THE THEORY OF MEASURE

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CHAPTER 3

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At the turn of the nineteenth century it was quite apparent to mathematicians that the properties of continuous functions and Riemann's theory of integration were not rich enough to solve many scientific problems. The inadequacies of the continuous functions led them to search for different classes of functions that would provide solutions to a variety of problems.

Around the beginning of the twentieth century, the theory of measure was originated. At that time, it was realized that to get a better understanding of the structure of functions it was necessary to make a thorough study of the subsets of Euclidean spaces. To study these sets, it became clear that the classical notions of length, area, and volume needed to be generalized. The search for devising ways of assigning a concept of a "measure" to a given set of points has its roots in that period.

E. Borel [4] in 1898 was the first to establish a measure theory on the subsets of the real numbers known today as Borel sets. Soon after (in 1902), H. Lebesgue [21] presented his pioneering work on Lebesgue measure, and a little later (around 1918), C. Carathéodory introduced and studied the properties of outer measures. From then on a rapid development of the theory of measure, which included among its contributors the most prominent mathematicians of the first half of the twentieth century, followed.

This chapter discusses in detail the theory of measure. We start our study by introducing the concept of a semiring of sets and then proceed by studying the properties of measures on semirings. The important notion of the outer measure is introduced and studied next. It is followed by a detailed investigation of the measurable sets and measurable functions. Our attention is then turned to the properties of simple and step functions and to the basic properties of the Lebesgue measure. The chapter culminates with an investigation of convergence in measure and a discussion on abstract measurability properties.

12. SEMIRINGS AND ALGEBRAS OF SETS

In this section the notion of a semiring of sets is introduced and its properties are studied. A semiring of sets is the simplest family of sets for which a measure theory can be built. It turns out that most "reasonable" collections of sets satisfy the semiring properties.

Definition 12.1. *Let X be a nonempty set. A collection $\mathcal{S}$ of subsets of X is called a* **semiring** *if it satisfies the following properties:*

1. *The empty set belongs to $\mathcal{S}$; that is $\emptyset \in \mathcal{S}$.*
2. *If $A, B \in \mathcal{S}$; then $A \cap B \in \mathcal{S}$; that is, $\mathcal{S}$ is closed under finite intersections.*
3. *The set difference of any two sets of $\mathcal{S}$ can be written as a finite union of pairwise disjoint members of $\mathcal{S}$. That is, for every $A, B \in \mathcal{S}$; there exist $C_1, \ldots, C_n$ in $\mathcal{S}$ (depending on A and B) such that $A \setminus B = \bigcup_{i=1}^{n} C_i$ and $C_i \cap C_j = \emptyset$ if $i \neq j$.*

Now, let $\mathcal{S}$ be a semiring of subsets of X. A subset A of X is called a **σ-set** with respect to $\mathcal{S}$ (or simply a σ-set) if there exists a disjoint sequence $\{A_n\}$ of $\mathcal{S}$ (i.e., $A_n \cap A_m = \emptyset$ if $n \neq m$) such that $A = \bigcup_{n=1}^{\infty} A_n$. If $A = \bigcup_{i=1}^{n} A_i$ with $A_1, \ldots, A_n \in \mathcal{S}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then A is a σ-set. To see this, put $A_i = \emptyset$ for $i > n$. It follows from Definition 12.1 that $A \setminus B$ is a σ-set for every pair A and B in $\mathcal{S}$.

Some basic properties of σ-sets are included in the next theorem.

Theorem 12.2. *For a semiring $\mathcal{S}$, the following statements hold:*

1. *If $A \in \mathcal{S}$ and $A_1, \ldots, A_n \in \mathcal{S}$, then $A \setminus \bigcup_{i=1}^{n} A_i$ can be written as a finite union of disjoint sets of $\mathcal{S}$ (and hence, it is a σ-set).*
2. *For every sequence $\{A_n\}$ of $\mathcal{S}$, the set $A = \bigcup_{n=1}^{\infty} A_n$ is a σ-set.*
3. *Countable unions and finite intersections of σ-sets are σ-sets.*

Proof. (1) We use induction on n. For $n = 1$, the statement is true from the definition of the semiring. Now, assume the statement true for some n. Let $A \in \mathcal{S}$, and let $A_1, \ldots, A_n, A_{n+1} \in \mathcal{S}$. By the induction hypothesis, there exist $B_1, \ldots, B_k \in \mathcal{S}$ such that $B = A \setminus \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{k} B_i$ and $B_i \cap B_j = \emptyset$ if $i \neq j$. Consequently,

$$A \setminus \bigcup_{i=1}^{n+1} A_i = B \setminus A_{n+1} = \bigcup_{i=1}^{k} (B_i \setminus A_{n+1}).$$

By property (3) of Definition 12.1, each $B \setminus A_{n+1}$ can be written as a finite union of disjoint sets of $\mathcal{S}$. Since $B_i \cap B_j = \emptyset$ if $i \neq j$, it easily follows that $A \setminus \bigcup_{i=1}^{n+1} A_i$ can be written as a finite union of disjoint sets of $\mathcal{S}$. This completes the induction and the proof of (1).

