Lecture 1 :

The Wilson-Sommerfeld Quantization Rule

The success of the Bohr model, as measured by its agreement with experiment, was certainly very striking, but it only accentuated the mysterious nature of the postulates on which the model was based. One of the biggest mysteries was the question of the relation between Bohr's quantization of the orbital angular momentum of an electron moving in a circular orbit and Planck's quantization of the total energy of an entity, such as an electron, executing simple harmonic motion. In 1916 some light was shed upon this by Wilson and Sommerfeld, who enunciated a set of rules for the quantization of any physical system for which the coordinates are periodic functions of time. These rules included both the Planck and the Bohr quantization as special cases. These rules can be stated as follows: For any physical system in which the coordinates are periodic functions of time, there exists a quantum condition for each coordinate. These quantum conditions are :

$$\oint p_q \, dq = n_q h \qquad \text{----------} \qquad (1)$$

where q is one of the coordinates, p_q is the momentum associated with that coordinate, n_q is a quantum number which takes on integral values, and means that the integration is taken over one period of the coordinate q. The meaning of these rules can best be illustrated in terms of some specific examples. Consider a one-dimensional simple harmonic oscillator. Its total energy can be written, in terms of position and momentum, as :

$$E = K + V = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \qquad \text{-------}(2)$$

& since $P = mv \qquad \rightarrow mv^2/2 = P^2/2m$ -------(3)

$$E = K + V = \frac{P_x^2}{2m} + \frac{Kx^2}{2}$$
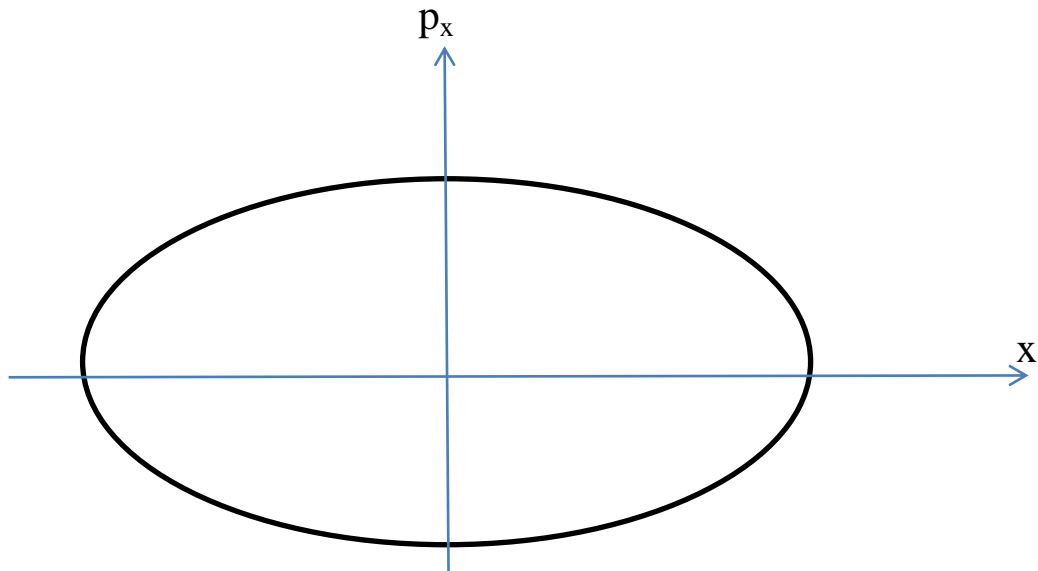
$$\frac{P_x^2}{2mE} + \frac{x^2}{2E/K} = 1 \qquad\qquad ----------(4)$$

The relation between p_x and x is the equation of an ellipse, i.e:

$$\frac{P_x^2}{b^2} + \frac{x^2}{a^2} = 1 \qquad ---------(5)$$

Where a and b are the semiaxes of the ellipse and by comparison the equation of ellipse with our equation, we find that :

$$a = \sqrt{2E/K} \qquad \text{and} \qquad b = \sqrt{2mE} \quad ------(6)$$

We call a space that represent p-q plane (phase space) and the plot diagram of the linear oscillator is a (phase diagram) that shown in figure(1):

Figure(1)

Phase space diagram of the linear motion oscillator S.H.O.

