by α-particle emission or by other forms of spontaneous fission into lighter elements. Another aim of this chapter is to understand why certain nuclei are stable against these decays and what determines the dominant decay modes of unstable nuclei. The problem of calculating the energies, spins and parities of nuclei is one of the most difficult problems of theoretical physics.

(2-5-1) Liquid-Drop Model

The liquid drop model of the nucleus, proposed by Bohr and derived by Von Weizsacker in 1935, was one of the earliest phenomenological successes constructed to account for the binding energy of a nucleus. Experiments revealed that nuclei were essentially spherical objects, with sizes that could be characterized by radii proportional to $A^{1/3}$, which suggested that nuclear densities were almost independent of nucleon number. This leads quite naturally to a model that envisions the nucleus as an incompressible liquid droplet, with nucleons playing the role analogous to molecules in a drop of normal liquid. In this picture, known as the liquid drop model, the individual quantum properties of nucleons are completely ignored. As in the case of a liquid drop, the nucleus is imagined as composed of a stable central core of nucleons for which the nuclear force is completely saturated (is based on the short range of nuclear forces), and a surface layer of nucleons that is not bound as tightly (forces not saturated). This weaker binding at the surface decreases the effective binding energy per nucleon (B/A), and provides a "surface tension", or an attraction of the surface nucleons towards the center. Nucleons interact strongly with their nearest neighbors, just as molecules do in a drop of water. Therefore, one can attempt to describe their properties by the corresponding quantities, i.e. the radius, the density, the surface tension and the volume energy, (see figure 2-3).

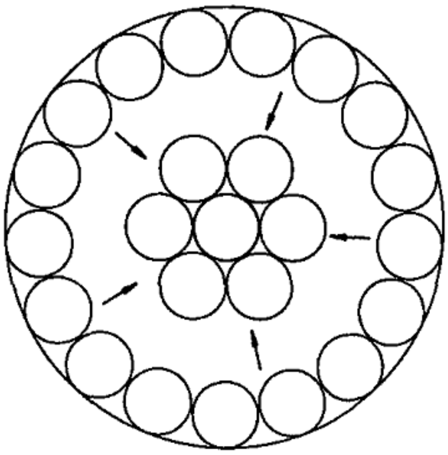

Figure (2-3): Surface layer and core of nucleus in the liquid drop model.

The essential assumptions are:

1- The nucleus consists of incompressible matter so that $R \sim A^{1/3}$.
2- The nuclear force is identical for every nucleon and in particular does not depend on whether it is a neutron or a proton.
3- The nuclear force saturates.

Semi-empirical mass formula

An excellent parameterization of the binding energies of nuclei in their ground state was proposed in 1935 by Bethe and Weizsacker. This formula relies on the liquid-drop analogy but also incorporates two quantum ingredients; one is an asymmetry energy which tends to favor equal numbers of protons and neutrons. The other is a pairing energy which favors configurations where two identical fermions are paired. The semi-empirical mass formula (SEMF) or Bethe-Weizsacker mass formula is:

$$B(A,Z) = a_v A - a_s A^{2/3} - a_c Z(Z-1)A^{-1/3} - a_a \frac{(A-2Z)^2}{A} + \delta a_p A^{-3/4} + \eta$$

or $B(A,Z)=T_v+ T_s+ T_c+ T_a+ T_p+ T_{sh}$

The coefficients a_i are chosen so as to give a good approximation to the observed binding energies. A good combination is the following:

Volume Term a_v = 15.5 MeV

Surface Term a_s = 16.8 MeV

Coulomb Term a_c = 0.72 MeV

Asymmetry Term a_a = 23 MeV

Pairing Term a_p = 34 MeV

Shell Term $T_{sh} = \eta = 1 \rightarrow 3$MeV

We will now study each term in the SEMF:

1- Volume term (T_v):

The first term is the volume term (a_vA), that describes how the binding energy is mostly proportional to A i.e. to the volume of nucleus, remember that the binding energy is a measure of the interaction among nucleons. Since nucleons are closely packed in the nucleus and the nuclear force has a very short range, each nucleon ends up interacting only with a few neighbors. This means that independently of the total number of nucleons, each one of them contribute in the same way. Thus the force is not proportional to the total number of nucleons one nucleon can interact with, but it's simply proportional to A.

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_o^3 A$$

where $R = R_o A^{1/3}$

$T_v \propto A$ or $T_v \propto V \Rightarrow T_v = a_v A$

The constant of proportionality is a fitting parameter that is found experimentally to be a_v = 15.5MeV

This value is smaller than the binding energy of the nucleons to their neighbors as determined by the strength of the nuclear (strong) interaction. The total binding energy is instead the difference between the interaction of a nucleon to its neighbor and the kinetic energy of the nucleon itself. As for electrons in an atom, the nucleons are fermions, thus they cannot all be in the same state with zero kinetic energy, but they will fill up all the kinetic energy levels according to Pauli's exclusion principle. This model, which takes into account the nuclear binding energy and the kinetic energy due to the filling of shells, indeed gives an accurate estimate for a_v.

For example T_v (^8Be) = 15.5x8=124MeV

2- Surface term (T_s):

The surface term ($-a_sA^{2/3}$), also based on the strong force, is a correction to the volume term. We explained the volume term as arising from the fact that each nucleon interacts with a constant number of nucleons, independent of A. While this is valid for nucleons deep within the nucleus, those nucleons on the surface of the nucleus have fewer nearest neighbors. This term is similar to surface forces that arise for example in droplets of liquids, a mechanism that creates surface tension in liquids. We can say that; whenever increasing of the nuclear surface area, the binding energy will decrease.

$$T_s \propto 4\pi R^2 \propto 4\pi R_o^2 A^{2/3} \Rightarrow T_s = a_s A^{2/3}$$

Where $4\pi R^2$ is the surface of the sphere, $R=R_oA^{1/3}$

Also the term must be subtracted from the volume term and we expect the coefficient as to have a similar order of magnitude as a_v. In fact a_s = 16.8MeV.