(2) Let $\{A_n\} \subseteq \mathcal{S}$. Put $A = \bigcup_{n=1}^{\infty} A_n$, and then write $A = \bigcup_{n=1}^{\infty} B_n$ with $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus \bigcup_{i=1}^{n} A_i$ for $n \geq 1$. Observe that $B_i \cap B_j = \emptyset$ if

$i \neq j$, and by statement (1) each B_i is a σ-set. It now follows easily that A is itself a σ-set.

(3) The proof follows from (2), and property (2) of Definition 12.1. ■

The proof of part (2) of the preceding theorem also guarantees the validity of the following useful result:

Lemma 12.3. *If $\{A_n\}$ is a sequence of sets in a semiring $\mathcal{S}$, then there exists a disjoint sequence $\{C_n\}$ of $\mathcal{S}$ such that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} C_n$ and for each n there exists some k with $C_n \subseteq A_k$.*

Some natural collections of sets happen to satisfy other properties that are stronger than those of a semiring. The "algebra of sets" is such a collection, and its definition follows.

Definition 12.4. *A nonempty collection $\mathcal{S}$ of subsets of a set X which is closed under finite intersections and complementation is called an* **algebra of sets** (*or simply an* **algebra**). *That is, $\mathcal{S}$ is an algebra whenever it satisfies the following properties:*

i. *If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$.*
ii. *If $A \in \mathcal{S}$, then $A^c \in \mathcal{S}$.*

Three basic properties of an algebra are included in the next theorem.

Theorem 12.5. *For an algebra of sets $\mathcal{S}$, the following statements hold:*

1. $\emptyset, X \in \mathcal{S}$.
2. *The algebra $\mathcal{S}$ is closed under finite unions and intersections.*
3. *The algebra $\mathcal{S}$ is a semiring.*

Proof. (1) Since $\mathcal{S}$ is nonempty there exists some $A \in \mathcal{S}$. Now, by hypothesis $A^c \in \mathcal{S}$, and so, $\emptyset = A \cap A^c \in \mathcal{S}$. Moreover, $X = \emptyset^c \in \mathcal{S}$.

(2) Let $A, B \in \mathcal{S}$. Then $A \cup B = (A^c \cap B^c)^c \in \mathcal{S}$, and the rest of the proof can be completed easily by induction.

(3) We have to verify only property (3) of Definition 12.1. But this is obvious in view of the identity $A \setminus B = A \cap B^c$. ■

We continue by illustrating the notions of semiring and algebra of sets with examples.

Example 12.6. For every nonempty set X, the collection $\mathcal{S} = \{\emptyset, X\}$ is an algebra of sets. This is the "smallest" (with respect to inclusion) possible algebra. ■

Example 12.7. For every nonempty set X, its power set $\mathcal{P}(X)$ (i.e., the collection of all subsets of X) forms an algebra. This is the "largest" possible algebra. ■

Example 12.8. Let $\mathcal{F}$ be a nonempty pairwise disjoint family of subsets of a set X. Then $\mathcal{S} = \mathcal{F} \cup \{\emptyset\}$ is a semiring of subsets of X. To see this, note first that $\emptyset \in \mathcal{S}$. Now, if $A, B \in \mathcal{S}$, then $A \cap B$ is either empty or equal to A. Likewise, $A \setminus B$ is either empty or equal to A. Thus, $A, B \in \mathcal{S}$ implies that $A \cap B$ and $A \setminus B$ both belong to $\mathcal{S}$, and so $\mathcal{S}$ is a semiring. ■

Example 12.9. If $a, b \in \mathbb{R}$, let us write $[a, b) = \emptyset$ if $a \geq b$ and (as usual) $[a, b) = \{x \in \mathbb{R}\colon a \leq x < b\}$ if $a < b$. Then the collection $\mathcal{S} = \{[a, b)\colon a, b \in \mathbb{R}\}$ is a semiring of subsets of $\mathbb{R}$, which is not an algebra (for instance, notice that $[0, 1) \cup [2, 3) \notin \mathcal{S}$). ■

The semiring of the previous example is very important because of its many applications. Its analogue in higher dimensions is presented next.

Example 12.10. Let $\mathcal{S}$ denote the collection of all subsets A of $\mathbb{R}^n$ for which there exist intervals $[a_1, b_1), \ldots, [a_n, b_n)$ such that $A = [a_1, b_1) \times \cdots \times [a_n, b_n)$. (If $a_i \geq b_i$ holds for some i, then $[a_i, b_i) = \emptyset$, and so $A = \emptyset$.) Then $\mathcal{S}$ is a semiring of subsets of $\mathbb{R}^n$. To see this, note first that only the third property of the semiring definition needs verification; the other two are trivial. The proof is based upon the following identity among sets A, B, C, and D:

$$A \times B \setminus C \times D = [(A \setminus C) \times B] \cup [(A \cap C) \times (B \setminus D)], \qquad (\star)$$

where the sets of the union on the right-hand side are disjoint.

