الجامعة المستنصرية / كلية العلوم

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# Tutorial

## Modern Optics: Maxwell's Equations

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## Maxwell Equations:

$\nabla \cdot \mathbf{D} = \rho$ Gauss Law for electric field.

$\nabla \cdot \mathbf{B} = 0$ Gauss Law for magnetic field.

$\nabla \times \mathbf{E} = - \partial \mathbf{B} / \partial t$ Faraday's Law.

$\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t$ Ampere's Law.

Where:

D: electric displacement ($c/m^2$), E: electric field (V/m),

B: magnetic field (T or $V.s/m^2$),

ρ: Volume charge density ($c/m^3$),

## Maxwell's equations in free space are:

$\nabla \cdot \mathbf{D} = 0$,

$\nabla \cdot \mathbf{B} = 0$

$\nabla \times \mathbf{E} = - \partial \mathbf{B} / \partial t$

$\nabla \times \mathbf{H} = - \partial \mathbf{D} / \partial t$

Where: $\mathbf{J} = \sigma \mathbf{E}$

$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 n^2 \mathbf{E}$

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r}$$

$\mathbf{B} = \mu_0 \mathbf{H}$

σ: conductivity of medium, ε: electric permittivity ($8.85\times10^{-12}$F/m), $\mu_0$: magnetic permeability ($4\pi\times10^{-7}$H/m), n: reflective index of medium.

## Electromagnetic Waves:

- General wave equation

$\nabla \times \mathbf{E} = -\mu\, \partial \mathbf{H} / \partial t$

$\nabla \times \mathbf{H} = \sigma\, \mathbf{E}\, \mathbf{J} + \varepsilon\, \partial \mathbf{E} / \partial t$

By take ($\nabla\times$) for any of the two equations above:

$\nabla\times\nabla \times\mathbf{E} = -\mu\nabla \times (\partial \mathbf{H} / \partial t) = -\mu\, \partial / \partial t\, \mathbf{(\nabla\times H)} = \mathbf{\nabla(\nabla.E)} - \mathbf{\nabla^2 E}$

$\mathbf{\nabla(\nabla.E)} - \mathbf{\nabla^2 E} = -\mu\, \partial \mathbf{E} / \partial t - \mu\varepsilon\, \partial^2 \mathbf{E}/\partial t^2$

For a free source media,

$\nabla.\, \mathbf{E} = \nabla.\, (\mathbf{D} / \varepsilon) = 1/\varepsilon\, (\nabla.\, \mathbf{D}) = 0$

Thus:

$\mathbf{\nabla^2 E} - \mu\sigma\, \partial \mathbf{E} / \partial t - (1/v^2)\, \partial^2 \mathbf{E} / \partial t^2 = 0$ Wave eq. for free source.

Where: $v = 1/\sqrt{\mu\epsilon}$

Solve the wave eq. to obtain electric field equation as a function of space and time.

$\mathbf{E}(x,y,z,t)= \mathbf{E}(x,y,z,)\, e^{j\omega t}$

$\mathbf{E}(x,y,z,t)= \mathbf{E}(x,y,z,)\, \{\cos \omega t \text{ and } \sin \omega t\}$

To drive the wave equation 2nd time:

$\nabla^2 \mathbf{E} - \gamma^2\, \mathbf{E} = 0$

Where: $\gamma$ = Propagation constant for EM wave

$\gamma = |\gamma|\, c^{i\Psi} = \alpha + i\beta = (j\omega\mu\sigma - \omega^2\varepsilon\mu)^{1/2}$; $j=\sqrt{-1}$

$\Psi = \pi/2 - (1/2) \tan^{-1} (\sigma/\omega\varepsilon)$

$\alpha$= attenuation constant = $|\alpha| \cos\Psi$ (n/m)

K = Phase constant

$\beta = |\alpha| \sin\Psi$ (rad/m).

The same work to obtain wave equation for magnetic field

$\nabla^2 \mathbf{H} - \gamma^2 \mathbf{H} = 0$

## Electromagnetic Waves in Dielectric Medium:

By assuming homogenous free source medium ($\nabla . \mathbf{E}=0$), Maxwell's equations become:

$$\nabla \times E = -j\omega\mu H \qquad (1)$$

$$\nabla \times H = j\omega\varepsilon E \qquad (2)$$

Multiply ($\nabla\times$) both side of eq.1

$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = 0$

Where: $K = \omega \sqrt{\mu\epsilon}$ = Phase constant (rad/m).

$\nabla^2 \mathbf{E} = \mu\varepsilon\, \partial^2 \mathbf{E} / \partial t^2$

$\nabla^2 \mathbf{B} = \mu\varepsilon\, \partial^2 \mathbf{B} / \partial t^2 = 1/v^2\, \partial^2 \mathbf{E} / \partial t^2$

Where: $v = 1/\sqrt{\mu\epsilon}$

*v*: speed of wave (phase velocity), in vacuum the velocity $c=2.99\times10^8$ m/s.

## Plan wave solution:

To find simple harmonic wave for dielectric, the electric field function as a function of time can be write as:

$E_x(z,t)= E_a e^{-jkz}+ E_b e^{jkz}$

Or

$E_x(z,t)= E_a e^{j(\omega t-kz)} + E_b\, e^{j(\omega t+kz)}$

Where:

$E_a$ and $E_b$ amplitude constant.

a) **For electric field as a function of time z=0**

$E_x(0, t) = E_a e^{j\omega t}$

b) **For electric field as a function of space t=0**

$E_x(z,0) = E_b e^{-kz}$

Where:

$K=2\pi/\lambda= \omega\sqrt{\mu\epsilon}$

$v=1/\sqrt{\mu\epsilon}$ (m/s).

c) **For magnetic field**

$H_y(z,t) = H_a\, e^{j(\omega t-kz)}$

$H_a = E_a / \eta$; $\eta = |\mathbf{E}| / |\mathbf{H}|$

$\eta = w\, \mu / k = \sqrt{\mu\epsilon}$ $(\Omega)$

η: intrinsic impedance for medium

$\eta = 120\ \Omega = \eta_0 =$ in vacuum (free space), $\eta_0=376.7\Omega$

d) **For magnetic field by f (Hz) in dielectric medium and $\sigma = 0\ (\Omega m)^{-1}$, ε (F/m) and μ (H/m).**

$\lambda =1/f\sqrt{\mu\epsilon}$ , $v_p =1/\sqrt{\mu\epsilon}$

v: wave propagation velocity.

**Problem 1:** A light wave is traveling in a glass of refractive index (n=1.5). If the amplitude of electric field is (E=100 V/m). What is the amplitude of magnetic field **H**?

**Solution:**

$|\mathbf{H}| = n\, |\mathbf{E}| / \eta_0$

Where; $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

$\eta_0 = (4\pi\times10^{-7}(H/m)/8.85\times10^{-12}(f/m))^{1/2} = 376.7\ \Omega$

$|\mathbf{H}| = (1.5\times100)/376.7 = 0.398$ A/m

## Phase velocity ($v_p$) and group velocity ($v_g$):

**a) Non-dispersive medium:**

$K = \omega\sqrt{\mu\epsilon}$

$v_p = 1/\sqrt{\mu\epsilon}$

$v_g$= constant with frequency.

The relation between ω and β is shown in figure (1).

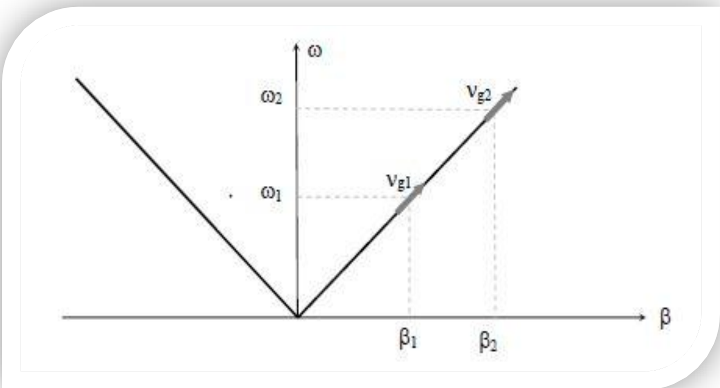



Figure (1): The relation between ω and β (β=k) for non-dispersive medium.

b) **Dispersive medium:**

$K = \xi\sqrt{\omega}$

$v_p = \sqrt{\frac{\omega}{\xi}}$

$v_g = 2\sqrt{\frac{\omega}{\xi}}$

The relation between ω and β is shown in figure (2).

