Boundary-Value Problems (B.V.P)

Introduction:

In the problems considered so far, we have singled out a particular solution of a differential equation by specifying some suitable initial conditions. Another important class of problems called boundary-value problems, involves the determination of solutions of a differential equation which satisfy prescribed conditions at two given points. Such conditions are called boundary conditions.

A differential equation that has given conditions allows to find the specific function that satisfies a given DE. rather than a family of functions. These type of problems are called initial value problems (IVP).

If the given conditions are given at more than one point & the differential equation is of order 2 or greater, it is called a boundary value problem. (BVP).

A BVP problem can have none, one, or many solutions.

Examples of homogeneous Boundary-Value Problems.

1- Find a solution to the BVP problem $\frac{d^2y}{dx^2} - y = 0$; $y(0) = 0$, $y(1) = 1$ if we know $y(x) = c_1 e^{x} + c_2 e^{-x}$ is a general solution to the differential equation.

solution// $y(0)=0$

$y(0) = c_1 e^0 + c_2 e^0 = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$y(1) = 1 \Rightarrow c_1 e^1 + c_2 e^{-1} = 1 \Rightarrow -c_2 e + c_2 e^{-1} = 1$

$c_2 (e^{-1} - e) = 1 \Rightarrow c_2 = \frac{1}{e^{-1} - e}, \; c_1 = \frac{-1}{e^{-1} - e}$

$\therefore y(x) = \frac{-1}{e^{-1} - e} e^x + \frac{1}{e^{-1} - e} e^{-x}$

$y(x) = \frac{e^{-x} - e^x}{e^{-1} - e}$

2. Find a solution to the initial value problem $y'' + 4y = 0$; $y(0) = 1$, $y'(\pi) = 2$ if we know $y(x) = c_1 \cos(2x) + c_2 \sin(2x)$ is a general solution to the differential equation.

solution//

$y(0) = 1 \Rightarrow c_1 \overset{=1}{\cos(0)} + c_2 \overset{=0}{\sin(0)} = 1 \Rightarrow c_1 = 1$

$y'(x) = -2 \sin(2x) + 2 c_2 \cos(2x)$

$y'(\frac{\pi}{2}) = -2 \overset{=0}{\sin(\pi)} + 2 c_2 \cos(\pi) = 2$

$-2 c_2 = 2 \Rightarrow c_2 = -1$

$\therefore y(x) = \cos(2x) - \sin(2x)$.

3. Solve the boundary-value problem $y'' = y'$; $y(0) = 1$, $y(1) = 2$. Reminder $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

solution//

$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m_1 = 0, m_2 = 1$

$\therefore y(x) = c_1 e^{0x} + c_2 e^{x} = c_1 + c_2 e^x$

$y(0) = 1 \Rightarrow 1 = c_1 + c_2 e^0 \Rightarrow c_1 = 1 - c_2$

$y(1) = 2 \Rightarrow 2 = c_1 + c_2 e^1 \Rightarrow 2 = 1 - c_2 + c_2 e^1 \Rightarrow c_2 = \frac{1}{e-1}$

$c_1 = 1 - \frac{1}{e-1} = \frac{e-2}{e-1}$

$\Rightarrow y(x) = \frac{e-2}{e-1} + \frac{e^x}{e-1}$

4. Solve the following B.V.P. problem a) $y'' + 4y = 0$

$y(0) = -2, \quad y(\frac{\pi}{4}) = 10$

Solution:

$m^2 + 4 = 0 \Rightarrow m = \mp 2i$

$\therefore y(x) = C_1 \cos(2x) + C_2 \sin(2x)$

$y(0) = -2 \Rightarrow -2 = C_1 \underbrace{\cos(0)}_{=1} + C_2 \underbrace{\sin(0)}_{=0} \Rightarrow -2 = C_1$

$y(\frac{\pi}{4}) = 0 \Rightarrow 10 = C_1 \underbrace{\cos\frac{\pi}{2}}_{=0} + C_2 \underbrace{\sin\frac{\pi}{2}}_{=1} \Rightarrow 10 = C_2$

$\therefore y(x) = -2\cos(2x) + 10\sin(2x)$

b) $y'' + 2y = 0 ; \; y(0) = 1, \; y(\pi) = 0$

$m^2 + 2 = 0 \Rightarrow m = \mp\sqrt{2}i$

$y(x) = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x)$

$y(0) = 1 \Rightarrow 1 = C_1$

$y(\pi) = 0 \Rightarrow 0 = \cos(\sqrt{2}\pi) + C_2 \sin(\sqrt{2}\pi)$

$C_2 = -\frac{\cos(\sqrt{2}\pi)}{\sin(\sqrt{2}\pi)} = -\cot(\sqrt{2}\pi)$

$\therefore y(x) = \cos(\sqrt{2}x) - \cot(\sqrt{2}\pi)\sin(\sqrt{2}x)$

5. Solve the following B.V.P. problem $y'' + 4y' + 4 = 0$;

$y(0) = 0, \; y(1) = 1$

solution:

$m^2 + 4m + 4 = 0 \Rightarrow (m+2)(m+2) = 0 \Rightarrow m_1 = -2, m_2 = -2$

$\therefore y(x) = e^{-2x}(C_1 x + C_2)$

$y(0) = 0 \Rightarrow 0 = e^0(0 + C_2) \Rightarrow C_2 = 0$

$y(1) = 1 \Rightarrow 1 = e^{-2\times 1}(C_1) \Rightarrow C_1 = e^{2}$

$\therefore y(x) = e^{-2x}(e^2 x) = e^{-2(x-1)}x$

Examples of nonhomogeneous boundary-value problems

Solve the following BVP. (initial value problem)

