# Classification of The Projective Line of Order Nineteen and its Application to Error-Correcting Codes

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Let $V(n+1,q)$ be the $n+1$ dimensional vector space over the Galois field $F_q$.

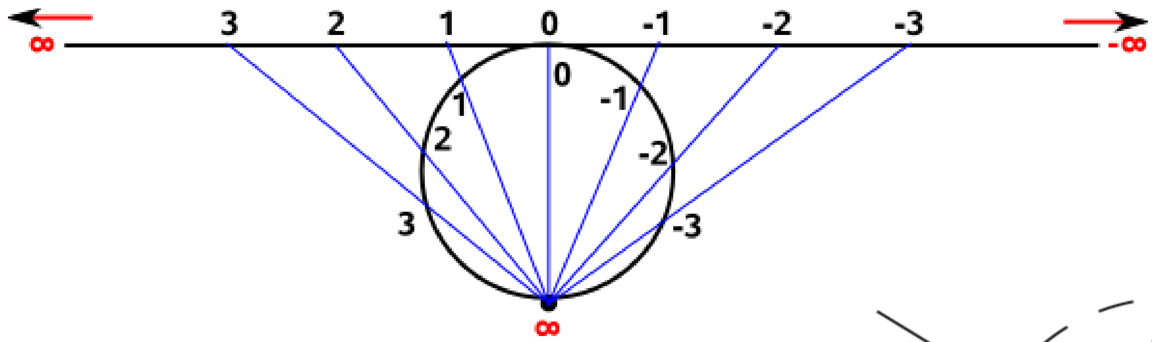
- Let $X=\langle x_0,\ldots,x_n\rangle, Y=\langle y_0,\ldots,y_n\rangle \in V\setminus\{0\}$ $X$ is equivalent to $Y$ if $X=tY$ for some $t$ in $F_q\setminus\{0\}$.

Then the set of equivalence classes is the *n-dimensional projective space* over $F_q$ and is denoted by $PG(n,q)$.

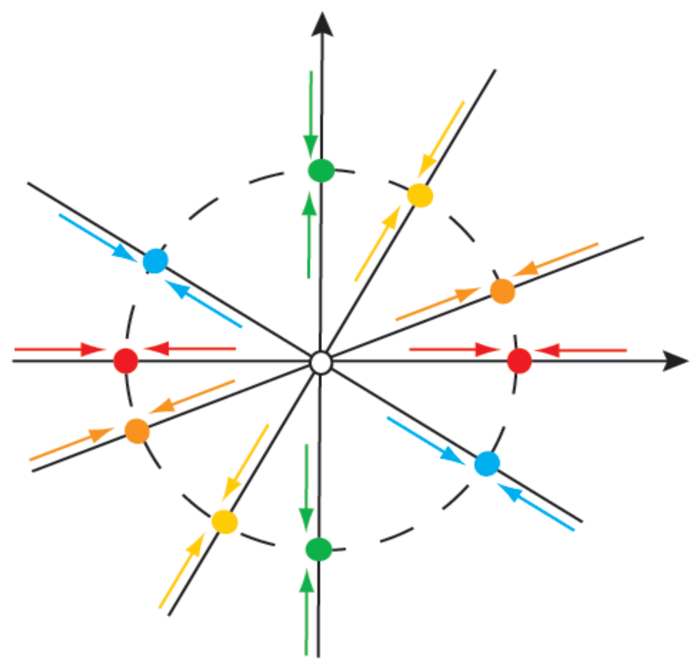
# Real Projective Line

We have Two descriptions of the real projective line, defined to be the space of lines in the Euclidean plane through a fixed point.