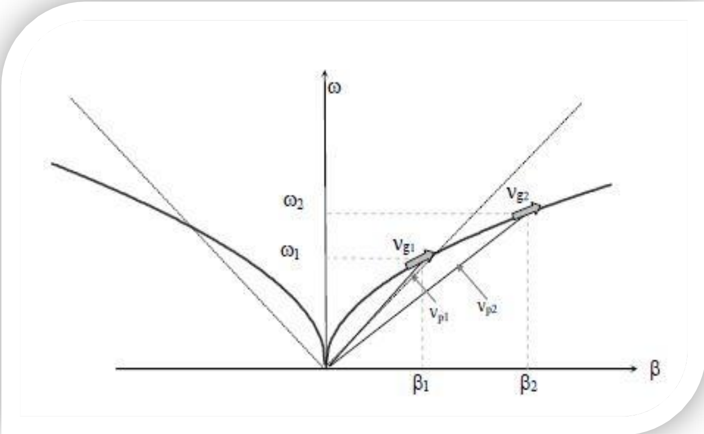



Figure (2): The relation between ω and β (β =k) for dispersive medium when ξ=2

**Problem 2:** By using the equation ( $v_p = \frac{\omega}{k}$ , $v_g = \frac{d\omega}{dk}$ ), prove equations:

$$v_g = v_p + k\frac{dv_p}{dk} \quad , v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

**Solution:**

$\omega = 2\pi f = (2\pi/\lambda)\,\lambda\, f = k\, v_p$

$$\frac{d\omega}{dk} = v_g = \frac{d(kv_p)}{dk} = v_p + k\frac{dv_p}{dk}$$

$$dk = d\left(\frac{2\pi}{\lambda}\right) = \frac{-2\pi}{\lambda^2}d\lambda$$

$$\frac{d\omega}{dk} = v_g = v_p + \left(\frac{2\pi}{\lambda}\right)\frac{dv_p}{\frac{-2\pi}{\lambda^2}}$$

$$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

## Electromagnetic wave in terms of energy density, pointing vector, intensity, momentum, and radiation pressure

**1- Energy density in electric and magnetic field of free space is given by:**

$$\boldsymbol{U_E} = \frac{1}{2}(\boldsymbol{D}.\boldsymbol{E}) = \frac{1}{2}\epsilon_\circ \boldsymbol{E}^2$$

$$\boldsymbol{U_B} = \frac{1}{2}(\boldsymbol{B}.\boldsymbol{H}) = \frac{1}{2}\frac{\boldsymbol{B}^2}{\mu_\circ}$$

Since, $\mathbf{E} = C\,\mathbf{B} = \frac{1}{\sqrt{\mu_\circ \epsilon_\circ}}\boldsymbol{B}$

$\mathbf{U_E = U_B}$

Therefore; the total energy density $U = \epsilon_\circ \boldsymbol{E}^2 = \frac{\boldsymbol{B}^2}{\mu_\circ} = \sqrt{\frac{\epsilon_\circ}{\mu_\circ}}\boldsymbol{EB}$ (J/m$^3$)

### 2- Poynting Vector ($\boldsymbol{S}$)

Poynting vector represents the directional energy flux density (the rate of energy transfer per unit area) of an electromagnetic field. Poynting vector ($\boldsymbol{S}$) defined as the cross product of the electric field and magnetic field

$$\boldsymbol{S} = \frac{1}{A}\frac{dU}{dt} = \frac{1}{A}\,uA\frac{dx}{dt} = \mathrm{U}\,c,$$

Where:

$$\boldsymbol{E} = c\boldsymbol{B}$$

$$\boldsymbol{S} = Uc = \frac{\boldsymbol{EB}}{\mu_\circ}$$

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

### 3- Intensity (I) of the wave

In plane wave: $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ (J/m$^2$.sec)

Where $\boldsymbol{E} = E_\circ \cos(kz - \omega t)\,\hat{x}$

$\boldsymbol{B} = B_\circ \cos(kz - \omega t)\,\hat{y} \; = (E_\circ \times B_\circ) cos^2(kz - \omega t)$

$\boldsymbol{S} = c\epsilon_\circ E_\circ^2 cos^2(kz - \omega t)$

$\varepsilon = \epsilon_\circ E_\circ^2 cos^2(kz - \omega t)$

The average value $cos^2(kz - wt)$ over one period is equal to ( $\frac{1}{2}$ )

$$|\boldsymbol{S}|_{ave} = \frac{1}{2} c\epsilon_\circ E_\circ^2$$

**4- Momentum (P)**

The linear momentum carried by an EM wave is related to energy as,

$$\nabla.\boldsymbol{E} = \frac{\rho}{\epsilon_\circ}$$

$$\nabla \times \boldsymbol{B} = \mu_\circ J + \epsilon_\circ \mu_\circ \frac{d\boldsymbol{E}}{dt}$$

$$F1 = \epsilon_\circ (\nabla.\boldsymbol{E})\boldsymbol{E} + \frac{1}{\mu_\circ}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \epsilon_\circ \frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}$$

Now consider; $\frac{\partial(\boldsymbol{E}\times\boldsymbol{B})}{\partial t} = \frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} + \boldsymbol{E} \times \frac{\partial B}{\partial t}$

$$\overrightarrow{F_1} + \epsilon_\circ \frac{\partial(\boldsymbol{E} \times \boldsymbol{B})}{\partial t} = \epsilon_\circ (\nabla.\boldsymbol{E})\boldsymbol{E} - \epsilon_\circ \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) + \frac{1}{\mu_\circ}(\nabla.\boldsymbol{B})\boldsymbol{B} - \frac{1}{\mu_\circ}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

But; $\boldsymbol{\nabla}.\boldsymbol{B} = 0$, $\;\boldsymbol{\nabla} \times \boldsymbol{E} = \frac{\partial \boldsymbol{B}}{\partial t}$

Therefore;

$$F_{total} + \frac{\partial}{\partial x}\int_v \epsilon_\circ (\boldsymbol{E} \times \boldsymbol{B}) dv = \int_v (r.h.s) dv$$

$$P_{field} = \int_v \epsilon_\circ (\boldsymbol{E} \times \boldsymbol{B}) dv = \frac{1}{c^2} \int \boldsymbol{S}\ dv$$

$g = \frac{S}{c^2}$ Momentum density

$P = \frac{U}{c}$ The momentum conservation of EM field.

**5- Radiation pressure**

The pressure from an EM wave, for example the radiation pressure from sun light hitting earth is ($6\text{x}10^8$kgm/s$^2$), can be expressed as:

$$\frac{F}{A} = \frac{S}{c} = u$$

Radiation pressure = $\frac{|S|}{C} = \frac{|E \times B|}{\mu_\circ c} = \frac{E^2}{\mu_\circ c^2}$ (kgm/s$^2$)

**Problem 3:** The electromagnetic plane wave propagates in the air free source and magnetic field given by: $\boldsymbol{H_z} = H_\circ e^{j(\omega t - 5\pi y)}$ A/m

Find: a) frequency of EM wave b) Electric field

**Solution:**

a) $k = 5\pi = \omega\sqrt{\mu_\circ \epsilon_\circ} = 2\pi f / 3 \times 10^8$

f=750 MHz

b) $\nabla \times H = j\omega\varepsilon_\circ E$

$\boldsymbol{E_x} = -\eta_\circ \vec{H}_\circ e^{j(\omega t - 5\pi y)}$ V/m

Where $\eta_\circ = 120\pi$ Ω

By using (*sin*) sinusoidal function

$$\boldsymbol{E_x} = -120\,\pi\, H_\circ \sin(2\pi \times 750 \times 10^6 t - 5\pi y) \qquad \text{V/m}$$

$$\boldsymbol{H_z} = H_\circ \sin(2\pi \times 750 \times 10^6 t - 5\pi y) \qquad \text{A/m}$$

**Problem 4:** What is rms of electric field of radiation?

a) 60 watt light at distance (1m)

b) of sunlight at the earth surface (the solar constant R=1.94 cal/cm$^2$)

c) 10 watt laser focused to spot (1m) in diameter.

**Solution:**

a) $\boldsymbol{S} = \frac{Power}{Area} = \frac{60}{\pi(1)^2} = \frac{60}{\pi}$ w/m$^2$

$$\boldsymbol{S} = \boldsymbol{EH} = EnE\left(\frac{\epsilon_\circ}{\mu_\circ}\right)^{1/2}$$

$$\boldsymbol{S} = n\boldsymbol{E^2}\left(\frac{\epsilon_\circ}{\mu_\circ}\right)^{1/2} \qquad for\;\; n = 1$$

$$\boldsymbol{E}^2 = \boldsymbol{S}\left(\frac{\mu_\circ}{\epsilon_\circ}\right)^{1/2} = 7.198 \times 10^3$$

$$\boldsymbol{E} = 84.8 \quad V/m$$

b) R = 1.94 (cal/cm$^2$. minute)

$cal \to j$ $\qquad\qquad$ $cal = 4.18\,J$

$cm^2 \to m^2$ $\qquad\qquad$ $cm^2 = 10^{-4} cm^2$

$min \to \sec$ $\qquad\qquad$ $min = 60\; sec$

$R = 1.94 * 4.18 * 10^4 . \frac{1}{60}$ $\text{w/m}^2$ = $1.352 * 10^3$ w/m$^2$

S = R

**Note:** The Poynting vector is equal to the magnitude of energy flow per unit area per unit time and equal to the solar constant

$$E^2 = S(\frac{\mu_\circ}{\epsilon_\circ})^{1/2} = 1.352 \times 10^3 (\frac{9\pi\times10^{-7}}{8.85\times10^{-12}})^{1/2}$$

$$= 5.094 \times 10^5 \ (V/m)^2 \ = \ 713.7 \quad V/m$$

c) radius = $\frac{d}{2} = 0.5 \ \times 10^{-6} m$

$$S = \frac{power}{\pi r^2} = 1.27 \times 10^{13} \quad w/m^2$$

$$E^2 = S\ (\frac{\mu_\circ}{\epsilon_\circ})^{1/2} = 4.78 \times 10^{15} \quad (V/m)^2$$

$$E = 6.917 \times 10^7 \ \ V/m$$

**Problem 5:** A radio station transmits 10 kW signal at a frequency of 100 MHz for simplicity assume that it radiates as a point source. At a distance of 1 km from the antenna, find: a) the amplitude of electric and magnetic field strengths and b) the energy incident normally on a square plate of side 10 cm in 5 min.