1. $y'' + 9y = \cos x$, $y'(0) = 5$, $y(\frac{\pi}{2}) = -\frac{5}{3}$

Solution:

$m^2 + 9 = 0 \Rightarrow m = \mp 3i$

$y_h(x) = C_1 \cos(3x) + C_2 \sin(3x)$

$y_p = A\cos x \Rightarrow y_p' = -A\sin x$, $y_p'' = -A\cos x$

$y_p'' + 9y_p = \cos x$

$-A\cos x + 9A\cos x = \cos x \Rightarrow 8A\cos x = \cos x \Rightarrow A = \frac{1}{8}$

$\therefore y_p = \frac{1}{8}\cos x$

$y(x) = C_1\cos(3x) + C_2\sin(3x) + \frac{1}{8}\cos x$

$y(\frac{\pi}{2}) = -\frac{5}{3} \Rightarrow -\frac{5}{3} = C_1(0) + C_2(-1) + \frac{1}{8}(0) \Rightarrow C_2 = \frac{5}{3}$

$y'(x) = -3C_1\sin(3x) + 3C_2\cos(3x) - \frac{1}{8}\sin x$

$y'(0) = 5 \Rightarrow 5 = -C_1(0) + C_2(1) + \frac{1}{8}(0) \Rightarrow C_2 = \frac{5}{3} \Rightarrow y(x) = y_h + y_p$

$\therefore y(x) = C_1\cos(3x) + \frac{5}{3}\sin(3x) + \frac{1}{8}\cos x$,

and there will be infinitely many solutions to the BVP.

2. $y'' + 3y' - 4y = 2e^x$, $y(0) = 1$, $y'(0) = 2$

Solution:

$m^2 + 3m - 4 = 0 \Rightarrow (m+4)(m-1) = 0 \Rightarrow m_1 = -4, m_2 = 1$

$\therefore y_h(x) = C_1 e^{-4x} + C_2 e^x$

$y_p = Axe^x \Rightarrow y_p' = Ae^x + Axe^x$, $y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$

$y'' + 3y' - 4y = 2e^x$

$2Ae^x + \cancel{Axe^x} + 3Ae^x + \cancel{3Axe^x} - \cancel{4Axe^x} = 2e^x$

$5Ae^x = 2e^x \Rightarrow A = \frac{2}{5} \Rightarrow y_p = \frac{2}{5}xe^x$

$\therefore y(x) = c_1 e^{-4x} + c_2 e^{x} + \frac{2}{5} x e^{x}$

$y(0)=1 \Rightarrow 1 = c_1 + c_2 \Rightarrow c_1 = 1 - c_2$

$y'(x) = -4c_1 e^{-4x} + c_2 e^{x} + \frac{2}{5} e^{x} + \frac{2}{5} x e^{x}$

$y'(0)=2 \Rightarrow 2 = -4(1-c_2) + c_2 + \frac{2}{5}$

$4+2 = 5c_2 + \frac{2}{5} \Rightarrow 5c_2 = 6 - \frac{2}{5} = \frac{28}{5} \Rightarrow c_2 = \frac{28}{25}$

$\therefore c_1 = 1 - c_2 = 1 - \frac{28}{25} = \frac{-3}{25} \Rightarrow y(x) = y_h + y_p$

$\therefore y(x) = \frac{-3}{25} e^{-4x} + \frac{28}{25} e^{x} + \frac{2}{5} x e^{x}$

3. $y'' + y = x^2 e^{x}$; $y(0)=0$, $y'(0)=1$

solution:

$m^2+1=0 \Rightarrow m = \mp i \Rightarrow y_h = c_1 \cos x + c_2 \sin x$

$y_p = (Ax^2 + Bx + C) e^{x}$

$y_p' = (Ax^2+Bx+C)e^{x} + e^{x}(2Ax+B) =$

$y_p'' = (Ax^2+Bx+C)e^{x} + e^{x}(2Ax+B) + e^{x}(2A) + (2Ax+B)e^{x}$

$y_p'' + y = x^2 e^{x}$

$\{2Ax^2 + (4A+2B)x + (2A+2B+2C)\} e^{x} = x^2 e^{x}$

$2A = 1$, $4A+2B=0$, $2A+2B+2C=0$

$A = \frac{1}{2}$, $B = -1$, $C = \frac{1}{2}$

$y_p(x) = (\frac{1}{2}x^2 - x + \frac{1}{2}) e^{x}$

$y(x) = y_h + y_p$

$y(x) = c_1 \cos x + c_2 \sin x + (\frac{1}{2}x^2 - x + \frac{1}{2}) e^{x}$

$y(0)=0 \Rightarrow 0 = c_1 + \frac{1}{2} \Rightarrow c_1 = -\frac{1}{2}$

$y'(x) = -c_1 \sin x + c_2 \cos x + (\frac{1}{2}x^2 - x + \frac{1}{2}) e^{x} + e^{x}(x-1)$

$y'(0)=1 \Rightarrow 1 = c_2 + \frac{1}{2} - 1 \Rightarrow c_2 = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore y(x) = -\frac{1}{2}\cos x + \frac{3}{2}\sin x + (\frac{1}{2}x^2 - x + \frac{1}{2}) e^{x}$

Excercise 6

Solve the following BVP.

1. $y'' + 4y = 0$; $y(0) = -2$, $y(2\pi) = -2$
2. $y'' + 25y = 0$, $y'(0) = 6$, $y'(\pi) = -9$
3. $y'' + 2y' - 3y = 1 + xe^{x}$; $y(0) = 0$, $y'(1) = 1$
4. $y'' + 4y' + 4y = e^{-2x} + \sin 2x$, $y(0) = 0$, $y'(0) = 1$
5. $y'' + 3y' + 2y = x^{2}$, $y(0) = 0$, $y'(2) = 0$