Al-Mustansiriyah Univ.
College of Science
Physics Dept.

الجامعة المستنصرية
كلية العلوم
قسم الفيزياء

حسابات جونز و ستوكس

Jones & Stokes calculations

تقرير من اعداد طلبة الدكتوراه (2015-2016)

حسام صبيح فليح و سارة حسين علي

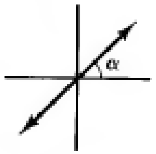
بإشراف

أ.د. علي عبد داود الزكي

Jones vectors & matrices

SUMMARY OF JONES VECTORS

I. Linear Polarization ($\Delta\varphi = m\pi$)

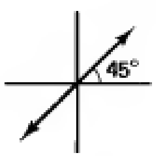
General: 

$$E_0 = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

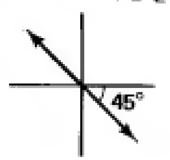
Vertical: $E_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Horizontal: $E_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

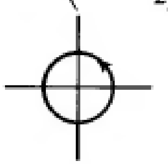
At $+45°$: $E_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



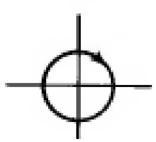
At $-45°$: $E_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



II. Circular Polarization $\left(\Delta\phi = \frac{\pi}{2}\right)$

Left: 

$$E_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Right: 

$$E_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

JONES MATRICES FOR SOME LINEAR OPTICAL ELEMENTS

Optical Element		*Jones Matrix*
Linear polarizer	Transmission axis horizontal	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
	Transmission axis vertical	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
	Transmission axis at $\pm 45°$	$\frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$
Quarter-wave plate	Fast axis vertical	$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$
	Fast axis horizontal	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
	Fast axis at $\pm 45°$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}$
Half-wave plate	Fast axis either vertical or horizontal	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Isotropic phase retarder		$\begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{bmatrix}$
Relative phase changer		$\begin{bmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{bmatrix}$
Circular polarizer	Right	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
	Left	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$

SUMMARY OF JONES MATRICES

I. Linear polarizers

horizontal $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ vertical $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ at 45° to horizontal $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

II. Phase retarders

QWP, SA vertical $e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ QWP, SA horizontal $e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$

HWP, SA vertical $e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ HWP, SA horizontal $e^{i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

III. Rotator

Rotator $(\theta \rightarrow \theta + \beta)$ $\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

Solved examples

(1) An optical filter can be described by a Jones matrix : $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Obtain the form of the emerging light for each of the following incident beams:

(a) A plane beam polarized by an angle θ horizontally: $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

(b) A left-circularly polarized beam. (c) A right-circularly polarized beam. (d) From the above, identify the filter.

Solution:

(a) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\ (\theta-\alpha) \\ -\sin\ (\theta-\alpha) \end{bmatrix}$

(b) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos\alpha + i\ sin\ \alpha \\ -\mathrm{i}\sin\alpha + i\ cos\ \alpha \end{bmatrix} = \begin{bmatrix} e^{i\alpha} \\ ie^{i\alpha} \end{bmatrix} = e^{i\alpha} \begin{bmatrix} 1 \\ i \end{bmatrix}$

(c) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos\alpha - i\ sin\ \alpha \\ -\mathrm{i}\sin\alpha - i\ cos\ \alpha \end{bmatrix} = \begin{bmatrix} e^{-i\alpha} \\ -ie^{-i\alpha} \end{bmatrix} = e^{-i\alpha} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

(d) The filter is rotator.

(2) An optical filter can be described by a Jones matrix : $\begin{bmatrix} cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & sin^2\,\alpha \end{bmatrix}$

(a) Obtain the form of the emerging beam when the incident light is plane polarized at angle θ to the horizontal: $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

(b) Deduce from the result of part (a) the nature of the filter.

Solution:

(a) $\begin{bmatrix} cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & sin^2\,\alpha \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} cos^2\alpha\cos\theta + \cos\alpha\sin\alpha\sin\theta \\ \cos\alpha\sin\alpha\cos\theta + sin^2\alpha\sin\theta \end{bmatrix}$

$$= \cos(\theta\text{-}\alpha) \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

(b) Emerging beam is polarized at angle θ to the horizontal and its amplitude is reduced by a factor [cos (θ - α)]. This is exactly what an ideal linear polarizer would do if its transmission axis were oriented at α to the horizontal (recall Malus's law).

(3) Show by means of the Jones calculus that circularly polarized light is produced by sending light through a linear polarizer and a quarter-wave plate only in the right order.

Solution:

Let a quarter-wave plate is : $\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$ & a linear polarizer (at 45°) is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then:

$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ which is circularly polarized light.

Note that the required condition or the right order that is the linear polarizer is at 45°.

(4) Verify that a circular polarizer whose Jones matrix is $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ is completely transparent to one type of circularly polarized light and opaque to the opposite circular polarization. (Note: This is not the same type of polarizer as a linear polarizer followed by a quarter-wave plate.)