**Solution:**

a) $\boldsymbol{S}_{av} = \frac{power}{4\pi r^2} = \frac{E_0^2}{2\mu_0\, c}$

$$E_0^2 = \frac{10000}{4\pi(1000)^2} \times 2 \times 4\pi \times 10^{-7} \ \times 3 \times \ 10^8$$

$$E_{o} = 0.775 \quad V/m$$

$$B_o \ = 2.58 \times 10^{-9} \quad T$$

b) $\Delta U = \ S_{av} \ \Delta t \ = 2.4 \times 10^{-3}$ J

**Problem 6:** The electric field of an electromagnetic wave can be writing as:

**E** = 20 Cos [(6.28x10$^8$ $m^{-1}$ )$x - \omega t$]$j$ V/m

a) What is the frequency wave length? b) What is the frequency?
c) What is the magnetic field amplitude? d) In what direction is the magnetic field?

**Solution:**

a) $k = 2\pi/\lambda,\qquad \lambda = \frac{2\pi}{6.28\times10^{8}}$ m,

b) $K = \omega\sqrt{\mu_\circ\epsilon_\circ}$

$= 2\pi f\sqrt{\mu_\circ\epsilon_\circ}\qquad f = (\,75\,)\ MHz$

c) $|\boldsymbol{H}| = \frac{|\boldsymbol{E}|}{\eta}$

**H** = (0.14) A/m

$\eta = \sqrt{\frac{\mu_\circ}{\epsilon_\circ}}$

## Plane wave in conductor:

**Properties of the EM wave in a conducting medium:**

- free source
- permittivity ε (F/m)
- permeability $\mu$ (H/m)
- conductivity $\sigma$ $(\Omega\ \text{m})^{-1}$ , $\sigma \gg \omega\varepsilon$

**Maxwell's equations:**

$$\nabla\times\boldsymbol{E} = J\omega\mu\boldsymbol{H} \qquad (1)$$

$$\nabla\times\boldsymbol{H} = (\sigma + J\omega\epsilon)\boldsymbol{E} \approx \sigma\boldsymbol{E} \qquad (2)$$

Multiply (∇) by both side of equation (1)

$\boxed{\nabla^2\boldsymbol{E} + \gamma^2\boldsymbol{E} = 0}$ General equation

Where $\gamma = \sqrt{j\omega\mu\sigma} = 1 + j\sqrt{\frac{\omega\mu\sigma}{2}} = \alpha + i\beta$

$\beta = \sqrt{\pi f \mu \sigma}\ \ rad/m$ , $\alpha = \sqrt{\pi f \mu \sigma}$ m$^{-1}$

If the plane wave in x-direction is:

$\boldsymbol{E} = E_x\ (z,t)\ a_x$

$$\frac{d^2 E_x}{dz^2} + \gamma^2 E_x = 0 \qquad (3)$$

Solving equation (3)

$E_x(z,t) = E^+\ e^{-\alpha z}\ e^{j(\omega t - \beta z)} + E^-\ e^{+\alpha z}\ e^{j(\omega t + \beta z)}$

Take the first term only and using sinusoidal function, to obtain

$E_x(z,t) = E^+ \sin(\omega t - \beta z)$

The Electric field will be vanished as an exponential decay ($e^{-\alpha z}$) as shown in figure (3).

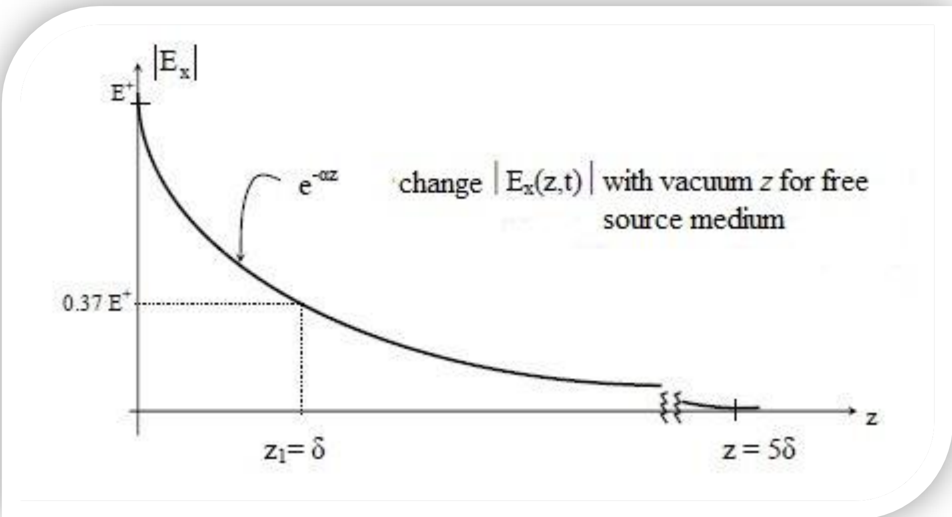



Figure (3): shows the change of electric field $|E_x(z,t)|$ with vacuum $z$ for free source medium

**Note**: the value of electric field in conductor medium at Z=0 is

$$E^{+}e^{-1} = 0.37E^{+} \ in \ z = z_1 = S$$

Where

$\delta$= skin depth (penetration depth)

Represented the space constant gave value as follow:

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \text{(m)}$$

$$\delta \ \ \alpha \ \ \frac{1}{f}$$

Table (1): skin depth ($\delta$) in medium for copper (Cu) and water from difference frequencies

| $\delta$ (Sea water ) | $\delta$ $(copper)$ | F |
|---|---|---|
| $\infty$ | $\infty$ | 0 DC |
| 1.25 m | 6.7 mm | 100 Hz power line frequency |
| 0.25 m | 0.07mm | 1MHz broadcast frequency |
| 25 mm | 6.7$\mu$m | 100MHz TV frequency |
| 7.9 mm | 2.1$\mu$m | 1GHz mobile frequency |
| 2.5 mm | 0.67$\mu$m | 10GHz satellite frequency |

## Discussion about the table of skin depth:

1. No change in electric field at equesurface potential in vacuum.
2. Electric field vanishes after skin depth Z =5 $\delta$.
3. Water considered good conductor at high frequencies.
4. skin depth at low frequencies go to hundred meters (100Hz or less )
5. The magnetic field eq.

$$H_y = H^{+} \ e^{-\alpha z} \ e^{j(\omega t - \beta z)} \qquad \text{A/m}$$

$$\eta_c = \sqrt{\frac{j\omega\mu}{\sigma}} = R_c + jx_c$$

$$H^{+} = \frac{E^{+}}{\eta_c}$$

$R_c = X_c = \sqrt{\frac{\pi f \mu}{\sigma}}$ $(\Omega)$ Impedence for conductor

Therefore magnetic field vanishes inside conductor

6. The impedance for small conductor medium approximate zero for matters (Copper, Gold, Aluminum, Fe …). Therefore consider the conductor material for EM wave.
7. To change impedance with frequency $f$ increase when $\eta_c$ increase ( $\eta \propto f$).
8. Finally, the EM wave will be found only in very short distances in conductor, not longer than ($5\,\delta$) i.e. EM wave evanescent inside conductor, and the phase velocity for evanescent wave in conductor:

$$\boxed{V_p = \sqrt{\frac{2\omega}{\mu\sigma}}} \quad m/s$$

$$\boxed{V_g = 2\sqrt{\frac{2\omega}{\mu\sigma}}} \quad \text{m/s}$$

the phase velocity inside Copper

$$v_p = 0.4\sqrt{f} \quad m/s \quad , \quad v_g = 0.8\sqrt{f} \quad m/s$$

The field in conductor media:

$E_x(z) = E^{+} e^{-\gamma z}$ V/m

$H_y(z) = H^{+} e^{-\gamma z}$ $A/m$

$J_x(z) = J^{+} e^{-\gamma z}$ $A/m^2$

**Problem 7:** If the value of electric field for EM wave inside water 100mV/m, find the skin depth to arrive (100$\mu$V/m) low sign to 60dB where frequencies 1 kHz, 1 MHz, 1GHz, $\sigma = 4(\Omega m)^{-1}$