T_s for ^8Be=-16.8x$8^{2/3}$ =-67.2MeV

3- Coulomb term (T_c):

The third term $-a_c\, Z(Z-1)A^{-1/3}$ derives from the Coulomb interaction among protons, and of course is proportional to Z. This term is subtracted from the volume term since the Coulomb repulsion makes a nucleus containing many protons less favorable (more energetic). To find the form of the term and estimate the coefficient a_c, the nucleus is modeled as a uniformly charged sphere.

the volume is $V = \frac{4}{3}\pi R^3$

nuclear density is $\rho = \frac{Q}{V} = \frac{Ze}{\frac{4}{3}\pi R^3} = \frac{3Ze}{4\pi R^3}$

We assume that we have a sphere of radius r when collected the nucleon to get the nucleus with volume $V = \frac{4}{3}\pi r^3$.

$$q = V\rho = \frac{4}{3}\pi r^3 . \frac{3Ze}{4\pi R^3} = \frac{Zer^3}{R^3}$$

The potential energy (V_p) of such a charge distribution at the surface is:

$$V_p = \frac{Kq}{r} = \frac{KZer^2}{R^3}$$

We add a charge sample dq to the sphere to get a shell of thickness dr

$$dq = \rho dV = \frac{3Ze}{4\pi R^3} 4\pi r^2 dr = \frac{3Zer^2}{R^3} dr \quad , \quad dV = 4\pi r^2 dr$$

The required work to add this layer is

$$dW = V_p dq = \frac{KZer^2}{R^3} . \frac{3Zer^2}{R^3} dr = \frac{3KZ^2e^2r^4}{R^6} dr$$

To find the total work to forming the nucleus

$$W = \int_0^R dW = \frac{3KZ^2e^2}{R^6}\int_0^R r^4 dr \Rightarrow W = \frac{3KZ^2e^2}{5R}$$

We know that the proton is not repulsion with itself but with the other protons around it i.e. repulsion with Z-1 protons, then we can write:

$$W = \frac{3KZ^2e^2}{5R} - \frac{3Ke^2}{5R}Z = \frac{3Ke^2Z(Z-1)}{5R}$$

Using the empirical radius formula $R = R_o A^{1/3}$, $R_o=1.2x10^{-15}m$, $e=1.6x10^{-19}C$ and $K=9x10^9 Nm^2/C^2$

$$T_c \propto W \propto \frac{3Ke^2Z(Z-1)}{5R_oA^{1/3}} = a_c Z(Z-1)A^{-1/3}$$

This gives the shape of the Coulomb term. Then the constant a_c can be estimated from $a_c \approx \frac{3}{5}\frac{e^2}{4\pi\varepsilon_o R_o}$ with R_o=1.2fm, to be a_c=0.72MeV which is agreement with the experimental value.

$$T_c \text{ for } {}^8_4Be_4 = -\frac{0.72x4x3}{2} = -4.32MeV$$

4- Asymmetry term (T_a):

The Coulomb term seems to indicated that it would be favorable to have less protons in a nucleus and more neutrons. However, this is not the case of the liquid-drop model in order to explain the fact that we have roughly the same number of neutrons and protons in stable nuclei. There is a correction term in the SEMF which tries to take into account the symmetry in protons and neutrons. This correction (and the following one) can only be explained by a more complex model of the nucleus, the shell model, together with the quantum-mechanical exclusion principle. If we were to add more neutrons, they will have to be more energetic, thus increasing the total energy of the nucleus. This increase more than the Coulomb repulsion, so that it is more favorable to have an approximately equal number of protons and neutrons. $(A-2Z)^2$ The shape of the symmetry term is $\frac{(A-2Z)^2}{A}$. It can be more easily understood by considering the fact that this term goes to zero for A = 2Z and its effect is smaller for larger A (while for smaller nuclei the symmetry effect is more important). i.e. for isobars of Z=N=A/2 (symmetry) has been more stability than isobars of Z≠N (Anti symmetry or Asymmetry) which reduce of the binding energy.

Asymmetry define as the difference between binding energy for two isobar, one have Z=N and the other Z≠N.

$|T_a| = B(A, Z=N) - B(A, Z\neq N)$

The coefficient is a_a = 23MeV

$$T_a \text{ for } {}^{16}_{7}N_9 = -23\frac{(16-14)^2}{16} = -5.75\text{MeV} \quad , T_a = 0 \text{ for } {}^{16}_{8}O_8$$

5- Pairing term (T_p):

This term is linked to the physical evidence that like-nucleons tend to pair off. Then it means that the binding energy is greater ($\delta > 0$) if we have an even-even nucleus, where all the neutrons and all the protons are paired-off. If we have a nucleus with both an odd number of neutrons and of protons, it is thus favorable to convert one of the protons into a neutrons or vice-versa. Thus, with all other factor constant, we have to subtract ($\delta<0$) a term from the binding energy for odd-odd configurations. Finally, for even-odd configurations we do not expect any influence from this pairing energy ($\delta = 0$). The pairing term is then:

$$+\delta a_p A^{-3/4} = \begin{cases} +a_p A^{-3/4} & \text{for even} - \text{even} \\ 0 & \text{for even} - \text{odd (odd A)} \\ -a_p A^{-3/4} & \text{for odd} - \text{odd} \end{cases}$$

with a_p = 34MeV

Assume that A is the mass number for even-odd nucleus, then A+1 represent an even-even nucleus and A-1 for odd-odd, the pairing term is written as:

$$T_p = \frac{B(A-1)+B(A+1)}{2} - B(A)$$

$$T_p \text{ for } {}^{16}_{8}O_8 = +34x16^{-3/4} = 4.25\text{MeV}, \quad T_p({}^{7}_{3}Li_4) = 0, \quad T_p({}^{6}_{3}Li_3) = -8.87\text{MeV}$$