For the proof, use induction on n. For $n = 1$ the result is straightforward. Assume it now true for some n. We have to show that any set of the form

$$[a_1, b_1) \times \cdots \times [a_n, b_n) \times [a_{n+1}, b_{n+1}) \setminus [c_1, d_1) \times \cdots \times [c_n, d_n) \times [c_{n+1}, d_{n+1})$$

can be written as a finite union of disjoint sets from the $(n + 1)$-dimensional collection $\mathcal{S}$. But this can be easily shown by letting

$$\begin{aligned} A &= [a_1, b_1) \times \cdots \times [a_n, b_n), \qquad & B &= [a_{n+1}, b_{n+1}), \\ C &= [c_1, d_1) \times \cdots \times [c_n, d_n), \qquad & D &= [c_{n+1}, d_{n+1}) \end{aligned}$$

in $(\star)$ and using the induction hypothesis. ■

An intermediate notion between semirings and algebras is that of a ring of sets. A **ring of sets** (or simply a **ring**) is a nonempty collection of subsets $\mathcal{R}$ of a set X satisfying these properties:

a. If $A, B \in \mathcal{R}$, then $A \cup B \in \mathcal{R}$.
b. If $A, B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}$.

Every ring $\mathcal{R}$ contains the empty set. Indeed, since $\mathcal{R}$ is nonempty, there exists $A \in \mathcal{R}$, and so $\emptyset = A \setminus A \in \mathcal{R}$. Clearly, every algebra of sets is a ring of sets. Also, a ring $\mathcal{R}$ is necessarily a semiring. Indeed, if $A, B \in \mathcal{R}$, then the relation $A \cap B = A \setminus (A \setminus B)$ shows that $A \cap B \in \mathcal{R}$.

Another useful concept is that of a σ-algebra of sets.

Definition 12.11. *An algebra $\mathcal{S}$ of subsets of some set X is called a* σ**-algebra** *if every union of a countable collection of members of $\mathcal{S}$ is again in $\mathcal{S}$. That is, in addition to $\mathcal{S}$ being an algebra, $\bigcup_{n=1}^{\infty} A_n$ belongs to $\mathcal{S}$ for every sequence $\{A_n\}$ of $\mathcal{S}$.*

By virtue of $\bigcap_{n=1}^{\infty} A_n = (\bigcup_{n=1}^{\infty} A_n^c)^c$, it easily follows that every σ-algebra of sets is also closed under countable intersections.

Every collection of subsets $\mathcal{F}$ of a nonempty set X is contained in a smallest σ-algebra (with respect to the inclusion relation). This σ-algebra is the intersection of all σ-algebras that contain $\mathcal{F}$ (notice that $\mathcal{P}(X)$ is one of them), is called the **σ-algebra generated** by $\mathcal{F}$.

An important σ-algebra of sets is the σ-algebra of all Borel sets of a topological space. Its definition is given next.

Definition 12.12. *The* **Borel sets** *of a topological space (X, τ) are the members of the σ-algebra generated by the open sets. The σ-algebra of all Borel sets of (X, τ) will be denoted by $\mathcal{B}$.*

EXERCISES

1. If X is a topological space, then show that the collection

$$\mathcal{S} = \{C \cap O\colon C \text{ closed and } O \text{ open}\} = \{C_1 \setminus C_2\colon C_1, C_2 \text{ closed sets}\}$$

is a semiring of subsets of X.

2. Let $\mathcal{S}$ be a semiring of subsets of a set X, and let $Y \subseteq X$. Show that $\mathcal{S}_Y = \{Y \cap A\colon A \in \mathcal{S}\}$ is a semiring of Y (called the **restriction semiring** of $\mathcal{S}$ to Y).
3. Let $\mathcal{S}$ be the collection of all subsets of $[0, 1)$ that can be written as finite unions of subsets of $[0, 1)$ of the form $[a, b)$. Show that $\mathcal{S}$ is an algebra of sets but not a σ-algebra.
4. Prove that the σ-sets of the semiring

$$\mathcal{S} = \{\, [a, b)\colon a, b \in \mathbb{R} \text{ and } a \leq b \,\}$$

form a topology for the real numbers.

5. Let $\mathcal{S}$ be a semiring of subsets of a nonempty set X. What additional requirements must be satisfied for $\mathcal{S}$ in order to be a base for a topology on X? (For the definition of a base, see Exercise [18] of Section [8].) Prove that if such is the case, then each member of $\mathcal{S}$ is both open and closed in this topology.