**Solution:**

$\alpha = attenution\ constant = \sqrt{\pi f \sigma \mu} = 4\pi\sqrt{10^{-7}\ f}$ neper/m

$|\boldsymbol{E}| = E_\circ\, e^{-\alpha d}$ Electric field as skin depth function

$$100 \times 10^{-6} = 100 \times 10^{-3}\, e^{-\alpha d}$$

$$d = \frac{ln(1000)}{\alpha}$$

| F | S |
|---|---|
| 1KHz | 55m |
| 1MHz | 174cm |
| 1GHz | 55mm |

**Problem 8:** The EM wave has frequency 300 MHz propagated in vacuum source and electric field: $\boldsymbol{E} = E_1(-\frac{1}{\sqrt{2}}a_x + \frac{1}{\sqrt{2}}a_y)e^{[j(\omega t+\sqrt{2}\pi x+kyY)]}$

Find: (a) $K_y$ (b) wavelength for (x, y, z) (c) $\vec{H}$ magnetic field

**Solution:**

$$K = \frac{2\pi}{\lambda}\sqrt{K_x^2 + K_y^2} = \sqrt{2\pi + K_y^2}$$

$$K_y = \sqrt{2}\,\pi\frac{rad}{m} = K_x$$

And $\lambda x = \lambda y = \sqrt{2}\, m\,,\ \lambda z = \infty$

$$H = -\frac{1}{j\omega\mu}\nabla \times \mathrm{E} = \frac{E_1}{\eta_0}e^{[j(\omega t+\sqrt{2}\,\pi(x+y))]}A/m$$

Where

$\eta_0 = 120\pi$, to rewrite the fields in sinusoidal function

$$\boldsymbol{E} = \frac{E_1}{\sqrt{2}}(-a_x + a_y)\sin[\omega t + \sqrt{2}\pi(x+y)]\ V/m$$

$$\boldsymbol{H} = \frac{-E_1}{120\pi}\sin[\omega t + \sqrt{2}\pi(x+y)]\,a_z\quad A/m$$

## Energy flow, power and Poynting vector:

**By using Maxwell's equation to obtain:**

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{H}$$

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (-\mu \frac{d\boldsymbol{H}}{dt}) - \boldsymbol{E} \cdot (\sigma \mathbf{E} + \varepsilon \frac{d\boldsymbol{E}}{dt})$$

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\sigma |\boldsymbol{E}|^2 - \frac{\varepsilon}{2} \frac{d|\boldsymbol{E}|^2}{dt} - \frac{\mu}{2} \frac{d|\boldsymbol{H}|^2}{dt}$$

The following equation represents power density per volume, $W/m^3$ to integral:

$$P = -\iint \boldsymbol{E} \times \boldsymbol{H} \cdot ds = \iiint \left\{ \sigma |\boldsymbol{E}|^2 + \frac{1}{2} ( \varepsilon \frac{d|E|^2}{dt} + \mu \frac{d|\boldsymbol{H}|^2}{dt}) \right\} dV$$

Where

$\iiint \sigma |\boldsymbol{E}|^2$ Represent the lose power conductor medium per volume

$\frac{1}{2} \iiint \varepsilon \frac{d|\boldsymbol{E}|^2}{dt}$ Represent the average electric energy storage in medium per volume

$\frac{1}{2} \iiint \mu \frac{d|\boldsymbol{H}|^2}{dt}$ Represent the average magnetic energy storage in medium per volume

But the quantity; $\mathbf{E} \times \boldsymbol{H}$ is called Poynting vector

$\mathbf{S} = \mathbf{E} \times \mathbf{H}$   $\mathrm{W/m^2}$

Take the unit vector for all limits or boundaries:

$a_s = a_E \times a_H$ , then

$a_s = \frac{\boldsymbol{s}}{|s|}$ ,  $a_E = \frac{\boldsymbol{E}}{|E|}$ ,  $a_H = \frac{\boldsymbol{H}}{|H|}$

The average $\mathbf{S} = \nabla \cdot \boldsymbol{S} = \frac{1}{2} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}^*)$

Where

$$\boldsymbol{H} = H^{+}e^{j\omega t}\ , \qquad \boldsymbol{H}^{*} = H^{+}e^{j\omega t}$$

$$\nabla\cdot\boldsymbol{S}_{ave} = \frac{1}{2}\sigma|\boldsymbol{E}|^2 + \frac{j}{2}(\mu|\boldsymbol{H}|^2 - \varepsilon|\boldsymbol{E}|^2) \qquad \text{(A)}$$

By using dispersive theorem and take integral of equation (A)

$$\hat{P} = \frac{1}{2}\,VI^{*} = \frac{1}{2}\,|Z|^{*} = \frac{1}{2}\,|I|^2$$

$$Z = R + jx = R + j(xl + xc) \qquad \Omega$$

$$R = \frac{1}{|I|^2}\iiint \sigma|\boldsymbol{E}|^2\,dV$$

$$x = \frac{1}{|I|^2}\iiint \mu|\boldsymbol{H}|^2 - \varepsilon|\boldsymbol{E}|^2)dV$$

$$E_x = E_0\sin(\omega t - \beta z)$$

$$H_y = H_0\sin(\omega t - \beta z)$$

$$\omega = \varepsilon E_0^2 sin^2(\omega t - \beta z)$$

$$V_g = \frac{\mathrm{s}}{\mathrm{V}} = \frac{1}{\sqrt{\mu\varepsilon a_z}} \qquad \text{(m/s)} \qquad \text{Group velocity}$$

## The energy and momentum radiation pressure of EM wave in conductor medium:

$$U_E = \frac{1}{2}\epsilon E^2\ , \qquad U_H = \frac{1}{2}\mu H^2 \qquad , \qquad U_{total} = \frac{1}{V}EH$$

Where; $v = \frac{1}{\sqrt{\epsilon\mu}}$

Linear momentum $= \frac{\text{absorb energy}}{\upsilon}$

Radiation pressure = $\frac{\boldsymbol{F}}{A} = \frac{|\boldsymbol{s}|}{\upsilon}$

$\vec{P} = \boldsymbol{E} \times \boldsymbol{H}$

$S = \boldsymbol{E} \times \boldsymbol{H}$

Energy flux $\Longrightarrow S_{ave} = u_v = \frac{EB}{2\mu}$

**Boundary conditions in non-continuity of Maxwell equations is shown in figure (4).**

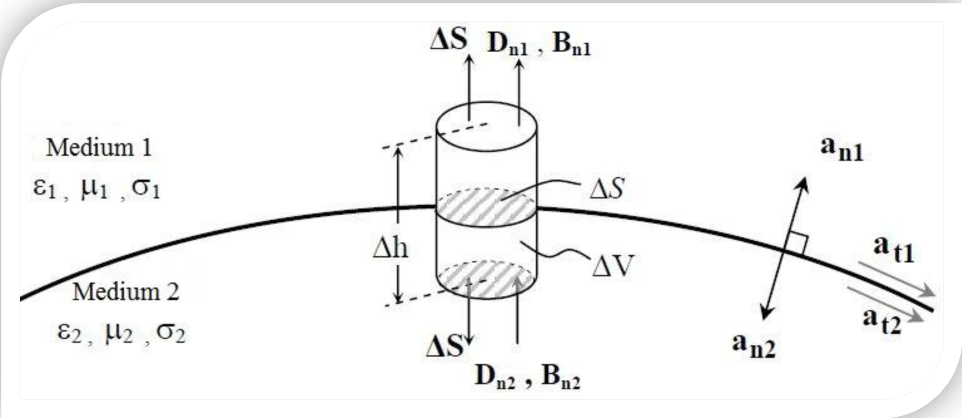



Figure (4): Boundary conditions in non-continuity of Maxwell equations.

Conductivity media $\sigma_1 = \sigma_2 \neq \infty, J_s = M_s = 0, q_{es} = q_{ms} = 0$

**In general:**

- Normal electric field density: $\hat{n}.(D_2 - D_1) = q_{es}$, $\hat{n}. (D_2 - D_1) = 0$
- Normal magnetic flux density: $\hat{n}.(B_2 - B_1) = q_{ms}$, $\hat{n}.(B_2 - B_1) = 0$
- Tangential $\vec{E}$ intensity: $\hat{n}.(E_2 - E_1) = M_s$, $\hat{n} \times(E_2 - E_1) = 0$
- Tangential $\vec{B}$ intensity: $\hat{n} \times(H_2 - H_1) = J_s$, $\times(H_2 - H_1) = 0$

## Maxwell's equations and Boundary conditions:

$\nabla \times \boldsymbol{E}$ = - jω$\boldsymbol{B}$, $\nabla \times \boldsymbol{H}$ = J + jωΔ , ∇.$\mathbf{D}$ =$\rho_v$,∇.$\mathbf{B} = 0$

**1- Normal field :**

$$\iint \boldsymbol{D}.ds = \iiint \rho_v dv$$

When $\Delta h \Rightarrow 0$ ➡ $\boldsymbol{D}n_1 - \boldsymbol{D}n_2 = \rho_s$

Non conductivity in Dn in interface which light travel between the two media equal to surface charge density in interface, while for $\boldsymbol{B}$:

$$\iint \boldsymbol{B}\,.\,ds = 0$$

$B_{n1} = B_{n2}$ i.e no magnetic charge

**2 - Tangential fields:**

$$\int \boldsymbol{E}.dl = \int \boldsymbol{B}.ds$$

$E_{t1} = E_{t2}$

Continuous electric field in interface i.e no magnetic current for H-field:

$$\int \boldsymbol{H}.dl = \iint J.ds + \iint \frac{d\boldsymbol{D}}{dt}.ds$$

$H_{t1} - H_{t2} = K$

K: linear current density, non-continuing magnetic field in interface medium.