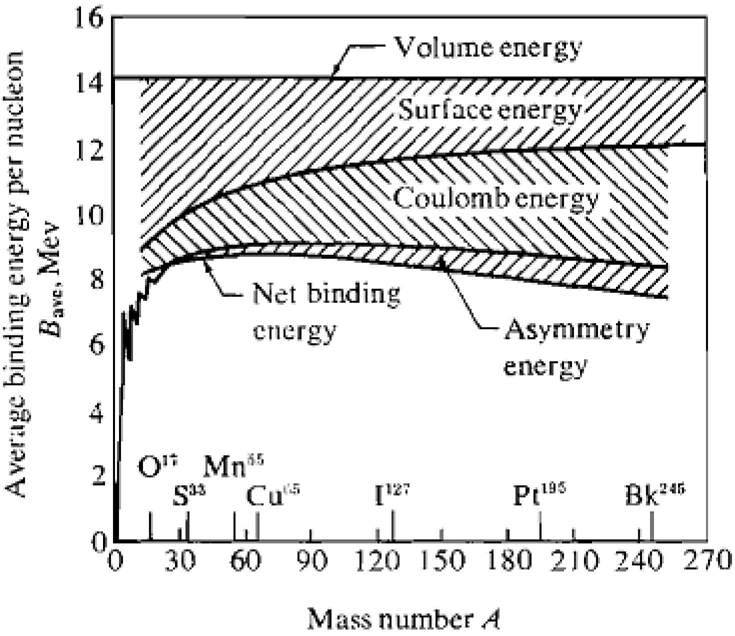

Figure (2-4): summary of liquid-drop model treatment of average binding energy.

6- Shell term (T_v):

There is found experimentally that for N=Z nuclei which has a double magic number (2,8,20,28,50,82,…) has been a very stability, very high nuclear binding energy and high abundance, then for one magic number N or Z, then for nearest of magic number which due to increases in binding energy.

$T_{sh}=1\rightarrow 3$MeV

$T_{sh}=3$ for double magic number (N and Z = magic number) like ${}^{4}_{2}He_{2}$, ${}^{16}_{8}O_{8}$

$T_{sh}=2$ for single magic number (N or Z = magic number and the other is near of magic number) like ${}^{15}_{7}N_{8}$, ${}^{15}_{8}O_{7}$

$T_{sh}=1$ for single magic number (N or Z = magic number and the other is far from magic number) like ${}^{18}_{8}O_{10}$

$T_{sh}=0$ for no magic number of N and Z, like ${}^{16}_{7}N_{9}$

H.W.: find the binding energy for ${}^{6}_{3}Li$, ${}^{8}_{4}Be$, ${}^{17}_{8}O$, ${}^{208}_{82}Pb$. Using:

a) mass formula, b) Weizsacker formula. Were the atomic mass of ^{6}Li=6.015124u, ^{8}Be=8.02502u, ^{17}O=17.00453u and ^{208}Pb=208.04754u

Mass parabolas

With a little rearrangement of SEMF Eq.:

$$B(A,Z) = a_v A - a_s A^{2/3} - a_c Z(Z-1)A^{-1/3} - a_a \frac{(A-2Z)^2}{A} + \delta a_p A^{-3/4} + \eta$$

From the formula of binding energy depending on the mass:

$M_N(A,Z)=Zm_p+Nm_n-B(A,Z)$

we can write the mass of a nucleus in the following way:

$$M_N(A,Z) = Zm_p + Nm_n - [a_v A - a_s A^{2/3} - a_c Z(Z-1)A^{-1/3} - a_a \frac{(A-2Z)^2}{A} + \delta a_p A^{-3/4} + \eta]$$

It's clear that for any value of A (A = constant), this equation represent of a parabola equation as $M(A,Z)c^2=xA+yZ+zZ^2\pm\delta$.

The minimum mass occurs for $Z = Z_o$ (usually not an integer). The plot of Z versus A or N gives the line of greatest nuclear stability. Setting $\partial(Mc^2)/\partial Z = 0$ yields:

$$Z_A = \frac{A/2}{1+0.25(a_c/a_a)A^{2/3}} = \frac{A/2}{1+0.0078A^{2/3}}$$

For odd-A isobars, δ = 0. and therefore the equation gives a single parabola, which is shown in Fig. (2-5) for a typical case.

It is clear from Fig. (2-5a) that for odd-A nuclides there can be only one (stable) isobar for which both these conditions do not occur. Note that:

$M(A,Z) > M(A,Z+1)$	beta (electron) decay takes place from Z to $Z+1$
$M(A,Z) > M(A,Z-1)$	electron capture and perhaps positron decay[1] takes place from Z to $Z-1$

For even-A isobars, two parabolas are generated by above equation, differing in mass by 26. A typical case is given in Fig. (2-5b). Depending on the curvature of the parabolas and the separation 26, there can be several stable even-even isobars. Three is the largest number found in nature. There should be no stable odd-odd nuclides. The exceptional cases H^2, Li^6, B^{10}, and N^{14} are caused by rapid variations of the nuclear binding energy for very light nuclides, because of nuclear structure effects which are not included in the liquid-drop model. Figure (2-5b) shows that for certain odd-odd nuclides both conditions are met so that electron and positron decay from the identical nuclide are possible and do indeed occur.