Solution:

Jones vector of right hand circularly polarized light is : $\begin{bmatrix} 1 \\ -i \end{bmatrix}$;

Jones vector of left hand circularly polarized light is : $\begin{bmatrix} 1 \\ i \end{bmatrix}$;

Jones matrix of right hand circularly polarized light is : $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$; then :

- $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -i \end{bmatrix}$ which is right hand circularly polarized light with double amplitude (transparent) .

- $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 - 1 \\ -i + i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ No light (opaque) .

(5) Linearly polarized light whose Jones vector is $\begin{bmatrix}1\\0\end{bmatrix}$ (horizontally polarized) is sent through a train of two linear polarizers. The first is oriented with its transmission axis at 45 degrees and the second has its transmission axis vertical. Show that the emerging light is linearly polarized in the vertical direction; that is, the plane of polarization has been rotated by 90 degrees.

Solution:

- Jones vector of linearly (horizontally) polarized light is $\begin{bmatrix}1\\0\end{bmatrix}$, and Jones matrix of linearly (tilted at 45 degrees) polarized light is $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$:

$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ which is the Jones vector of linearly (tilted at 45 degrees) polarized light.

- Jones matrix of linearly (vertically) polarized light is $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$:

$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ which is the Jones vector of linearly (vertically) polarized light.

(6) Use the Jones method to find the result when a horizontal linear polarizer acts on: (a) a wave polarized in the x, y plane at angle θ to the x axis; and (b) circularly polarized waves of either hand. In each case, compare the initial and final irradiance.

Solution:

(a) Horizontal linear polarizer acts on rotated linearly polarized wave:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$$

The result is a wave polarized along the x axis, but with irradiance reduced by a factor of $\cos^2\theta$.

(b) Horizontal linear polarizer acts on circularly polarized wave:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \pm i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The final wave is linearly polarized along the x-axis, but with half the original irradiance of the incident wave.

Stokes Parameters

The Jones vectors apply <u>only to fully polarized light</u>. For partial polarization the appropriate analysis uses the Stokes parameters, which are functions of irradiance rather than fields. If the irradiance is measured through four different analyzers, (i) passing all states (but transmitting only half of each), (ii) and (iii) linear analyzers with axes at angles 0° and 45°, (iv) a circular analyzer, which measure respectively $I_0 I_1 I_2 I_3$, the Stokes parameters are :

$$S_0 = 2I_0$$

$$S_1 = 2I_1 - 2I_0$$

$$S_2 = 2I_2 - 2I_0$$

$$S_3 = 2I_3 - 2I_0.$$

For partially polarized light with polarized and un-polarized components of irradiance I_p and I_u we define the degree of polarization P as:

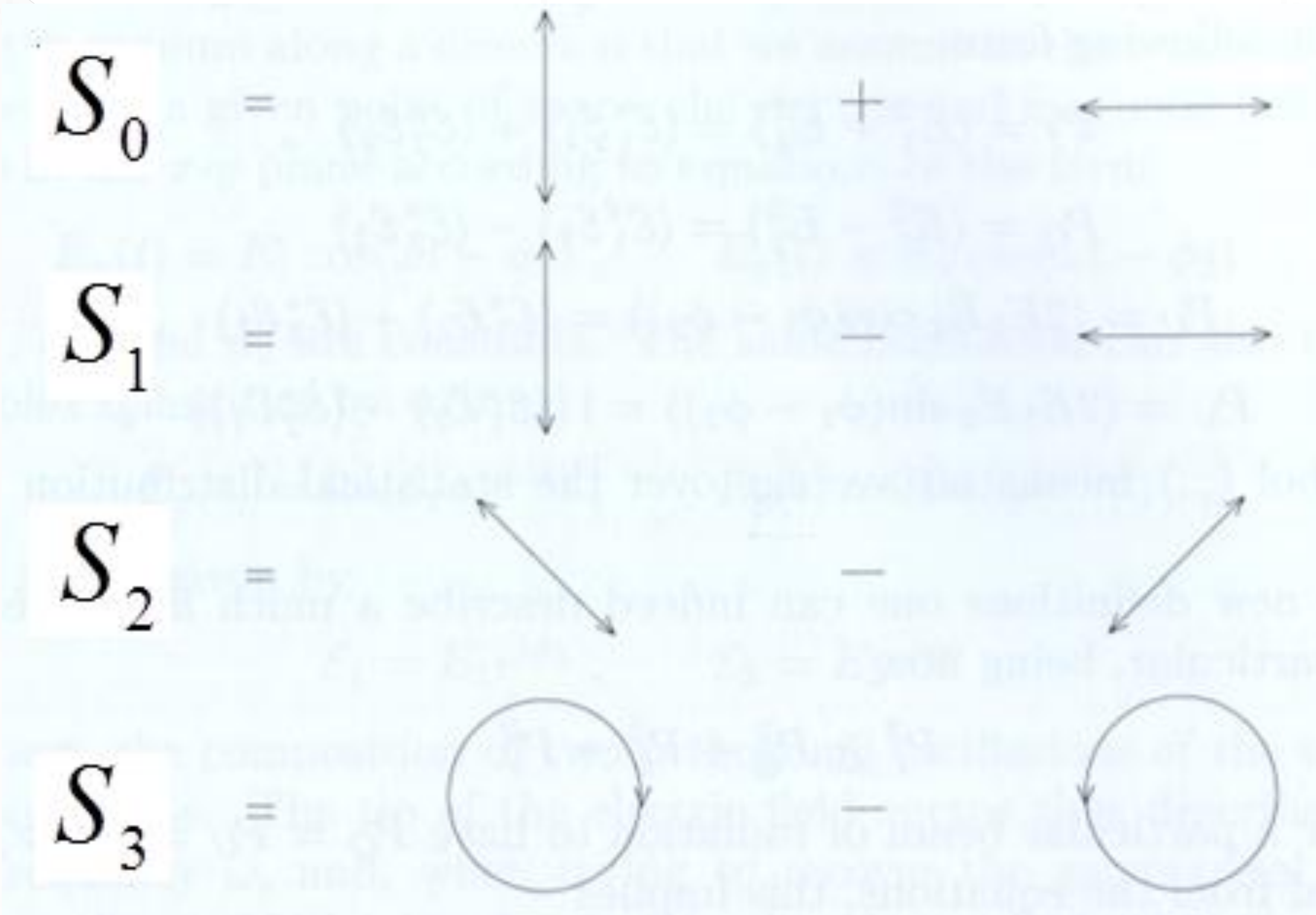
$$P = \frac{I_{\mathrm{p}}}{I_{\mathrm{u}} + I_{\mathrm{p}}} = \frac{(S_1^2 + S_2^2 + S_3^2)^{1/2}}{S_0}.$$