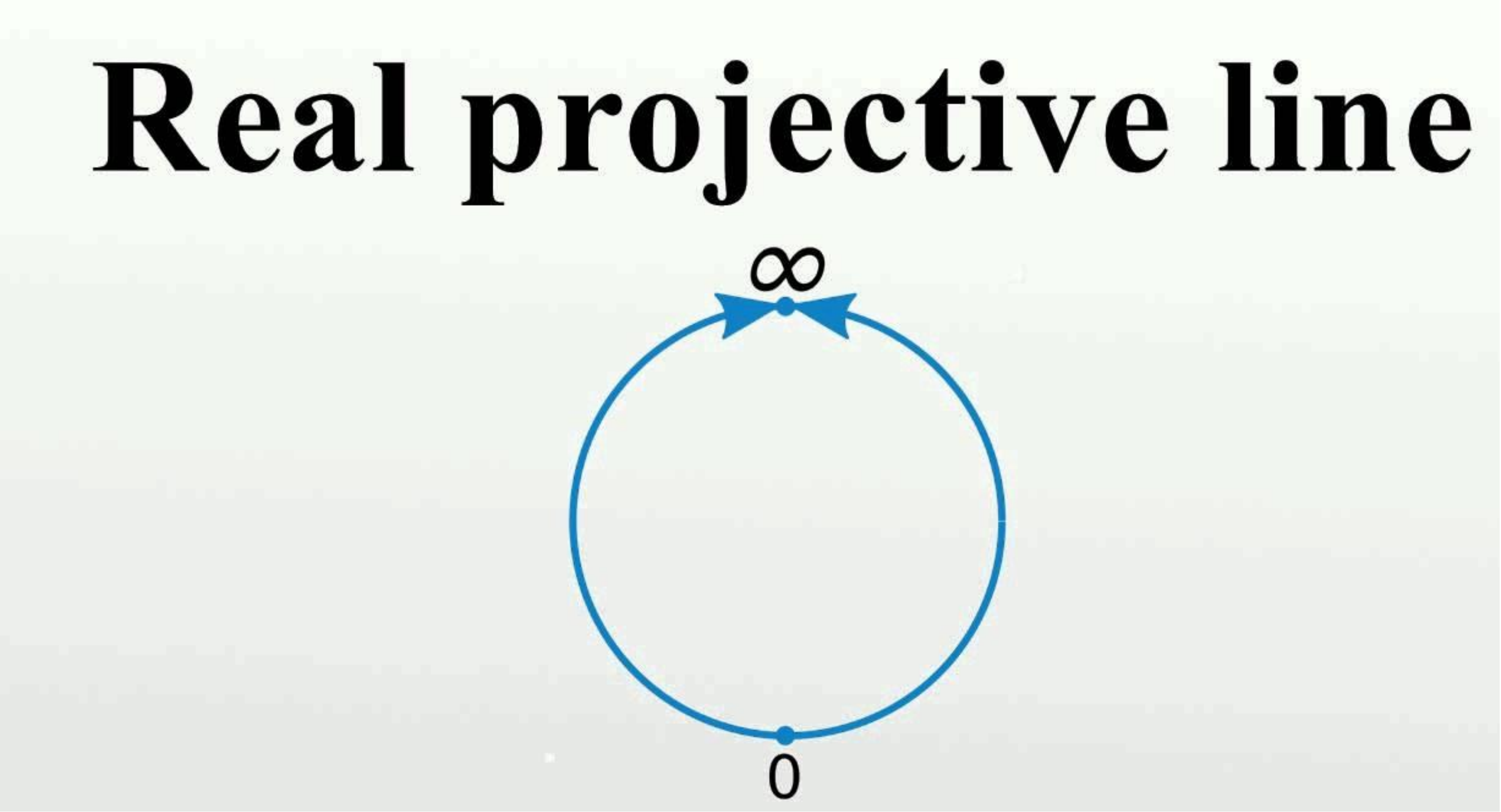
1- A line together with a point at infinity.




2- All lines in the plane through the origin point except (0,0) .



# The projective lines looks like a circle.



# Algebraic representation of Projective line

When the field is finite of order $q$

Each point $\mathbf{P}(t_0, t_1)$ with $t_1 \neq 0$ in $PG(1,q)$ is determined by the non-homogeneous coordinate $t_0/t_1$; the coordinate for $\mathbf{P}(1, 0)$ is $\infty$. So, the points of $PG(1,q)$ can be represented by the set $F_q \cup \{\infty\}$.