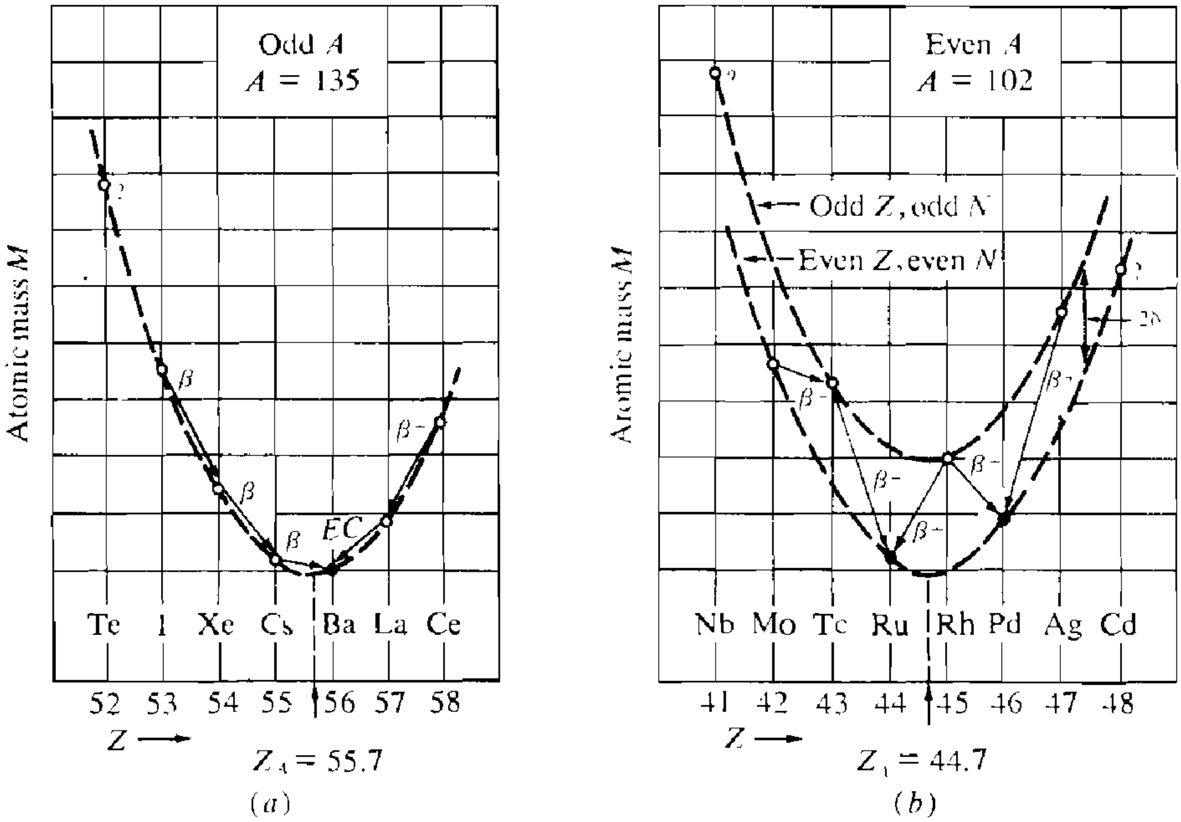

Figure (2-5): mass parabola for isobars. (a) odd A nuclei. (b) even A nuclei. Full circles represent stable nuclides and open circles radioactive nuclides.

(2-5-2) Nuclear Shell Model

The nuclear shell model is based on the analogous model for the orbital structure of atomic electrons in atoms. Although the liquid drop model of the nucleus has proved to be quite successful for predicting subtle variations in the mass of nuclides with slightly different mass and atomic numbers, it avoids any mention of the internal arrangement of the nucleons in the nucleus. We have observed that: 1- there are an abnormally high number of stable nuclides whose proton and/or neutron numbers equals the magic numbers 2,8,20,28,50,82,126. 2-Further evidence for such magic numbers is provided by the very high binding energy of nuclei with both Z and

N being magic, and 3- the abnormally high or low alpha and beta particle energies emitted by radioactive nuclei according to whether the daughter or parent nucleus has a magic number of neutrons. Similarly, 4- nuclides with a magic number of neutrons are observed to have a relatively low probability of absorbing an extra neutron, i.e. they have lowest of absorption cross sections for neutrons (neutron-capture cross sections).

To explain such nuclear systematics and the internal structure of the nucleus, a shell model of the nucleus has been developed. This model uses Schrodinger's wave equation or quantum mechanics to describe the energetics of the nucleons in a nucleus in a manner analogous to that used to describe the discrete energy states of electrons around the nucleus. This model assumes:

1. Each nucleon moves independently in the nucleus uninfluenced by the motion of the other nucleons.
2. Each nucleon moves in a potential well which is constant from the center of the nucleus to its edge where it increases rapidly by several tens of MeV.

When the model's quantum-mechanical wave equation is solved numerically, the nucleons are found to distribute themselves into a number of energy levels. There is a set of energy levels for protons and an independent set of levels for neutrons. Filled shells are indicated by large gaps between adjacent energy levels and are computed to occur at the experimentally observed values of 2, 8, 20, 28, 50, 82, and 126 neutrons or protons. Such closed shells are analogous to the closed shells of orbital electrons. However, the shell model has been useful to obtain such results that predicts the magic numbers and particularly useful in predicting several properties of the nucleus, including (1) the total angular momentum of a nucleus, (2) characteristics of isomeric transitions, which are governed by large changes in nuclear angular momentum, (3) the characteristics of beta decay and gamma decay, and (4) the magnetic moments of nuclei.

Single-particle shell model

The basic assumption of any shell model is that despite the strong overall attraction between nucleons which provides the binding energy considered in previous section, the motion of each nucleon is practically independent of that of any other nucleon. This apparent contradiction is resolved by effects of the Pauli Exclusion Principle. If all inter-nucleon couplings (called residual interactions) are ignored, we call the model the single-particle shell model. In terms of Schrodinger's equation, each nucleon is then assumed to move in the same potential. The potential is spherical in the simplest case, but there is good evidence that for nucleon numbers far from closed shells the potential should have an ellipsoidal shape. This condition will be considered later.

This model depends on two quantum numbers, the radial (total) quantum number n and the orbital quantum number ℓ. In nuclear physics each state is specified by n and ℓ. Also for ℓ = 0, 1, 2, 3, 4, 5, we use the spectroscopic letters s, p, d, f, g, h, respectively. A state denoted by 2p therefore means that n = 2, ℓ = 1.

