Maxwell's Equations

1- Electrostatics

$$\Phi = N = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(4\pi r^2) = \frac{q}{\epsilon_0}$$

$$\Phi = \int_s \vec{E}\cdot d\vec{A}$$ «The flux»

$$= \frac{q}{\epsilon_0}$$

$$\sum_{i=1}^{n} q_i = \int \rho \, dV$$

$$\therefore \int_s \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0}\int \rho \, dV$$

$$\because \vec{\nabla}\cdot\vec{E} = \lim_{\Delta V \to 0} \frac{\int_s \vec{E}\cdot d\vec{A}}{\Delta V}$$

$$\therefore \int_V \vec{\nabla}\cdot\vec{E} \, dV = \int_s \vec{E}\cdot dA$$

$$= \frac{1}{\epsilon_0}\int \rho \, dV$$

$$\boxed{\therefore \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$$ Maxwell 1st Eq.

Magnetostatics

$$\Phi = N = \int_s \vec{B}\cdot d\vec{A} = \int_V \vec{\nabla}\cdot\vec{B}\, dV$$

$$\because \quad \vec{B} = \frac{\mu_0}{4\pi}\ \frac{q(\vec{v}\times\vec{r})}{|\vec{r}|^3}$$

$$\therefore \quad \vec{\nabla}\cdot\vec{B} = \frac{\mu_0 q}{4\pi}\vec{\nabla}\cdot\frac{(\vec{v}\times\vec{r})}{|\vec{r}|^3}$$

By using.

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

we get

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$ Maxwell 2^{nd} Eq.

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}\overbrace{\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)}^{=0} + \hat{j}\underbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)}_{=0} + \hat{k}\underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial}{\partial y}A_x\right)}_{=0}$$

also $\overset{=0}{\vec{\nabla}\times\vec{B}} = 0$

Varying Electric Field

$$\because \quad \vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$$

and $\vec{\nabla}\times\vec{B} = \mu_0(\vec{J}+\vec{D})$ $\qquad \vec{E} = \frac{\vec{D}}{\epsilon_0}$

$$\because \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = \vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{\epsilon_0}\vec{\nabla}\cdot\vec{D}$$

$$\text{i.e} \quad \vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t} - \frac{D}{\epsilon_0}\right) = 0$$

$$\therefore \quad \frac{\partial\vec{E}}{\partial t} = \frac{\vec{D}}{\epsilon_0} \quad \Rightarrow \vec{D} = \epsilon_0\frac{d\vec{E}}{dt}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 J + \mu_0\epsilon_0\frac{\partial E}{\partial t}$$

But $c^2 = \frac{1}{\epsilon_0\mu_0}$

$$\boxed{\therefore \quad \vec{\nabla}\times\vec{B} = \frac{J}{\epsilon_0 c^2} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$$

Maxwell's 3rd eq.

Varying Magnetic Field

$$e.m.f\ (\mathcal{E}) = -\frac{dN}{dt}$$

$$\oint = \mathcal{E} = \int_s \vec{E}\cdot d\vec{a}$$

$$\vec{\nabla}\times\vec{E} = \lim_{A\to 0}\frac{\int \vec{E}\cdot d\vec{a}}{A}$$

$$\therefore \int_s \vec{E}\cdot da = \int \vec{\nabla}\times\vec{E}\cdot d\vec{A} = -\frac{dN}{dt}$$ «Stokes The.»

$$\because N = \int_A \vec{B}\cdot d\vec{A}$$

$$\frac{dN}{dt} = \int \frac{d\vec{B}}{dt}\cdot d\vec{A}$$

$$\therefore \int_A \vec{\nabla}\times\vec{E}\cdot dA = -\int_A \frac{d\vec{B}}{dt}\cdot d\vec{A}$$

$$\therefore \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}}$$ «4^{th} MaxWells Eq.»

...ple 1

...f $\vec{E} = E_{max} \sin(\omega t - \alpha z)\hat{a}_y$

Find $\vec{D}$, $\vec{B}$ and $\vec{H}$

sol

$$D = \epsilon_0 \vec{E} = \epsilon_0 E_{max} \sin(\omega t - \alpha z)\hat{a}_y$$

$\because \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Maxwell 4th eq.