**Example:** Codings of a path avoiding an enemy territory

NNWNNWWSSWWNNNNWWN

Three ways to encode the safe route from Mohammad to Ahmmed are:

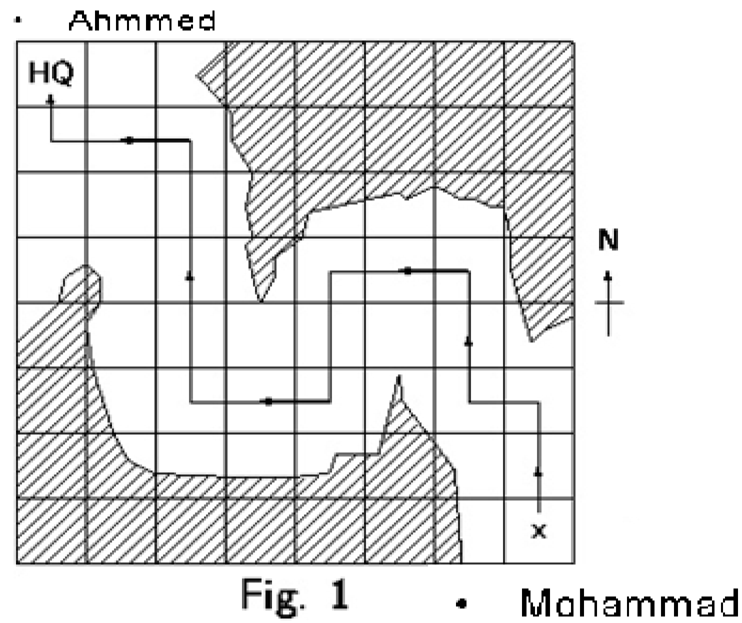


Fig. 1

1. $C_1$ = {N=00, W=01, E=10, S=11}

00 00 01 00 00 01 01 11 11 01 01 00 00 00 00 01 01 00

Any error in the code word would be a disaster.

N=00 ⟶ 10

2. $C_2$ = {000, 011, 101, 110}

A single error in encoding each of symbols N, W, S, E could be detected.

N=000 ⟶ 100 (000 or 101 or 110)

3. $C_3 = \{00000, 01101, 10110, 11011\}$

A single error in decoding each of symbols N, W, S, E could be corrected.

$N = 00000 \longrightarrow 11000$

$d(N, 11000) = 2$

$d(W, 11000) = 3$

$d(E, 11000) = 3$

$d(S, 11000) = 2$

$N = 00000 \longrightarrow 10000$ decode as N.

We can generate $C_3$ by the matrix $G$

$$G = \begin{bmatrix} W \\ E \end{bmatrix} = \begin{bmatrix} 01101 \\ 10110 \end{bmatrix}$$

- A *projective linear code* $C$, $[n,k,d]_q$– code, is a subspace of $V(n+1,q)$ where $k = \dim C$,

  $d = d(C) = \min\{|\{i \mid x_i \neq y_i\}|\}$

  $d \leq n-k+1$

and any two columns of a generator matrix $G$ are linearly independent, where $G$ is a $k \times n$ matrix having as rows the vectors of a basis of $C$. If $d = n-k+1$, then $C$ is called *maximum distance separable codes (MDS code)*.

- A code with minimum distance at least $d = 2e+1$ can correct up to $e$ errors. This type of code is called an *e-error correcting code*.

# Relation Between Projective Code and Projective Space

**Theorem:** There exists a projective $[n,k,d]_q$ -code if and only if there exists an $(n\ ;\ n-d)$-arc in $PG(k-1,q)$.

If $K=2$ then $n-d \le n-(n-k+1)=k-1=1$.

We have $(n\ ;\ 1)$ -arc in $PG(1,q)$; that is, we have $n$ distinct point.

## Theorem: (The Fundamental Theorem of Projective Geometry)

If $\{P_0, \ldots, P_{n+1}\}$ and $\{P'_0, \ldots, P'_{n+1}\}$ are both subsets of $PG(n,q)$ of cardinality $n+2$ such that no $n+1$ points chosen from the same set lie in a *hyperplane*, then there exists a unique projectivity $\mathfrak{I}$ such that

$P'_i = P_i\mathfrak{I}$ for $i = 0,1,\ldots,n+1.$

For $n=1$, simplifies: there is a unique projectivity of $PG(1,q)$ transforming any three distinct points on a line to any other three.