The simplest useful potentials are an infinite square well potential of radius R

$$V = \begin{cases} 0 & r < R \\ \infty & r = R \end{cases}$$

or a harmonic oscillator potential

$$V = \tfrac{1}{2} m_0 \omega^2 r^2$$

where ω is the frequency of oscillation of the particle of mass m_o. More realistic potentials are a finite square well potential as:

$$V = \begin{cases} -V_0 & r \leq R \\ 0 & r > R \end{cases}$$

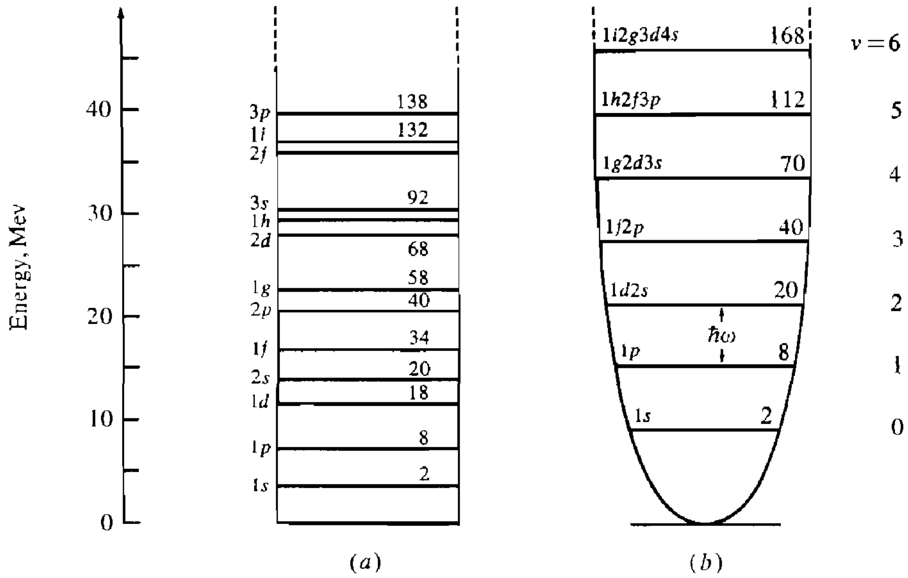

Figure (2-6): Energy levels of nucleons (a) in an infinite spherical square-well potential, (b) in a harmonic oscillator potential. The spectroscopic notation (n, ℓ) and the total occupation number up to any particular level are given.

Spin-Orbit coupling shell model

A simple Coulomb potential is clearly not appropriate and we need some form that describes the effective potential of all the other nucleons. Since the strong nuclear force is short-ranged we would expect the potential to follow the form of the density distribution of nucleons in the nucleus. For medium and heavy nuclei, the Fermi distribution fits the data and the corresponding potential is called the Woods-Saxon form:

$$V_{\mathrm{central}}(r) = \frac{-V_0}{1 + \mathrm{e}^{(r-R)/a}}$$

However, although these potentials can be shown to offer an explanation for the lowest magic numbers, they do not work for the higher ones. This is true of all purely central potentials.

The crucial step in understanding the origin of the magic numbers was suggested that by analogy with atomic physics there should also be a spin–orbit part, so that the total potential is:

$$V_{\text{total}} = V_{\text{central}}(r) + V_{\ell s}(r)\mathbf{L}\cdot\mathbf{S}$$

Where L and S are the orbital and spin angular momentum operators for a single nucleon and $V_{\ell s}(r)$ is an arbitrary function of the radial coordinate. This form for the total potential is the same as that used in atomic physics except for the presence of the function $V_{\ell s}(r)$. Once we have coupling between L and S then m_ℓ and m_s are no longer 'good' quantum numbers and we have to work with eigenstates of the total angular momentum vector J, defined by J=L+S. Squaring this, we have:

$$\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2)$$

and hence the expectation value of L.S, which we write as $\langle \ell s\rangle$, is:

$$\langle \ell s\rangle = \frac{\hbar^2}{2}[j(j+1) - \ell(\ell+1) - s(s+1)] = \begin{cases} \ell/2 & \text{for} \quad j = \ell + \frac{1}{2} \\ -(\ell+1)/2 & \text{for} \quad j = \ell - \frac{1}{2} \end{cases}$$

(We are always dealing with a single nucleon, so that s=1/2) The splitting between the two levels is thus:

$$\Delta E_{ls} = \frac{2\ell+1}{2}\hbar^2\langle V_{\ell s}\rangle$$

Experimentally, it is found that $V_{\ell s}(r)$ is negative, which means that the state with $j = \ell + \frac{1}{2}$ has a lower energy than the state with $j = \ell - \frac{1}{2}$. This is the opposite of the situation in atoms. Also, the splitting are substantial and increase linearly with ℓ. Hence for higher ℓ, crossings between levels can occur. Namely, for large ℓ, the splitting of any two neighboring degenerate levels can shift the $j = \ell - \frac{1}{2}$ state of the initial lower level to lie above the $j = \ell + \frac{1}{2}$ level of the previously higher level.

An example of the resulting splitting up to the 1G state is shown in Figure (2-7), where the usual atomic spectroscopic notation has been used, i.e. levels are written

($n\ell_j$) with S, P, D, F, G, ... : used for ℓ = 0, 1, 2, 3, 4, ... Magic numbers occur when there are particularly large gaps between groups of levels. Note that there is no restriction on the values of ℓ for a given n because, unlike in the atomic case, the strong nuclear potential is not Coulomb-like.

The configuration of a real nuclide (which of course has both neutrons and protons) describes the filling of its energy levels (sub-shells), for protons and for neutrons, in order, with the notation $(n\ell_j)^k$ for each sub-shell, where k is the occupancy of the given sub-shell. Sometimes, for brevity, the completely filled sub-shells are not listed, and if the highest sub-shell is nearly filled, k can be given as a negative number, indicating how far from being filled that sub-shell is.