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_{max}\sin(\omega t - \alpha z) & 0 \end{vmatrix} = -\frac{\partial \vec{B}}{\partial t}$$

$$\hat{a}_x\left[\alpha E_{max} \cos(\omega t - \alpha z)\right] = -\frac{\partial \vec{B}}{\partial t}$$

$$\therefore \vec{B} = \int \alpha E_{max} \cos(\omega t - \alpha z)\, dt$$

$$= -\frac{\alpha E_{max}}{\omega} \sin(\omega t - \alpha z)\, \hat{a}_x$$

$$\vec{B} = \mu_0 \vec{H} \Rightarrow \vec{H} = \frac{-\alpha E_{max}}{\omega \mu_0} \sin(\omega t - \alpha z)\hat{a}_x$$

$$\vec{B} = \mu_0 \vec{H} \rightarrow H = \frac{\alpha E_{max}}{\omega \mu_0} \sin(\omega t - \alpha z)\hat{a}_x$$

Example-2 If $\vec{H} = \vec{H}_{max}\, e^{j(\omega t+\beta z)}\, a_x$ in free space.

Find $\vec{E}$.

Solution

$$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_{max}\, e^{j(\omega t+\beta z)} & 0 & 0 \end{vmatrix} = \frac{\partial \vec{D}}{\partial t}$$

$$\frac{\partial}{\partial z}\left(\vec{H}_{max}\, e^{j(\omega t+\beta z)}\, a_y\right) = \frac{\partial \vec{D}}{\partial t}$$

$$j\beta \vec{H}_{max}\, e^{j(\omega t+\beta z)}\, a_y = \frac{\partial \vec{D}}{\partial t}$$

$$\therefore \vec{D} = \frac{\beta H_{max}}{\omega}\, e^{j(\omega t+\beta z)}\, \hat{a}_y$$

$$E = \frac{\vec{D}}{\epsilon_0} \quad \text{and} \quad \vec{D} = \frac{\beta H_{max}}{\omega\,\epsilon_0}\, e$$

Example 3

If $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{E}}{\partial t}$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \cdot H = 0$$

$$\vec{\nabla} \cdot E = 0$$

Show that H $\vec{H}$ and $\vec{E}$ satisfies

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$$ "wave Equation"

Sol

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial H}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

$$= -\frac{\partial}{\partial t}\left(\frac{\partial \vec{E}}{\partial t}\right) = -\frac{\partial^2 E}{\partial t^2}$$

$$\because \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 E + \vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{E})}_{=0}$$

$$= -\nabla^2 \vec{E}$$

$$\therefore \nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$$

Similarly

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t}\right) = \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) = \frac{\partial}{\partial t}\left(-\frac{\partial \vec{H}}{\partial t}\right)$$

$$= -\frac{\partial^2 H}{\partial t^2}$$

But $\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = -\nabla^2 H + \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) = -\nabla^2 H$

$$\therefore \nabla^2 \vec{H} = \frac{\partial^2 H}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$

xample 4

$\vec{D} = \epsilon_0 \vec{E}$

$\vec{D} = \epsilon_0 E_{max} \sin(\omega t - \beta z)\, \hat{a}_y$

$\because \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ maxwell 4^{th} eq.

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_{mx} \sin(\omega t - \beta z)\, \hat{a}_x & 0 \end{vmatrix} = -\frac{\partial B}{\partial t}$$

$B = \mu_0 \vec{H}$

$\therefore H = \frac{-\beta E_{max}}{\omega \mu_0} \sin\omega(\omega t - \beta z)\, \hat{a}_x$

$$-\frac{dB}{dt} = \hat{a}_x \left(\frac{\partial 0}{\partial y} - \frac{\partial}{\partial z} E_{max} \sin(\omega t - \beta z) \right) + 0 + 0$$

$$= -(-\hat{a}_x E_{max} \beta \cos(\omega t - \beta z)$$

$$-\frac{dB}{dt} = \hat{a}_x E_{max} \beta \cos(\omega t - \beta z)$$

$$B = -\int \hat{a}_x E_{max} \beta \cos(\omega t - \beta z)\, dt$$

$$= \hat{a}_x \frac{E_{max}\, \beta}{\omega} \sin(\omega t - \beta z)$$

$$H = \frac{\vec{B}}{\mu_0} = \frac{E_{mx}\, \beta}{\omega \mu_0} \sin(\omega t - \beta z)\, \hat{a}_x$$