**Problem 9:** If EM wave in free space with frequency 300 MHz, $\boldsymbol{E} = 4\, e^{j(\omega t - kz)}$ find (1) wavelength and phase velocity $v_p$ (2) H field (3) average Poynting vector ***Sav*** and power from rectangle in plane z=0 and about point (0,0), (5,0), (5,10), (0,10)

**Solution:**

1) $k = \omega\sqrt{\mu_\circ \epsilon_\circ} = 2\pi * 3 * 10^8 = \frac{2\pi}{\lambda}$ ➡ $\lambda$=1m

$Vp = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_\circ \epsilon_\circ}} = 3 * 10^8\ m/sec$

2) $\boldsymbol{\nabla} \mathbf{x} \boldsymbol{E} = -j\omega\mu\boldsymbol{H}$

$$\boldsymbol{H} = -\frac{4}{120\pi}\ e^{j(\omega t - kz)}$$

3) $\mathbf{S}av = 1/2\ \boldsymbol{E}\mathrm{x}\boldsymbol{H}^* = \left(\frac{1}{2}\right)\left[4ay * \left(\frac{4}{120\pi}\ ax\right)\right] = \frac{1}{15\pi}\ az\frac{w}{m2}$

$$p = \iint Sav\ ds = \int_0^5 \int_0^{10} Sav\ dx\ dy\ dz\ = \frac{10}{3\pi}\ w$$

**Problem10:** In medium dispersive in plane wave have $\sigma, \mu, \omega$ and f , verify $\sigma \gg \omega\epsilon$ and E to z direction , E=10 v/m about x=0 . Find:

1) **H**, **E** for a medium.

2) Sav, P for rectangle 5*10 $m^2$ in *zy* plane and slides parallel to *ZY* axes about x=0 and x=$\delta, \delta$ is skin depth.

**Solution:**

1) $\mathbf{E}_z = 10\ e^{-\alpha x} e^{j(\omega t - \beta z)}$

$$\alpha = \beta = \sqrt{\pi\mu f\sigma} = \frac{1}{\delta}$$

$$\boldsymbol{H} = \frac{|\boldsymbol{E}|}{\eta_c} ay = \frac{10}{\eta_c}\ e^{-\alpha x} e^{j(\omega t - \beta x)}\ ay \qquad \frac{A}{m}$$

R=X=$\sqrt{\pi f\mu \backslash \sigma}\ \Omega$ ; $\eta_c = R + jx$

2) Sav = 1/2 $\boldsymbol{E}\mathbf{x}\boldsymbol{H}^*$ = 1/2 10 $e^{-\alpha x} * \frac{-10}{R - jx}\ e^{-\alpha x}\ ay\ \frac{w}{m2}$

$$= \frac{50}{R\sqrt{2} - 45} e^{-2\alpha x}\ ax \frac{w}{m3}$$

$$p\ (\delta = x) = \frac{2500}{2R}\ (j + 1)e^{-2}\ W$$

$$p(x = 0) = \frac{45}{R\sqrt{2}}\ 5 * 10 = \frac{2500\ (j + 1)}{R\ \sqrt{2}\ \sqrt{2}}\ w$$

**Problem 11:** The time harmonic complex field inside source free conducting pipe for rectangular filled with free space which is shown in bellow figure is given by:

$$\boldsymbol{E} = ay\, E_0 \sin\frac{\pi}{a}x\ e^{-j\beta_0 z}\ 0 \le x \le a\,, 0 \le v \le b$$

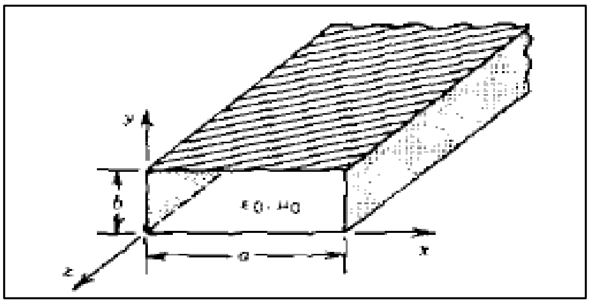
$$\beta z = \beta_0 \sqrt{1 - (\frac{\lambda_0}{2a})^{\,2}}$$

$E_0$ is constant , $\beta_0 = \frac{2\pi}{\lambda_0} = \omega\sqrt{\mu_\circ \varepsilon_\circ}$ along z- axis determine

$a$) The corresponding complex magnetic field.

$b$) Supplied complex power.

$c$) $Sav$




**Solution:**

$$\underline{E} = \hat{a}_y E_o \sin\left(\frac{\pi}{a}x\right) e^{-j\beta_z z}$$

a. $$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left(-\hat{a}_x\frac{\partial E_y}{\partial z} + \hat{a}_z\frac{\partial E_y}{\partial x}\right) = -\hat{a}_x\frac{\beta_z}{\omega\mu_o}E_o\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} - \hat{a}_z\frac{E_o}{j\omega\mu_o}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

b. $$P_s = -\frac{1}{2}\iiint_V \left(\underline{H}^*\cdot\underline{M}_i^{\,0} + \underline{E}\cdot\underline{J}_i^{*\,0}\right)dv = 0$$

c. $$P_e = \oiint_S \left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)\cdot d\underline{s}$$

$$\frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2}\hat{a}_y E_y \times\left(-\hat{a}_x H_x^* - \hat{a}_z H_z^*\right) = \frac{1}{2}\left(\hat{a}_z E_y H_x^* - \hat{a}_x E_y H_z^*\right)$$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \hat{a}_z\frac{\beta_z}{2\omega\mu_o}|E_o|^2\sin^2\left(\frac{\pi}{a}x\right) + \hat{a}_x\frac{|E_o|^2}{j2\omega\mu_o}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)$$

**Note:** All the above for plane EM wave in single medium without experiencing reflection and refraction in homogenous, free source and linear medium.

## Electromagnetic theory:

The phenomena of reflection and refraction of light from standing point of EM theory:

### Fresnel relation:

Fresnel relation is shown in figure (5).

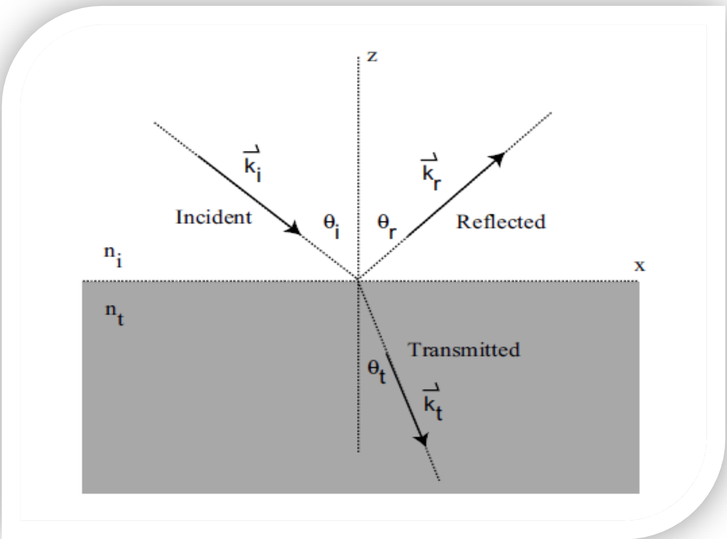



Figure (5): represent Fresnel relation.

The space time dependence of these three waves, aside from constant amplitude factors is given by:

$e^{i(k_i r-\omega t)}$ incident

$e^{i(k_r r-\omega t)}$ reflected

$e^{i(k_t r-\omega t)}$ transmitted

- The condition that $k_i r = k_r r = k_t r$ is required at the interface for the boundary condition to be satisfied all the points along the interface at all times.