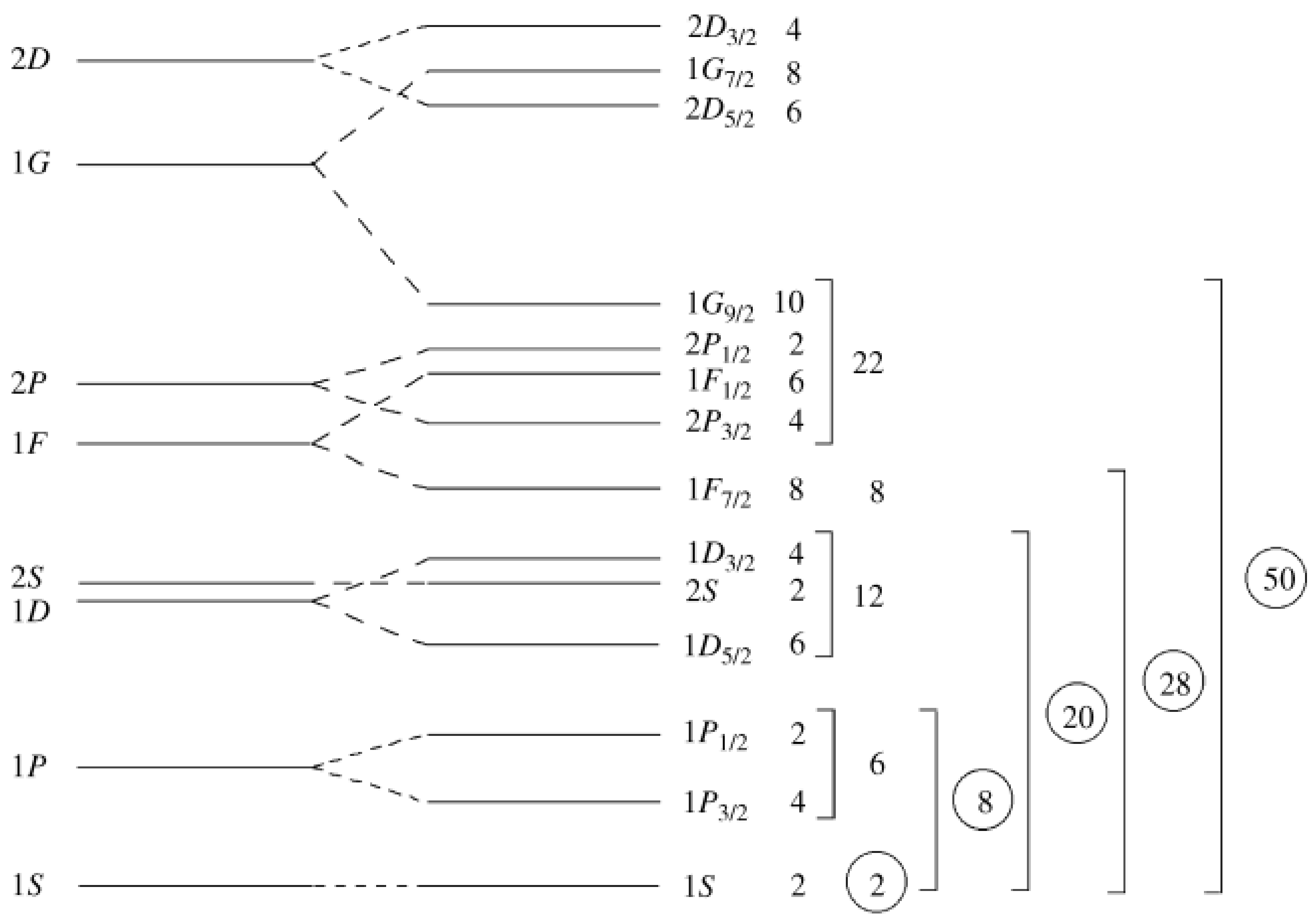

Figure (2-7): low-lying energy levels in a single-particle shell model using a Woods-Saxon potential plus spin-orbit term.

Using the ordering diagram above, and remembering that the maximum occupancy of each sub-shell is 2j+1, we predict, for example, the configuration for ${}^{17}_{8}\mathrm{O}$ to be:

$$(1s_{\frac{1}{2}})^2(1p_{\frac{3}{2}})^4(1p_{\frac{1}{2}})^2 \quad \text{for the protons}$$

$$(1s_{\frac{1}{2}})^2(1p_{\frac{3}{2}})^4(1p_{\frac{1}{2}})^2(1d_{\frac{5}{2}})^1 \quad \text{for the neutrons}$$

Notice that all the proton sub-shells are filled, and that all the neutrons are in filled sub-shells except for the last one, which is in a sub-shell on its own. Most of the ground state properties of ${}^{17}_{8}\mathrm{O}$ can therefore be found from just stating the neutron configuration as $(1\mathrm{d}_{\frac{5}{2}})^1$.

Although the spin-orbit shell model had one of the most stimulating effects on nuclear structure physics, the simple form given above cannot be sufficient. For example, the model cannot explain why even-even nuclei always have a zero ground-state spin, or more generally, why any even number of identical nucleons couples to zero ground-state spin. Evidently there is a (residual) nucleon-nucleon interaction which favors the pairing of nucleons with opposing angular momenta.

