Chapter Three

Multiplication Principle

EXAMPLE 3.1 Counting the Number of Possible Meals

The fixed-price dinner at Mabenka Restaurant provides the following choices:

Appetizer: soup or salad
Entree: baked chicken, broiled beef patty, baby beef liver, or roast beef au jus
Dessert: ice cream or cheesecake

How many different meals can be ordered?

SOLUTION Ordering such a meal requires three separate decisions:

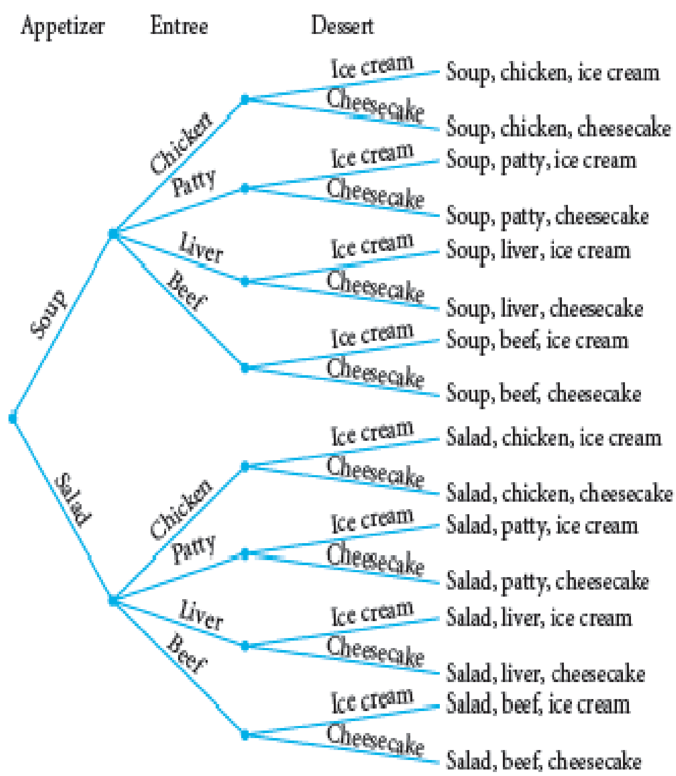
Choose an Appetizer	Choose an Entree	Choose a Dessert
2 choices	4 choices	2 choices

Look at the **tree diagram** in Figure 15. We see that, for each choice of appetizer, there are 4 choices of entrees. And for each of these $2 \cdot 4 = 8$ choices, there are 2 choices for dessert. A total of

$$2 \cdot 4 \cdot 2 = 16$$

different meals can be ordered.

FIGURE 1



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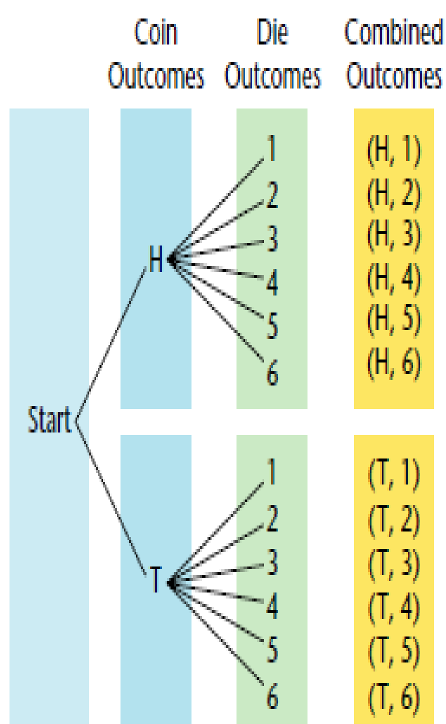
EXAMPLE 3.2 Combined Outcomes

Suppose we flip a coin and then throw a single die (Fig. 2). What are the possible combined outcomes?

SOLUTION One way to solve this problem is to use a **tree diagram:**

Fig 2



There are 12 possible combined outcomes—two ways in which the coin can come up followed by six ways in which the die can come up.

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Theorem 3.3: Multiplication Principle of Counting

1. If two operations O_1 and O_2 are performed in order with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

$$N_1 \cdot N_2$$

possible combined outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2, \ldots, O_n$, are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$, respectively, then there are

$$N_1 \cdot N_2 \cdot \cdots \cdot N_n$$

possible combined outcomes of the operations performed in the given order.

Example 3.4:

Computer-Generated Tests

Many universities and colleges are now using computer-assisted testing procedures. Suppose a screening test is to consist of five questions, and a computer stores five equivalent questions for the first test question, eight equivalent questions for the second, six for the third, five for the fourth, and ten for the fifth. How many different five-question tests can the computer select? Two tests are considered different if they differ in one or more questions.

SOLUTION

O_1:	Select the first question	N_1:	five ways
O_2:	Select the second question	N_2:	eight ways
O_3:	Select the third question	N_3:	six ways
O_4:	Select the fourth question	N_4:	five ways
O_5:	Select the fifth question	N_5:	ten ways

The computer can generate

$$5 \cdot 8 \cdot 6 \cdot 5 \cdot 10 = 12{,}000 \text{ different tests}$$

Example 3.5: Combination Locks

A particular type of combination lock has 100 numbers on it.

(i) How many sequences of three numbers can be formed to open the lock?
(ii) How many sequences can be formed if no number is repeated?

Solution:

(i) Each of the three numbers can be chosen in 100 ways. By the Multiplication Principle, there are

$$100 \cdot 100 \cdot 100 = 1000000$$

different sequences.