- The projection of these three wave vector on the interface are equal so that

$$k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$$

Where:

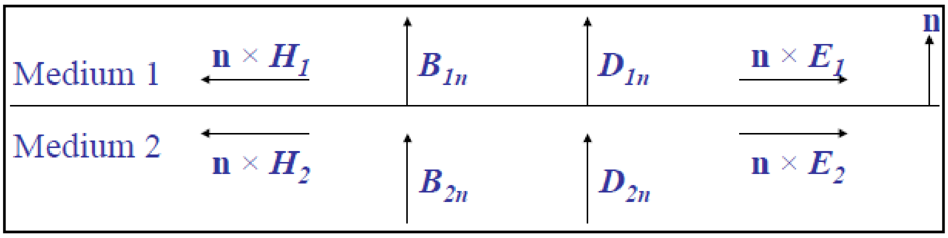
$\theta_i$ the angle of incident, $\theta_r$ the angle of reflection, $\theta_t$ the angle of transmission

Because $k_i = k_r$ and $k_i = \frac{n_1}{n_2}$

$\theta_i = \theta_r$ (Law of reflection)

$n_1 \sin\theta_i = n_2 \sin\theta_t$ (Snell's Law)

## Boundary conditions:




These two situations of Fresnel Formula:

**1- S polarization** $\vec{E} \perp$ plan of incident

(Also known as σ, N or TE polarization: no)

**2-P polarization: no** $\vec{E}$// plan of incident

(Also called π or TM polarization )

### 1-TE-polarization (S-wave)

The incidence of vertical polarized wave on medium is shown in figure (6).

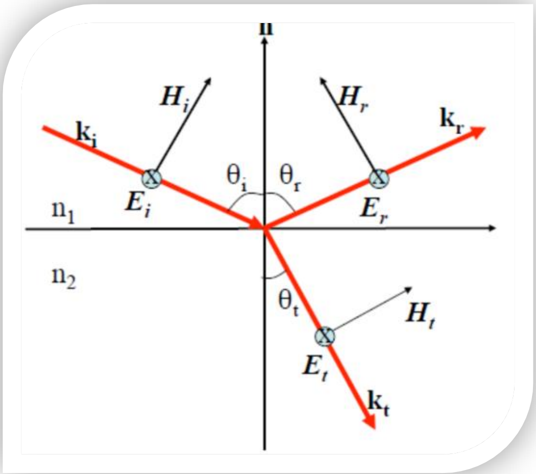



Figure (6): The incidence of vertical polarized wave on medium.

The reflection coefficient, and the transmission coefficient, t, of TE electric field is given by Fresnel equations:

$$r_s = {}^{E_r}/_{E_i}$$ **E** ⊥ plan of incident

$$= \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i - n_2 \cos\theta_t} = \frac{n_1 \cos\theta_i - \sqrt{n_2^2 - n_1^2\, sin^2\theta_i}}{n_1 \cos\theta_i - \sqrt{n_2^2 - n_1^2 \sin^2\theta_i}}$$

$$t_s = {}^{E_t}/_{E_i} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i \sqrt{n_2^2 - n_1^2 \sin^2\theta_i}}$$

The intensity reflectance and transmittance, R and T, which are also known as reflectivity and transmissivity, are given by:

$$R_s = {}^{I_r}/_{I_i} = \left|\frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i - n_2 \cos\theta_t}\right|^2$$

$$T_s = {}^{I_t}/_{I_i} = 1 - R_s$$

## 2-TM-Polarization (P-wave)

The incidence of parallel polarized wave on medium is shown in figure (7).

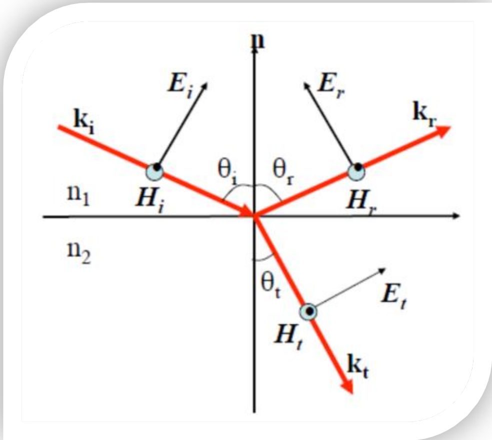



Figure (7): The incidence of vertical polarized wave on medium.

The reflection coefficient, r, and transmission coefficient, t, of the TM electric field are given by Fresnel equation:

$$r_p = {}^{E_r}/_{E_i} = \frac{-n_2\cos\theta_i + n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} = \frac{-n_2^2\cos\theta_i + n_1\sqrt{n_2^2 - n_1^2\sin\theta_i}}{n_2^2\cos\theta_i + n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_i}}$$

$$t_p = {}^{E_t}/_{E_i} = \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t} = \frac{2n_1 n_2\cos\theta_i}{n_2^2\cos\theta_i + n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_i}}$$

The intensity reflectance and transmittance for Tm polarization are given by:

$$R_p = {}^{I_r}/_{I_i} = \left|\frac{-n_2\cos\theta_i + n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}\right|^2$$

$$T_p = {}^{I_t}/_{I_i} = 1 - R_p$$

**Total Reflection:**

For $\theta_i > \theta_c$ $\sin\theta_i > \frac{n_2}{n_1}$ for S and P polarization :

$$|r_s| = \left|\frac{n_1\cos\theta_i - i\sqrt{n_1^2 sin^2\theta_i - n_2^2}}{n_1\cos\theta_i + i\sqrt{n_1^2\sin^2\theta_i - n_2^2}}\right| = 1$$

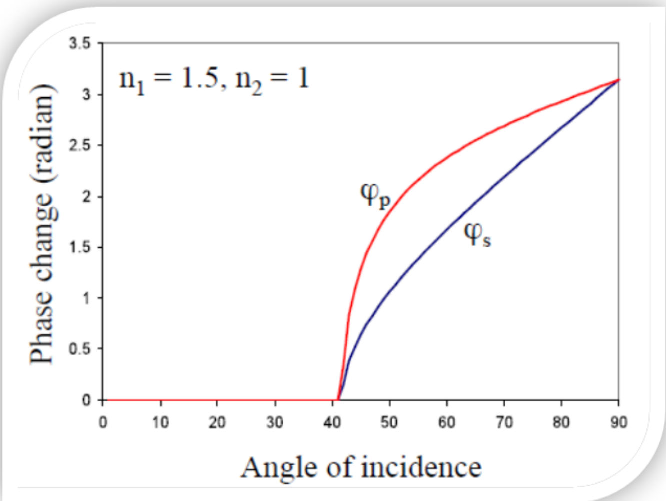
$$|r_p| = \frac{-n_2^2\cos\theta_i + i\,n_1\sqrt{n_1^2\sin^2\theta_i - n_2^2}}{n_2^2\cos\theta_i + i\,n_1\sqrt{n_1^2\sin^2\theta_i - n_2^2}}$$

**Problem 12** Two medium glass and air find

a) $\theta_B$, b) $\theta_c$ internal reflection as shown in figure (3)

**Solution:**

$$\theta_c = \sin^{-1}\frac{n_2}{n_1} \rightarrow \theta_{c=42^\circ} , \qquad \theta_B = \tan^{-1}\frac{n_2}{n_1} \rightarrow \theta_B = 34^\circ$$



## Phase change in total internal reflection:

$$r_s = {ae^{-i\alpha}}/{ae^{i\alpha}} = e^{-\varphi_s}$$

$$r_p = {-be^{-i\beta}}/{be^{i\beta}} = -e^{-i\varphi_p}$$

$$ae^{-i\alpha} = n_1 \cos\theta_i - i\sqrt{n_1^2 \sin^2\theta_i - n_2^2}$$

$$be^{-i\beta} = n_2^2 \cos\theta_i - in_1\sqrt{n_1^2 \sin^2\theta_i - n_2^2}$$

See that $\varphi_s = 2\alpha$ and $\varphi_p = 2\beta$ . Accordingly, $\tan\alpha = \tan\left(\frac{\varphi_s}{2}\right)$ and$\tan\beta = \tan\left(\frac{\varphi_p}{2}\right)$.

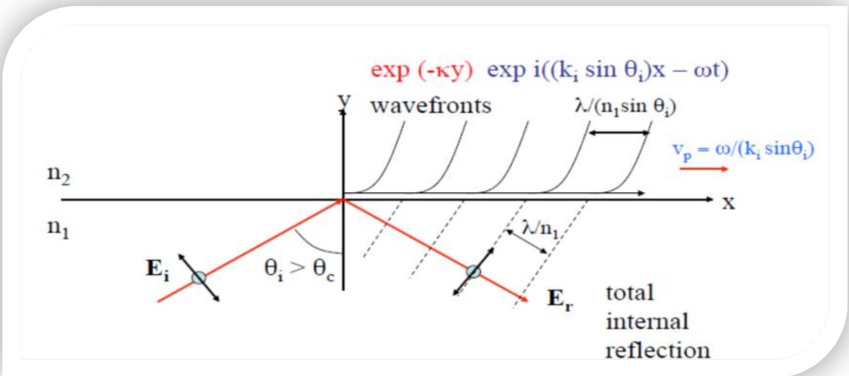
Therefore find the following expressions for the phase changes that occur in internal reflection:

$$\tan\left(\frac{\varphi_s}{2}\right) = \frac{\sqrt{n_1^2 \sin^2\theta_i - n_2^2}}{n_1 \cos\theta_i}$$

$$\tan\left(\frac{\varphi_p}{2}\right) = \frac{n_1\sqrt{n_1^2 \sin^2\theta_i - n_2^2}}{n_2^2 \cos\theta_i}$$

## Evanescent waves:

$$E_t = E_t e^{i(k_t \cdot r - \omega t)}$$



$$k_t \cdot r = k_t x \sin\theta_t + k_t y \cos\theta_t$$

$$= k_t x \left(\frac{n_1}{n_2}\right) \sin\theta_i + k_t y \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i}$$

$$= k_i x \sin\theta_i + i k_t y \sqrt{\left( n_1^2 \sin^2\theta_i / n_1^2 \right) - 1}$$

The wave function for the electric field of evanescent wave is:

$$E_{evan} = E_t \, e^{(-ay)} \, e^{i[(k_i \sin\theta_i)x - \omega t]}$$

Where $$a = k_t \sqrt{\left( n_1^2 \sin^2\theta_i / n_2^2 \right) - 1}$$

Evanescent wave amplitude normal to the interface drops exponentially is shown in figure (8).