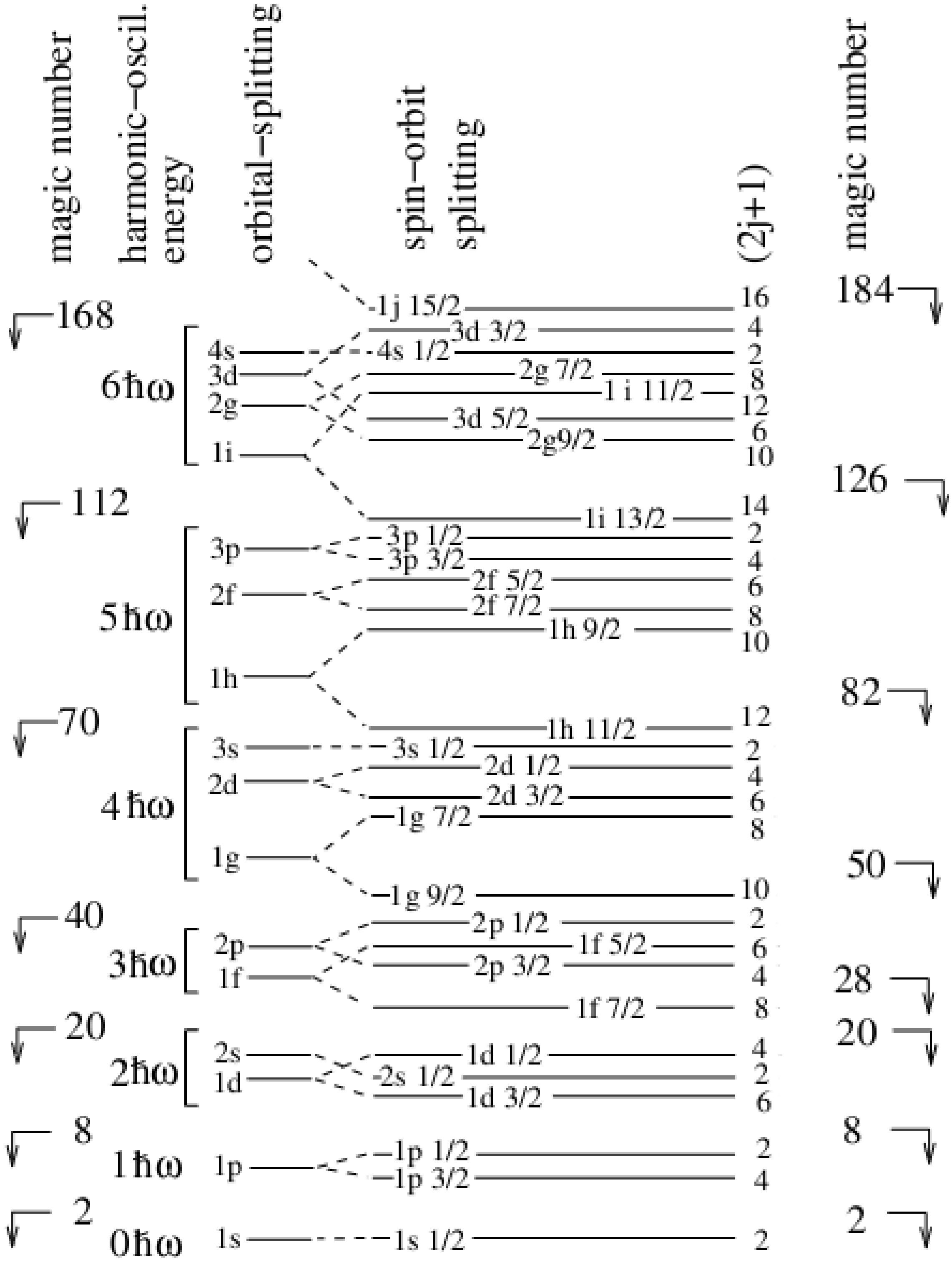

Figure (2-8): Nucleon orbitals in a model with a spin-orbit interaction. The two left columns show the magic numbers and energies for a pure harmonic potential. The splitting of different values of the orbital angular momentum ℓ can be arranged by modifying the central potential. Finally, the spin-orbit coupling splits the levels so that they depend on the relative orientation of the spin and orbital angular momentum. The number of nucleons per level (2j+1) and the resulting magic numbers are shown on the right.

Table (2-1): Arrangement of the nuclear shells and its distributions.

ℓ / n	0	1	2	3	4	5	6	shell	No. of nucleons
1	1s							s	2
2		1p						p	6
3	2s		1d					d	10
4		2p		1f				f	14
5	3s		2d		1g			g	18
6		3p		2f		1h		h	22
7	4s		3d		2g		1i	i	26

For example, ${}^{17}_{9}F_8$ and ${}^{17}_{8}O_9$ have one unpaired nucleon outside a doubly magic ${}^{16}_{8}O_8$ core. The above figure, tells us that the unpaired nucleon is in a $\ell=2$, j=5/2. The spin parity of the nucleus is predicted to be $5/2^+$ since the parity of the orbital is -1^{ℓ}. This agrees with observation. The first excited states of ${}^{17}_{9}F_8$ and ${}^{17}_{8}O_9$ corresponding to raising the unpaired nucleon to the next higher orbital, are predicted to be $1/2^+$, once again in agreement with observation.

On the other hand, ${}^{15}_{8}N_7$ and ${}^{15}_{8}O_7$ have one “hole” in their ${}^{16}_{8}O_8$ core, the ground state quantum numbers should then be the quantum numbers of the hole which are $\ell=1$ and j=1/2 according to above Figure. The quantum numbers of the ground state are then predicted to be $1/2^-$, in agreement with observation.

(2-5-3) Fermi Gas Model (Statistical Model or Uniform Particle Model)

This model supposes that, as a result of the strong nuclear compound between the nucleons, the movement of them cannot study alone, but we must study them statistically, i.e. it's give the average of the physical quantity from all of the nucleons.

In this model, the protons and neutrons that make up the nucleus are assumed to comprise two independent systems of nucleons, each freely moving inside the nuclear volume subject to the constraints of the Pauli principle. The potential felt by every nucleon is the superposition of the potentials due to all the other nucleons. In the case of neutrons this is assumed to be a finite-depth square well; for protons, the Coulomb potential modifies this. A sketch of the potential wells in both cases is shown in Figure (2-9).