(ii) If no number can be repeated, then there are 100 choices for the first number, only 99 for the second number, and 98 for the third number. By the Multiplication Principle, there are

$$100 \cdot 99 \cdot 98 = 970200$$

different sequences.

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Example 3.6:

Counting Code Words

How many three-letter code words are possible using the first eight letters of the alphabet if:

(A) No letter can be repeated? (B) Letters can be repeated?

(C) Adjacent letters cannot be alike?

SOLUTIONS

(A) No letter can be repeated.

O_1:	Select first letter	N_1:	eight ways	
O_2:	Select second letter	N_2:	seven ways	Because one letter has been used
O_3:	Select third letter	N_3:	six ways	Because two letters have been used

There are

$$8 \cdot 7 \cdot 6 = 336 \text{ possible code words}$$

(B) Letters can be repeated.

O_1:	Select first letter	N_1:	eight ways	
O_2:	Select second letter	N_2:	eight ways	Repeats are allowed.
O_3:	Select third letter	N_3:	eight ways	Repeats are allowed.

There are

$$8 \cdot 8 \cdot 8 = 8^3 = 512 \text{ possible code words}$$

(C) Adjacent letters cannot be alike.

O_1:	Select first letter	N_1:	eight ways	
O_2:	Select second letter	N_2:	seven ways	Cannot be the same as the first
O_3:	Select third letter	N_3:	seven ways	Cannot be the same as the second, but can be the same as the first

There are

$$8 \cdot 7 \cdot 7 = 392 \text{ possible code words}$$

Example 3.7:

Standard license plates in the state of South Carolina consist of three letters of the alphabet followed by three digits.

(a) The South Carolina system will allow how many possible license plates?
(b) Of these, how many will have all their digits distinct?
(c) How many will have distinct digits and distinct letters?

Solution:

(a) There are six positions on the license plate to be filled, the first three by letters and the last three by digits. Each of the first three positions can be filled in any one of 26 ways, while each of the remaining three positions can each be filled in any of 10 ways. The total number of license plates, by the Multiplication Principle, is then

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000.$$

(b) Here the tasks involved in filling the digit positions are slightly different. The first digit can be any one of 10, but the second digit can be only any one of 9 (we cannot duplicate the first digit); there are only 8 choices for the third digit (we cannot duplicate either the first or the second). By the Multiplication Principle, there are

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 9 \cdot 8 = 12{,}654{,}720$$

plates with no repeated digit.

(c) If the letters and digits are each to be distinct, then the total number of possible license plates is

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11{,}232{,}000$$

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Permutations and Combinations

In this sections we use the Multiplication Principle to discuss two general types of counting problems, called ***permutations*** **and** ***combinations***.

Definition 3.8 Factorial

The symbol $n!$, read as "*n factorial*," is defined as

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 \cdot 1 &&= 2 \\ 3! &= 3 \cdot 2 \cdot 1 &&= 6 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 &&= 24 \end{aligned}$$

and, in general, for $n \geq 1$ an integer,

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

Theorem 3.9: (Recursion Formula for n Factorial)

$$n! = n \cdot (n-1)!.$$

Examples 3.10:

(a) $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

(b) $\dfrac{5!}{4!} = \dfrac{5 \cdot \cancel{4!}}{\cancel{4!}} = 5$

(c) $\dfrac{52!}{5!47!} = \dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} = 2{,}598{,}960$

(d) $\dfrac{7!}{(7-5)!5!} = \dfrac{7!}{2!5!} = \dfrac{7 \cdot 6 \cdot \cancel{5!}}{2!\cancel{5!}} = \dfrac{7 \cdot 6}{2} = 21$

(e) $\dfrac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{50!} = \dfrac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45!} = \dfrac{1}{45!}$

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Permutations

Definition 3.11: A **permutation** is an ordered arrangement of r objects chosen from n objects.

We discuss three types of permutations:
1- The n objects are distinct (different), and repetition is allowed in the selection of r of them. [Distinct, with repetition]
2- The n objects are distinct (different), and repetition is not allowed in the selection of r of them, where [Distinct, without repetition]
3- The n objects are not distinct, and we use all of them in the arrangement. [Not distinct].

1- Permutations: Distinct, with Repetition

Example 3.12:
The International Airline Transportation Association (IATA) assigns three-letter codes to represent airport locations. For example, the airport code for Ft. Lauderdale, Florida, is FLL. Notice that repetition is allowed in forming this code. How many airport codes are possible?
Solution: We are choosing 3 letters from 26 letters and arranging them in order. In the ordered arrangement a letter may be repeated. This is an example of a permutation with repetition in which 3 objects are chosen from 26 distinct objects.
The task of counting the number of such arrangements consists of making three selections. Each selection requires choosing a letter of the alphabet (26 choices). By the Multiplication Principle, there are

$$26 \cdot 26 \cdot 26 = 26^3 = 17{,}576$$

different airport codes.

Theorem 3.13: Permutations: Distinct Objects, with Repetition
The number of ordered arrangements of r objects chosen from n objects, in which the n objects are distinct and repetition is allowed, is n^r.

2- Permutations: Distinct Objects, without Repetition

Example 3.14:
Suppose that we wish to establish a three-letter code using any of the 26 uppercase letters of the alphabet, but we require that no letter be used more than once. How many different three-letter codes are there?
Solution: Some of the possibilities are: ABC,ABD, ABZ,ACB, CBA, and so on. The task consists of making three selections. The first selection requires choosing from 26 letters. Because no letter can be used more than once, the second selection requires