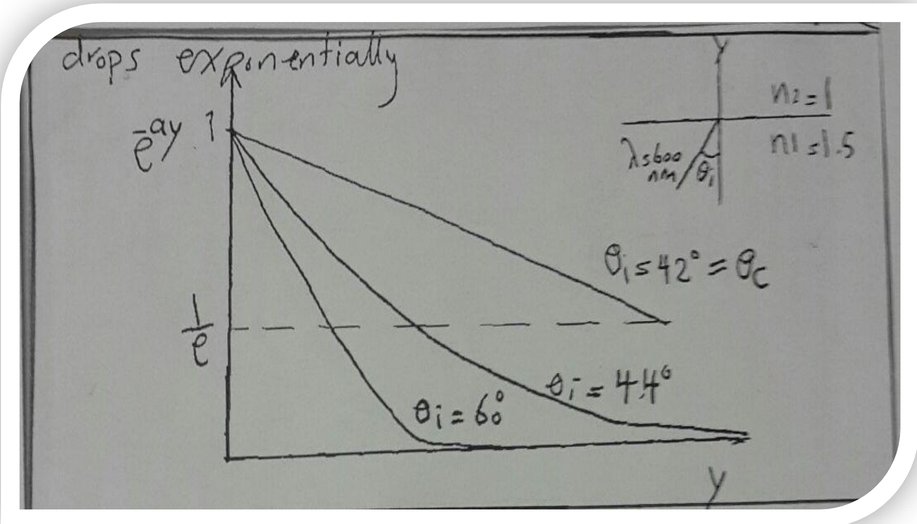



Figure (8): Evanescent wave amplitude normal to the interface drops exponentially

Skin depth ($\delta$): $$\delta = \frac{1}{k_t\sqrt{(\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1)}}$$

Where $K_t = K_2$

In conductor: $\delta = (2/\sigma\,\mu\,\omega)^{1/2}$

## Brewster's angle:

There is a particular angle at which the reflectivity is Zero for TE polarization light wave, and this angle is known (Brewster's angle).

$$r_p = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = 0, \text{ when } \theta_i + \theta_t = 90^o$$

$$\frac{n_t}{n_2} = \frac{\sin\theta_i}{\sin(90-\theta_i)} = \frac{\sin\theta_i}{\cos\theta_i}$$

$$\theta_B = \tan^{-1}(\frac{n_t}{n_i})$$

The power reflectivity and transmissivity for TE and TM polarization

$$A_i = A_r = A_o\cos\theta_i \quad , \quad A_t = A_o\cos\theta_t$$

Where α = i, r and t

$S_{\alpha} = 1/2(\varepsilon_{\alpha} / \mu_{\alpha})^{1/2} E^2$

Where $\mu_{\alpha} = \mu_{\circ}$

$\varepsilon_{\alpha} = n\,\alpha\,(\varepsilon_{\circ})\,1^{/2}$

$W_i = n_i / 2 = (\varepsilon_{\circ} / \mu_{\circ})^{1/2} E_i^2 A_0 \cos\theta_i$ incident wave

$W_r = n_r/2 = (\varepsilon_{\circ} / \mu_{\circ})\,1^{/2} E_r^2 A_0 \cos\theta_r$ reflected wave

$W_t = n_t/2 = (\varepsilon_{\circ} / \mu_{\circ})^{1/2} E_t^2 A_0 \cos\theta_t$ transmitted wave

The reflectivity and transmitted power for TE and TM polarization

R = $W_r / W_i = E_r^2 / E_i^2 = |r|^2$

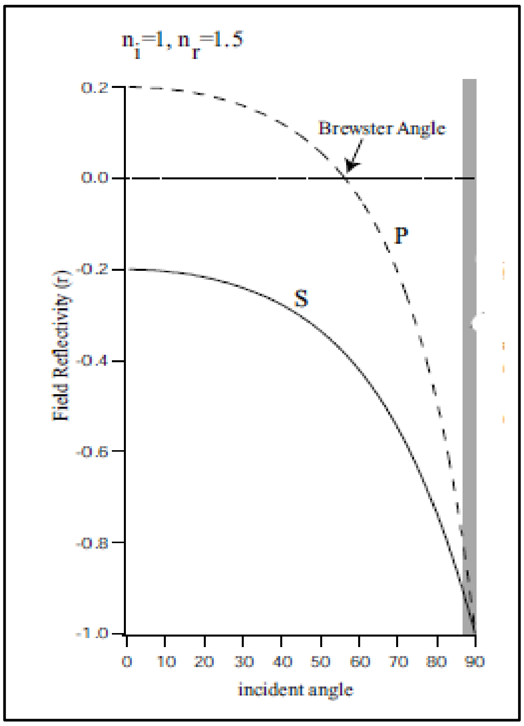
T= $W_t / W_i = n_t \cos\theta_t E_t^2 / n_i \cos\theta_i E_t^2 = n_t \cos\theta_t / n_i \cos\theta_i = |t|^2$

**Problem 13:** Draw the relation between $\theta_i$ and Rs, Rp

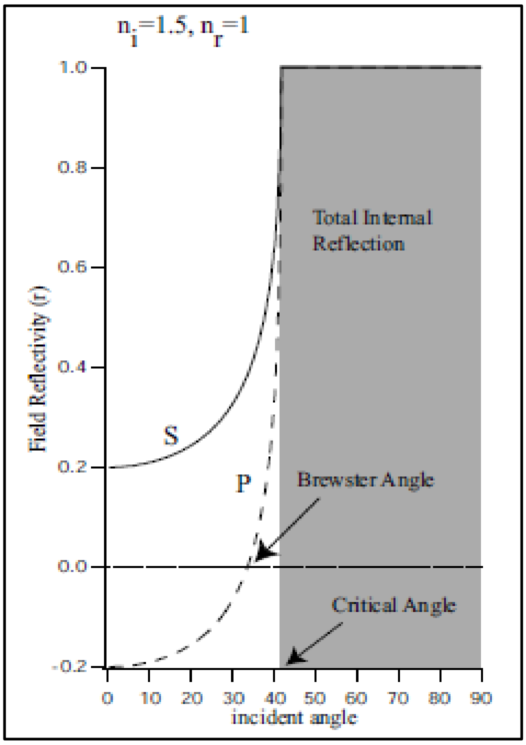
1- air- glass interface 2- glass- air interface

**Solution:**

For the incident light in coming from low refractive index material to high refractive index materials



2) For high refractive index material to low refractive index material:




## Normal incidence of EM waves:

**1) Optical wave at a dielectric interface**

**2) Optical wave at a conducting medium**

### 1) Optical wave at a dielectric interface:

Normal incidence in dielectric medium is shown in figure (9).

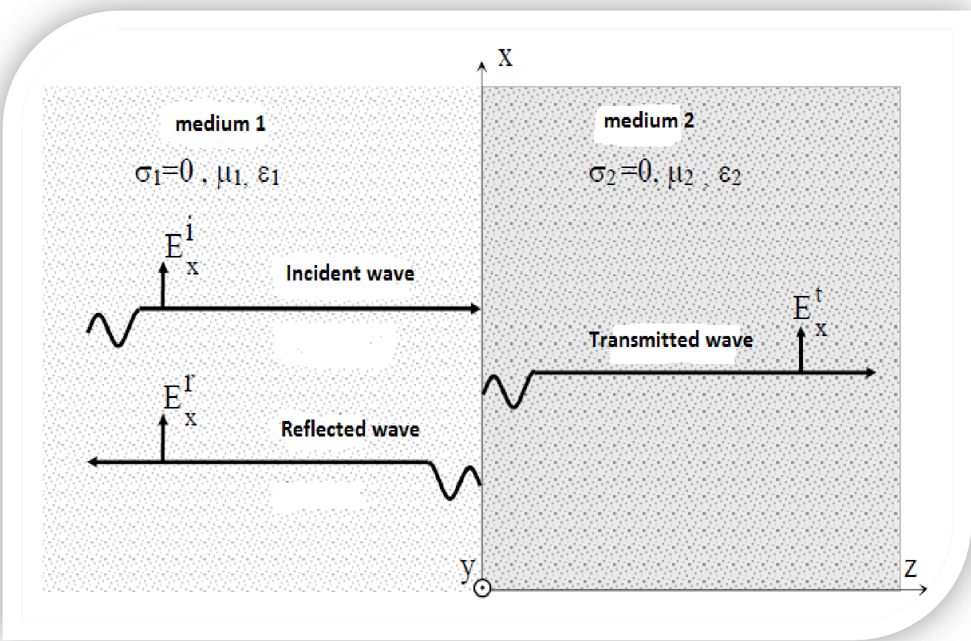



Figure (9): Normal incidence in dielectric medium is shown in figure (9).