To find the value of integration $\oint p_q \, dq$, which represents the left side of the quantization rule, we will use phase diagrame that represents the amount of linear motion p_x and displacement x . This geometric representation gives information about the amount of linear motion at any displacement. Note that the values of integration is the area bounded by curved inside the ellipse which is equal to :

$$\text{Area of ellipse} = \oint p_q \, dq = \pi \, ab$$

Now, by substitute the value of a and b :

$$a = \sqrt{2E/K} \qquad \text{and} \qquad b = \sqrt{2mE}$$

$$\oint p_q \, dq = \frac{2\pi E}{\sqrt{K/m}} \qquad \text{--------(7)}$$

Since : $$v = \frac{1}{2\pi}\sqrt{K/m} \qquad \text{--------(8)}$$

Where v is the frequency of the oscillation, so that :

$$\oint p_q \, dq = \frac{2\pi E}{2\pi v} \qquad \text{----------------------(9)}$$

By use the Wilson- Sommerfeld quantization rule:

$$\oint p_x \, dx = \frac{E}{v} = n_x h = nh \qquad \text{------(10)}$$
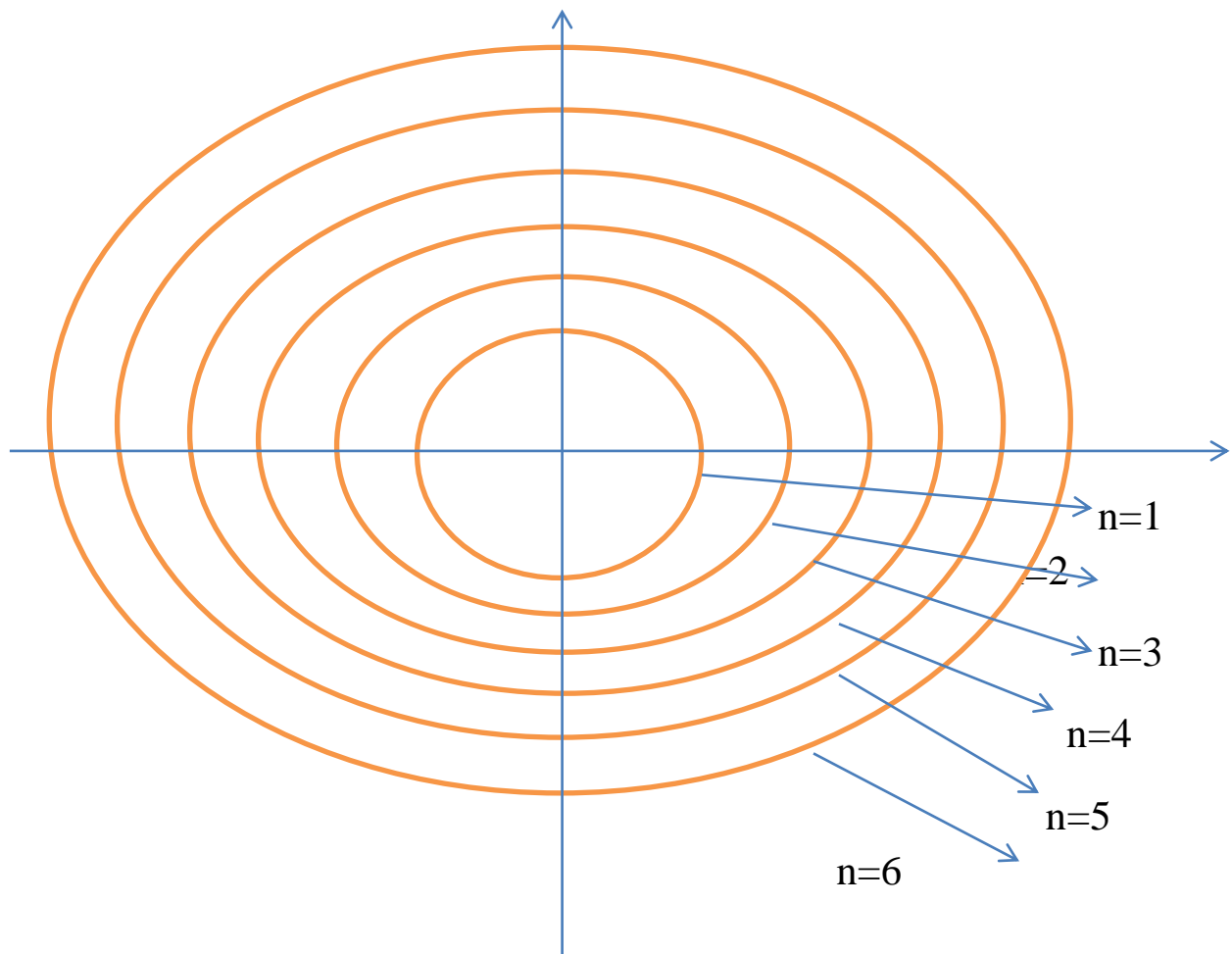
and the total energy of the body that moves simple harmonic motion in one dimension is: :
E = nhv   ------------(11)

which is identical with Planck's quantization rule.