For a given ground state nucleus, the energy levels will fill up from the bottom of the well. The energy of the highest level that is completely filled is called the Fermi level of energy E_F and has a momentum $p_F=(2ME_F)^{1/2}$, where M is the mass of the nucleon. Within the volume V, the number of states with a momentum between p and p+dp is given by the *density of states factor*:

$$n(p)\mathrm{d}p = \mathrm{d}n = \frac{4\pi V}{(2\pi\hbar)^3} p^2 \mathrm{d}p$$

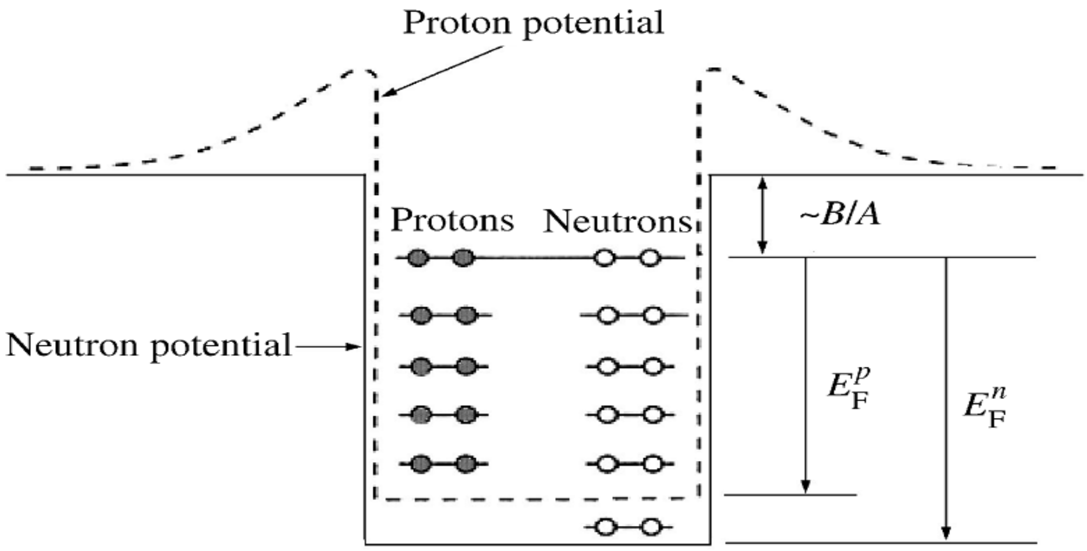

Figure (2-9): proton and neutron potentials and states in the fermi gas model.

Since every state can contain two fermions of the same species, we can have (using $n = 2\int_0^{p_F} dn$)

$$N = \frac{V(p_F^n)^3}{3\pi^2\hbar^3} \quad \text{and} \quad Z = \frac{V(p_F^p)^3}{3\pi^2\hbar^3}$$

For neutrons and protons, respectively, with a nuclear volume

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A$$

Where experimentally R_o=1.2fm, Assuming for the moment that the depths of the neutron and proton wells are the same, we find for a nucleus with Z=N=A/2, the Fermi momentum:

$$p_F = p_F^n = p_F^p = \frac{\hbar}{R_0}\left(\frac{9\pi}{8}\right)^{1/3} \approx 250\,\mathrm{MeV/c}.$$

Thus the nucleons move freely within the nucleus with quite large momenta. The Fermi energy is:

$$E_F = \frac{p_F^2}{2M} \approx 33\,\mathrm{MeV}.$$

The difference between the top of the well and the Fermi level is constant for most heavy nuclei and is just the average binding energy per nucleon B_{ave}=B/A=7–8 MeV. The depth of the potential and the Fermi energy are to a good approximation independent of the mass number A:

$V_o = E_F + B_{ave} \approx 40$MeV

Heavy nuclei generally have a surplus of neutrons. Since the Fermi levels of the protons and neutrons in a stable nucleus have to be equal (otherwise the nucleus can become more stable by β-decay) this implies that the depth of the potential well for the neutron gas has to be deeper than for the proton gas, as shown in Figure (2-9). Protons are therefore on average less tightly bound in nuclei than are neutrons.

(2-5-4) Collective Model

For each of the liquid drop model and shell model have a specific applications, all of them succeed in the interpretation of some phenomena and fails to explain other phenomena. So it became logical to consider each of these models is complementary to another in a single model called the collective model as a model that combines the two models. This model views the nucleus as having a hard core of nucleons in filled shells, as in the shell model, with outer valence nucleons that behave like the surface molecules of a liquid drop. In addition to the successes of each of the two models, this model has succeeded in formulating an equation to calculate the rotational energy levels to the even-even nuclei, i.e. the energy levels of deformed nuclei are very complicated, because there is often coupling between the various modes of excitation, but nevertheless many predictions of the collective model are confirmed experimentally.

$$E_{rot.} = \frac{\hbar^2}{2I} J(J+1)$$

Where, I is the moment of inertia to the nucleus.
J is the total angular momentum to the nucleus.

(2-5-5) Optical Model

The name comes from likening of the nucleus target as an optical lens, while the fallen particle represents the fallen optical wave. The most important achievements of the optical model are a description of the cross section for neutron absorption as a function of neutron's energy and of the mass number of the nucleus target. This model has been assumed that the total potential of the neutron and the nucleus target is a complex potential and can be written as:

$V=V_o+iV_1$

Where V_o is the real part to the total potential which represents the effect of the nucleus on the neutron.

V_o=-42MeV for $r \leq R$
=0 for $r>R$

While iV_1 is the imaginary part to the potential which represents the probability to creates the compound nuclei.

(2-5-6) Cluster Model (α-Particle Model)

This model supposes that the alpha particle represent the building block of the nucleus, it's clear that this model explain the emitting of alpha particles from the heavy nucleus, for examples:

$${}^{12}_{6}C_6 \equiv 3\alpha \quad , \quad {}^{16}_{8}O_8 \equiv 4\alpha \quad , \quad {}^{20}_{10}Ne_{10} \equiv 5\alpha$$