**Reflectivity at normal incident:**

- **Incident wave**

$$\mathbf{E}_x^i = E_1\, e^{j(\omega t - k_1 z)}$$

$$\mathbf{H}_y^i = H_1\, e^{j(\omega t - k_1 z)} = \frac{E_1}{\eta 1}\, e^{j(\omega t - k_1 z)}$$

$\eta_1 = (\mu_1 / \varepsilon_1)^{1/2}$ , $\eta_2 = (\mu_2 / \varepsilon_2)^{1/2}$

$K_1 = \omega\, (\varepsilon_1.\mu_1)^{1/2}$, $K_2 = \omega\, (\varepsilon_2.\mu_2)^{1/2}$

Where

η: characteristic impedance, k: wave number

- **Reflected wave**

$$\mathbf{E}_x^r = E_2\, e^{j(\omega t - k_2 z)}$$

$$\mathbf{H}_y^r = H_2\, e^{j(\omega t - k_2 z)} = \frac{E_2}{\eta 2}\, e^{j(\omega t - k_2 z)}$$

- **Transmitted wave**

$$\mathbf{E}_x^t = E_3\, e^{j(wt - k_2 z)}$$

$$\mathbf{H}_y^t = H_3\, e^{j(\omega t - k_1 z)} = \frac{E_3}{\eta 2}\, e^{j(\omega t - k_2 z)}$$

From B.Cs, get at z = 0

$E_{t1} = E_{t2}$ / $H_{t1} = H_{t2}$

$E_1 + E_2 = E_3$

Reflection coefficient $\Gamma = E_2 / E_1$ = reflected field / incident field

Transmitted coefficient $\tau = E_3 / E_1$ = transmitted field / incident field

$1 + \Gamma = \tau$

$1 - \Gamma = (\eta_1 / \eta_2)\, \tau$

$\tau = 2\eta_2 / (\eta_2 - \eta_1)$

**Problem 14:** Air interface for n1 =1 , n2 = 1.5 glass find reflectivity and transmitted at: 1) normal incident, 2) at $\theta_i = 45°$ , 3) Brewster angle, 4) critical angle .

**Solution:**

1) At normal incident cos $\theta_i$= cos $\theta_t$= 1

$$r = r_s = r_p = \frac{n_1 - n_2}{n_1 + n_2} = -0.21$$

And $R = |r|^2 = 4.3$ %

The glass window attenuates light due to reflection near normal in cadence by about twice this moment around 8%.

2) At θ; = 45° the reflectance for the two polarization are quite different

$\theta_t = \mathrm{Sin}^{-1}\left(\frac{n1}{n2} Sin\theta\ i\right) = 27.7°$

The reflectances are:

$$R_s = |r_s|^2 = \left|\frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i - n_2 \cos\theta_t}\right|^2 = 9.7\%$$

$$R_p = |r_p|^2 = \left|\frac{n_1 \cos\theta_t - n_2 \cos\theta_i}{n_1 \cos\theta_t - n_2 \cos\theta_i}\right|^2 = 0.94\%$$

Or about 10% and 1% respectively.

The $T_m$ polarization is much smaller because this.

Incidence angle is fairly close to the Brewster angle.

3) $\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = 56.7^0$

4) $\theta_c = \sin^{-1}\left(\frac{n_{air}}{n_{glass}}\right) = 41.1^0$

## 2) Optical wave at a conducting medium

- Reflection from a conductor

$$R = |r|^2 = \frac{n-1}{n+1} . \frac{n^*-1}{n^*+1} = \frac{(n-1)2_+(nk)2}{(n+1)2+(nk)2}$$

$$R = 1 - \frac{\boldsymbol{an}}{(\boldsymbol{n}+\mathbf{1})\mathbf{2}+(\boldsymbol{nk})\mathbf{2}}$$

Where $K = \omega\sqrt{M(\varepsilon - \frac{\tau\sigma}{\varphi})}$ , $n = n(1 - ik)$ , $n^* = n(1 + ik)$

$K = n\ \omega / c$

- Skin depth in conductor medium

$$\delta = \sqrt{\frac{2}{m\sigma\omega}}$$

- The filed reflection coefficient far TE and T'm polarization

$$r_s = \frac{\sqrt{1+\propto^2}\, cos\theta i - i\propto sin\theta i}{\sqrt{1+\propto^2} cos\theta i + i\propto sin\theta i} = e^{i\Phi s}$$

$$r_p = \frac{sin\theta i cos\theta i - i\propto\sqrt{1+\propto^2}}{Sin\theta i Cos\theta i + i\propto\sqrt{1+\propto^2}} = e^{i\Phi p}$$

- The phase of the light for TE and TM polarization are

$$\varphi_s = 2tan^{-1}\frac{\propto sin\theta i}{\sqrt{1+\propto^2}\;\; cos\theta i}$$

$$\varphi_p = 2tan^{-1}\frac{\propto\sqrt{1+\propto^2}}{Sin\theta i cos\theta i}$$

Where:

$$\propto = \sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1} \quad ;\ \theta c = sin^{-1}\left(\frac{nt}{ni}\right)$$

**Problem 15**: Compute the reflectance of Copper for wavelength $\lambda$ = 1mm and 1nm at normal incident.

**Solution:**

a) $R \cong 1 - (\frac{8\omega\epsilon_0}{\sigma})^{1/2}$

From table $\sigma_{cop} = 5.8 \times 10^7$ $(\Omega.m)^{-1}$

$$R \cong 1 - (\frac{8(2\pi f)\epsilon_0}{\sigma_{cop}})^{1/2}$$

$\omega = 2\pi f = 2\pi c/\lambda = 2\pi c/10^{-3} = 1.884 \times 10^{12}$ Hz

R=1-1.51 x $10^{-3}$ = 0.998

b) $\omega = \frac{2\pi c}{10^{-6}} = 1.88x10^{15} H_z$

R= 1- 0.0479=0.952

**Note:** in conducting materials reflectivity and absorbability are very high, this make the permeability for this materials very small. Hence the conducting materials are dark to light.

**Problem 16:** For Aluminum wave length λ = 5500Å , n=1.15 and K=3.2 find:

(a) Reflectance, (b) absorption coefficient, and (c) phase change on reflection at normal incident.

**Solution**:

a) $R = \frac{(1-n)2_{+k2}}{(1+n)2_{+k2}}$

$= \frac{(1-1.15)2_{+}(3.2)2}{(1+1.15)+(3.2)2} \cong 0.69$

b) $k = \frac{c}{\omega} B \rightarrow absorption\ coefficient$

$B = \frac{\omega}{c}\ k$

$\omega = 2\pi f = \pi \frac{c}{\lambda} = 3.4x10^{15} H_z$

$B = \frac{3.4x10^{15}}{3x10^{8}} (3.2) = 3.655\times 10^{7}$ $m^{-1}$

C) $\varphi_s = \tan^{-1}\left(\frac{2k}{1-n^2-k^2}\right) = \tan^{-1}\frac{2(3.2)}{12.625}$

$\varphi_s \cong 29^{\circ}$

**Problem 17**: An electric plane wave has a frequency of 90Hz and amplitude 0.85 V/m. Write the equation for the electric wave and the magnetic wave.

**Solution:**

$\omega = k\,c = 2\boldsymbol{\pi}f = 2\boldsymbol{\pi}$ x 90Hz = $180\boldsymbol{\pi}$

$K = \frac{\omega}{\boldsymbol{c}} = \frac{180\pi}{3x10^{8}m/s} = 60\pi x10^{-8}m^{-1}$

The electric plane wave

$E = E_0 \sin(kx - \omega t) = 0.85\ \sin(60\pi \times 10^{-8} \times -180\pi b)\ \frac{\boldsymbol{V}}{\boldsymbol{m}}$

The magnetic plane wave

$H = H_0 \sin(kx - \omega t) = \frac{E_0}{\eta_{\cdot}} \sin(kx - \omega t) = \frac{\mathbf{0.85}}{\mathbf{120}\boldsymbol{\pi}} \sin(60\pi \times 10^{8}x - 180\pi t)$ A/m

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