Note that the allowed energy states of oscillation are represented by a series of ellipses in phase space, the area enclosed between successive ellipses always being h as in fig.(2):

Figure(2)

We find that the classical situation corresponds to h →0, all values of E and hence all ellipses being allowed if that were true.

From the Wilson-Sommerfeld rule, (Eq.1), We can also deduce the Bohr quantization of orbital angular momentum(L = nh/2π) .

An electron moving in a circular orbit of radius r has an angular momentum, mvr = L, which is constant. The angular coordinate is θ, which is a periodic function of the time. That is θ versus t is a function increasing linearly from zero to 2π rad in one period and repeating this pattern in each succeeding period. The quantization rule:

$$\oint p_q \, dq = n_q h \quad , \quad \text{in this case becomes:}$$

$$\oint L d\theta = nh$$

$$\oint L d\theta = L \oint_0^{2\pi} d\theta = 2\pi L \quad \& \quad 2\pi L = nh \quad \text{so:}$$

$$L = \frac{nh}{2\pi} = n\hbar \quad ----- (12)$$

which is identical with Bohr's quantization law.

Physical Meaning of Second Bohr's Postulate :

A more physical interpretation of the Bohr quantization rule was given in 1924 by de Broglie. The Bohr quantization of angular momentum can be written as:

$$L = mvr = pr = nh/2\pi = n\hbar \qquad n = 1, 2, 3, ...$$

where p is the linear momentum of an electron in an allowed orbit of radius r. If we substitute into this equation the expression for p in terms of the corresponding de-Broglie wavelength($p=h/\lambda$),the Bohr equation becomes:

$$L = \frac{hr}{\lambda} = \frac{nh}{2\pi}$$

$$2\pi r = n\lambda \qquad n = 1,2,3 \qquad -------------- (13)$$
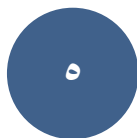
Thus the allowed orbits are those in which the circumference of the orbit can contain exactly an integral number of de Broglie wavelengths.

This requirement means that the electron moving at a constant speed in its orbit around the nucleus and have associated wave where wavelength specified under the de-Broglie hypothesis. When the electron complete cycle around the nucleus, the wave associated with the electron will restore itself. If the orbit circumference is equal to the integer of wavelength, this means that the resulting overlapping waves from the completed several courses around the nucleus will be in the same phase. If the orbit circumference is not equal to an integer of the wavelength, overlapped will cancel each other and will be accompanying wave in this case is zero. This means that there is no electron orbit and is not available for an electron to exist in it(see fig.3).
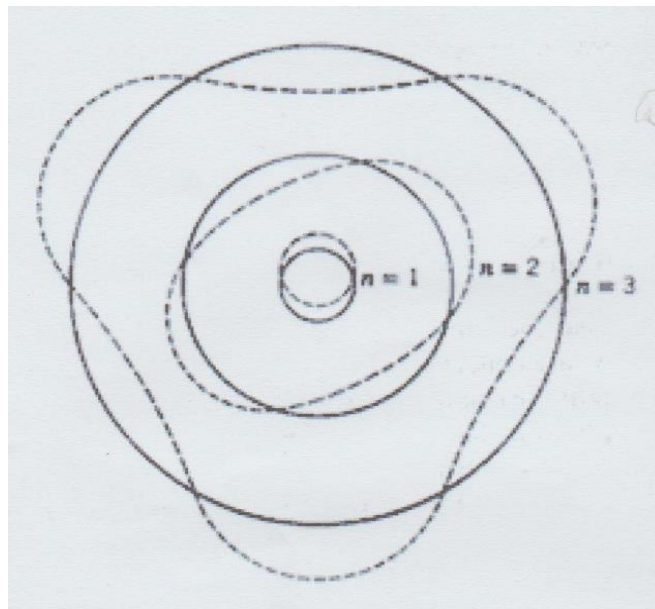
Figure(3)

Sommerfeld́s assumption—Energies & Orbits:

One of the important applications of the Wilson-Sommerfeld quantization rules is to the case of a hydrogen atom in which it was assumed that the electron could move in elliptical orbits. This was done by Sommerfeld in an attempt to explain the fine structure of the hydrogen spectrum. The fine structure is a splitting of the spectral lines, into several distinct components, which is found in all atomic spectra. It can be observed only by using equipment of very high resolution since the separation, in terms of reciprocal wavelength, between adjacent components of a single spectral line is of the order of 10^{-4} times the separation between adjacent lines. Sommerfeld first evaluated the size and shape of the allowed elliptical orbits, as well as the total energy of an electron moving in such an orbit using the formulas of classical mechanics. Describing the motion in terms of the polar coordinates r and θ by applied the quantization rule to fined :

$$\oint L d\theta = n_\theta h \quad \text{------------(14)}$$

$$\oint P_r dr = n_r h \quad \text{-------------(15)}$$

- The first condition yields the same restriction on the orbital angular momentum

$$L = n_\theta h \quad n_\theta = 1, 2, 3, \ldots\ldots$$

- The second condition leads to the following relation

$$L(a/b - 1) = n_r h \quad n_r = 1, 2, 3, \ldots\ldots$$

- By applying the condition of the mechanical stability,

where μ is the red mass of electron

$$\frac{1}{4\pi\varepsilon_o} \frac{Ze^2}{r^2} = \mu \frac{v^2}{r}$$

We get a third equation, and solving these three equations Sommerfeld calculate the shape and the size of the elliptical orbits and also the total energy.

The Results are

$$a = 4\pi\varepsilon_o \frac{n^2 \hbar^2}{\mu Z e^2}$$

$$b = a \frac{n_\theta}{n}$$

$$E = -\frac{\mu Z^2 e^4}{(4\pi\varepsilon_o)^2 2\hbar^2} \frac{1}{n^2}$$

$$a = 4\pi\varepsilon_o \frac{n^2 \hbar^2}{\mu Z e^2} \qquad b = a \frac{n_\theta}{n} \qquad E = -\frac{\mu Z^2 e^4}{(4\pi\varepsilon_o)^2 2\hbar^2 n^2} \frac{1}{n^2}$$

n: the quan number

Where defined

$$n \equiv n_\theta + n_r$$

Since n_θ= 1, 2, 3, …….. and n_r= 0, 1, 2, 3, ….., n can take on the values

$$n=1, 2, 3, \text{.......}$$

For a given value n, n_θ can be only the values

$$n_\theta=1, 2, 3, \text{........}, n$$

n is called the principal quantum number, and n_θ is called the azimuthal quantum number.

Note that from the equation $b = a\frac{n_\theta}{n}$ the shape of the orbit is determined by the ratio of the azimuthal quantum number and the principal quantum number. For n_θ = n the orbit is circles of radius a.

Degeneracy , Energies and Orbits:

- The figure below show the possible orbits for the first 3 values of n.
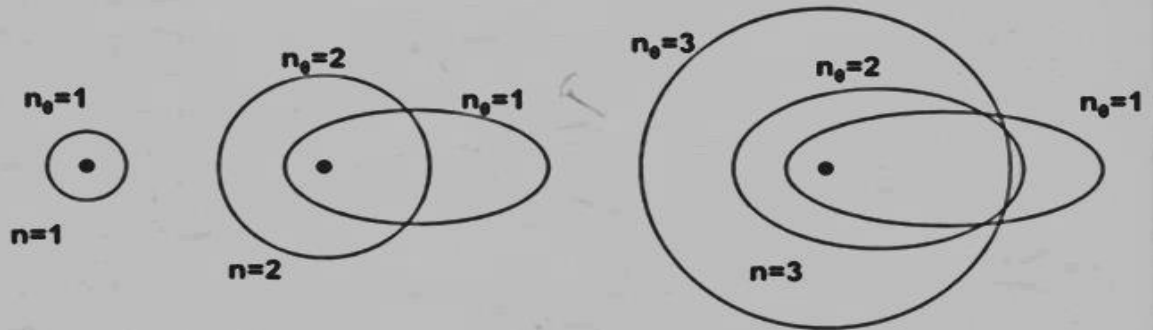
- For each quantum number n there are n different allowed orbits.
- The circular orbit is the orbit predicted by Bohr model.
- The different elliptical orbits of the quantum number n have the same energy.
- The total energy of the electron depends only on n.
- The several orbits with the same quantum number n is called degenerate.

The degeneracy in the total energy of an electron, following the orbits of very different shape but common n.