**Definition:** The cross-ratio $\lambda = \{P_1, P_2; P_3, P_4\}$ of four ordered points (tetrad) $P_1, P_2, P_3, P_4 \in PG(1,q)$ with coordinates $t_1, t_2, t_3, t_3$ is

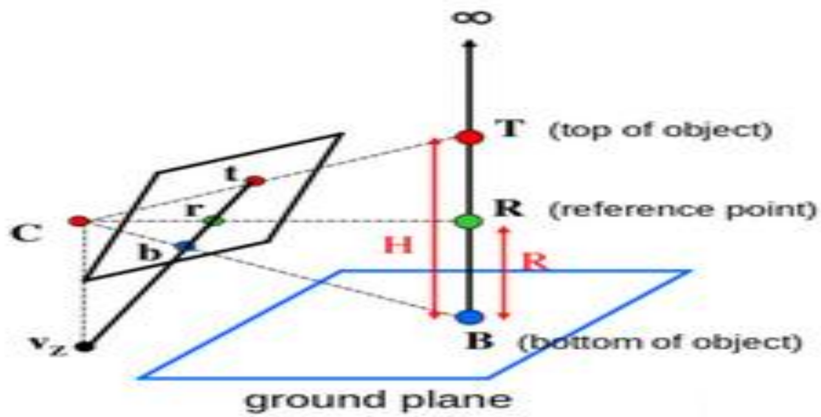
$$\lambda = \{P_1, P_2; P_3, P_4\} = \{t_1, t_2; t_3, t_4,\} = \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_4)(t_2 - t_3)}$$

Under all 24 permutations of $\{P_1, P_2; P_3, P_4\}$ the cross-ratio takes just the six values

$$\lambda, 1/\lambda, (1-\lambda), 1/(1-\lambda), (\lambda-1)/\lambda, \lambda/(\lambda-1).$$

The cross-ratio is quantity that does not change under projective transformations.