The Stokes parameters for two incoherent light beams are the sum of their individual Stokes parameters.

The Stokes parameters can be arranged in a Stokes vector:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\varepsilon \\ 2E_{0x}E_{0y}\sin\varepsilon \end{pmatrix} = \begin{pmatrix} \text{intensity} \\ I(0^\circ) - I(90^\circ) \\ I(45^\circ) - I(135^\circ) \\ I(RCP) - I(LCP) \end{pmatrix}$$

State of polarization	**Stokes parameters**	**Type of polarization**
Linear polarization	$S_1 \neq 0, S_2 \neq 0, S_3 = 0$ $S_1 \neq 0, S_2 = 0, S_3 = 0$	$S_1 > 0 \equiv \mathit{Horizontal\ LP}$ $S_1 < 0 \equiv$ Vertical LP $S_2 > 0 \equiv$ at 45 ° $S_2 < 0 \equiv$ at - 45 °
Circular polarization	$S_1 = 0, S_2 = 0, S_3 \neq 0$	$S_3 > 0 \equiv$ RHCP OR RHEP $S_3 < 0 \equiv$ LHCP OR LHEP $S_1 > 0 \equiv$ long axis *horizental* $S_1 < 0 \equiv$ long axis vertical
Elliptical polarization	$S_1 \neq 0, S_2 = 0, S_3 \neq 0$	
Fully polarized light	$S_0{}^2 = S_1{}^2 + S_2{}^2 + S_3{}^2$	
Partially polarized light	$S_0{}^2 > S_1{}^2 + S_2{}^2 + S_3{}^2$	
Un-polarized light	$S_1 = S_2 = S_3 = 0$	


$S_0 =$
+
$S_1 =$
−
$S_2 =$
−
$S_3 =$
−

(7) What are the polarization states of the two independent (incoherent) light beams with Stokes parameters (1,-1, 0, 0) and (3, 0, 0,-2), and of their sum?

Solution:

The first is fully linearly polarized (vertically, i.e. at 90°)

$$P = \frac{I_p}{I_u + I_p} = \frac{(S_1^2 + S_2^2 + S_3^2)^{1/2}}{S_0}. \quad = 1$$

The second is partially left-hand circularly polarized (P = 0.67) .

Their sum (4,-1, 0,-2) is partially elliptically polarized (long axis vertical) with $P = \sqrt{5}/4 = 0.56$.

(8) Two incoherent light beams represented by (1, 1,0,0) and (3,0,0, 3) are superimposed.

(a) Describe in detail the polarization states of each of these. (b) Determine the resulting Stokes parameters of the combined beam and describe its polarization state. (c) What is its degree of polarization? (d) What is the resulting light produced by overlapping the incoherent beams (1,1,0,0) and (1,- I,0,0) .

Solution:

(a) The first is fully linearly horizontally polarized light, and the second is fully right-hand circularly polarized light.

(b) Their sum (4, 1, 0,3) is partially right-hand elliptically polarized (long axis horizontal) light.

(c) $P = \dfrac{I_p}{I_u + I_p} = \dfrac{\left(S_1^2 + S_2^2 + S_3^2\right)^{1/2}}{S_0}.$ = 0.8 .

(d) The resulting light (2,0,0,0) is un-polarized light .

المصادر

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