Q1) Show that every finite integral domain is a field.

Q2) **(a)** Show that the set $K = \{\begin{pmatrix} a & b \\ -3b & a \end{pmatrix} : a, b \in \mathbb{Q}\}$ is a field with respect to matrix addition and multiplication.

(b) Show that $K$is isomorphic to the field $\mathbb{Q}(i\sqrt{3}) = \{a + bi\sqrt{3}: a, b \in \mathbb{Q}\}$.

Q3) Let D be an integral domain, let φ be the monomorphism from D into $Q(D)$ such that $\varphi(a) = \frac{a}{1}$, and let K be a field with the property that there is a monomorphism θ from D into K. Prove that, there exists a monomorphism $\psi: Q(D) \to K$ such that $\psi \circ \varphi = \theta$.

Q4) Consider the group G of order 8 given by the multiplication table

.	e	a	b	c	p	q	r	s
e	e	a	b	c	p	q	r	s
a	a	b	c	e	q	r	s	p

b	b	c	e	a	r	s	p	q
c	c	e	a	b	s	p	q	r
p	p	s	r	q	e	c	b	a
q	q	p	s	r	a	e	c	b
r	r	q	p	s	b	a	e	c
s	s	r	q	p	c	b	a	e

(a) Show that $B = \{e, b\}$ and $Q = \{e, q\}$ are subgroups.

(b) List the left and right cosets of B and of Q, and deduce that B is normal and Q is not.

(c) Let H be the group given by the table

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	e	z	y

y	y	z	e	x
z	z	y	x	e

Describe a homomorphism φ from G onto H with kernel B.

Q5) Prove that, every Euclidean domain is a principal ideal domain.

Q6) Let $R = \{a + bi\sqrt{3}: a, b \in \mathbb{Z}\}$.

(a) Show that R is a subring of $\mathbb{C}$.

(b) Show that the map $\varphi: R \to \mathbb{Z}$ given by $\varphi(a + bi\sqrt{3}) = a^2 + 3b^2$ preserves multiplication: for all u, v in R, $\varphi(uv) = \varphi(u)\varphi(v)$. Show also that $\varphi(u) > 3$ unless $u \in \{0,1,-1\}$.

(c) Show that the units of R are 1 and -1.

(d) Show that $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$ are irreducible, and deduce that R is not a unique factorization domain.

Q7) Show that, even if K is a field, $K[X, Y]$ is not a principal ideal domain.

Q8) Show that $3X^4 - 7X + 5$ is irreducible over $\mathbb{Q}$.

Q9) Let $L: K$ be a field extension such that $[L: K]$ is a prime number. Show that there is no subfield E of L such that $K \subset E \subset L$.

Q10) Let α be a root in $\mathbb{C}$ of the polynomial $X^2 + 2X + 5$. Express the element $\frac{\alpha^3+\alpha-2}{\alpha^2-3}$ of $\mathbb{Q}(\alpha)$ as a linear combination of the basis $\{1, \alpha\}$.

Q11) Show that the polynomial $X^3 + X + 1$ is irreducible over $\mathbb{Z}_2 = \{0,1\}$, and let α be the element $X + \langle X^3 + X + 1\rangle$ in the field $K = \mathbb{Z}_2[X]/\langle X^3 + X + 1\rangle$. List the 8 elements of K, and show that $K\backslash\{0\}$ is a cyclic group of order 7, generated by α.

Q12) Describe a ruler and compasses construction for the bisection of an angle.

Q13) Describe ruler and compasses constructions for the angle $\frac{\pi}{3}$.

Q14) Show that splitting field of $X^4 + 3$ over $\mathbb{Q}$ is $\mathbb{Q}(i, \alpha\sqrt{2})$, where $\alpha = \sqrt[4]{3}$. What is its degree over $\mathbb{Q}$?

Q15) Let $A = \{a_1,\ a_2,\ \dots\ ,\ a_n\}$ be a finite subset of a commutative ring R. Then the set $Ra_1 + Ra_2 + \cdots + Ra_n$ is the smallest ideal of R containing A.

Q16) Let D be a principal ideal domain, let p be an irreducible element in D, and let $a, b \in D$. Show that, if $p\backslash ab$ implies that $p\backslash a$ or $p\backslash b$.

Q17) Let $L: K$ and $M: L$ be field extensions, and $[M: K]$ be finite. Show that, if $[M: K] = [L: K]$, then $M = L$.

Q18) Show that $f(X) = X^3 + X + 1$ is irreducible over $\mathbb{Q}$. let α be a root of f in $\mathbb{C}$. Express $\frac{1}{\alpha}$ and $\frac{1}{\alpha+1}$ as linear combinations of $\{1, \alpha, \alpha^2\}$.

Q19) Let K be a field of characteristic 0, and suppose that $X^4 - 16X^2 + 4$ is irreducible over K. Let α be the element $X + \langle X^4 - 16X^2 + 4 \rangle$ in the field $L = K[X]/\langle X^4 - 16X^2 + 4 \rangle$. Determine the minimum polynomial $\alpha^3 - 14\alpha$.

Q20) Show how to construct a square equal in area to a given parallelogram.

Q21) Describe ruler and compasses constructions for the angle $\frac{\pi}{4}$.

Q22) Determine the splitting fields over $\mathbb{Q}$ of $X^4 - 5X^2 + 6$, and find their degree over $\mathbb{Q}$.

Q23) Let n be a positive integer. Prove that, the residue class ring $\mathbb{Z}_n = \mathbb{Z}/\langle n\rangle$ is a field if and only if n is prime.

Q24) Show that $g = 7X^4 + 10X^3 - 2X^2 + 4X - 5$ is irreducible over $\mathbb{Q}$.

Q25) Let $L: K$ and $M: L$ be field extensions, and $[M: K]$ be finite. Show that, if $[M: L] = [M: K]$, then $L = K$.

Q26) Determine the minimum polynomial of $\sqrt{1+\sqrt{2}}$ over $\mathbb{Q}$. What is its minimum polynomial over $\mathbb{Q}[\sqrt{2}]$?

Q27) Let K be a field of characteristic 0, and suppose that $X^4 - 16X^2 + 4$ is irreducible over K. Let α be the element $X + \langle X^4 - 16X^2 + 4 \rangle$ in the field $L = K[X]/\langle X^4 - 16X^2 + 4 \rangle$. Determine the minimum polynomial $\alpha^3 - 18\alpha$.

Q28) Construct a square equal in area to a given rectangle.

Q29) Describe ruler and compasses constructions for the angle $\frac{\pi}{6}$.

Q30) Determine the splitting fields over $\mathbb{Q}$ of $X^4 - 1$, and find their degree over $\mathbb{Q}$.