# Measuring height



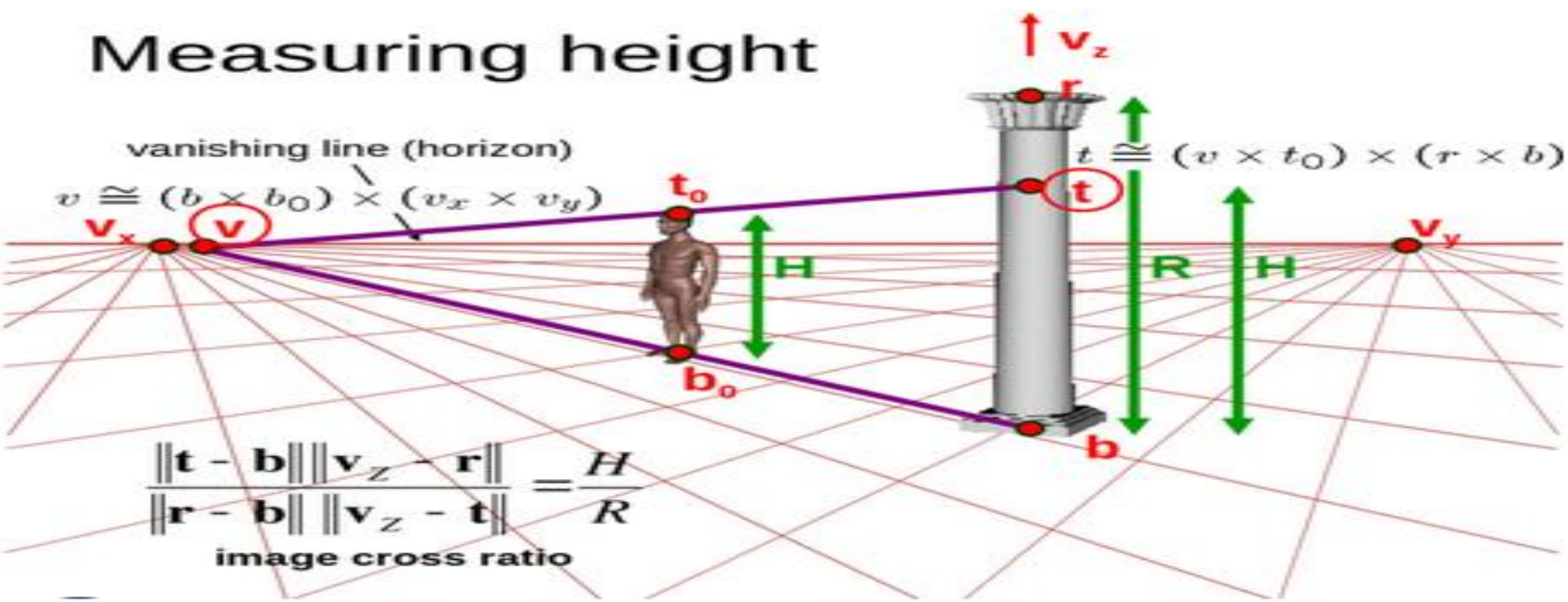

$$\frac{\|\mathbf{T}-\mathbf{B}\|\,\|\infty-\mathbf{R}\|}{\|\mathbf{R}-\mathbf{B}\|\,\|\infty-\mathbf{T}\|}=\frac{H}{R}$$

**scene cross ratio**

$$\frac{\|\mathbf{t}-\mathbf{b}\|\,\|\mathbf{v}_Z-\mathbf{r}\|}{\|\mathbf{r}-\mathbf{b}\|\,\|\mathbf{v}_Z-\mathbf{t}\|}=\frac{H}{R}$$

**image cross ratio**

# Measuring height



Let T be a tetrad with cross-ratio $\lambda$. Then T is called

- *harmonic* $(H)$ if $\lambda = 1/\lambda$ or $\lambda = \lambda/(\lambda - 1)$ or $\lambda = (1-\lambda)$
- *equianharmonic* $(E)$ if $\lambda = 1/(1-\lambda)$ or equivalently $\lambda = (\lambda - 1)/\lambda$.
- *neither harmonic nor equianharmonic* $(N)$ if the cross-ratio is another value.

The aims of this research are to answer the following questions in $PG(1,19)$:

1- How many projectively inequivalent $k$-sets in $PG(1,19)$ are there and what is the stabilizer group of each one?

2- Where $D_i$ is $10$-set and $D_i^c$ its complement.

Are $D_i$ and $D_i^c$ equivalent? What is the group of projectivities of $PG(1,19)$ of the partition?

3- Does the projective line split into five disjoint harmonic tetrads, five equianharmonic tetrads, and five tetrads of type $N_1$ or five tetrads of type $N_2$?

Also, the relation between a projective MDS codes of dimension two and $k$-sets on $PG(1,19)$ is given.

# Classification of The Projective Line $PG(1,19)$

- On $PG(1,19)$, there are 20 points.
- There is a harmonic tetrad $H=\{\infty,0,1,a\}$

  for $a=-1,2,-9.$
- There is an equianharmonic tetrad $E=\{\infty,0,1,b\}$

  for $b=-7,8.$
- There are two tetrads of type $N$

  $N_1 = \{\infty,0,1,c\}$ for $c=-2,3,-6,7,-8,9;$

  $N_2 = \{\infty,0,1,d\}$ for $d=-3,4,-4,5,-5,6.$

**Theorem:** On $PG(1,19)$, the number of inequivalent $k$-sets are

(1) Four projectively distinct 4-sets.

(2) Five projectively distinct 5-sets.

(3) Thirteen projectively distinct 6-sets.

(4) Eighteen projectively distinct 7-sets.

(5) Thirty one projectively distinct 8-sets.

(5) Thirty three projectively distinct 9-sets.

(6) Forty four projectively distinct 10-sets.

Theorem: The projective line $PG(1,19)$ has

(1) twenty eight projectively distinct partitions into two equivalent 10-sets;

(2) sixteen projectively distinct partitions into two inequivalent 10-sets.

Theorem: The projective line $PG(1,19)$ split into five disjoint harmonic, equianharmonic, tetrads of type $N_1$ and five tetrads of type $N_2$.

# The End