Finite Projective Geometry and Its Applications

Postgraduate Curse

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Course Title: Finite Projective Geometry and Its Applications

Level: Post Graduate

Syllabus of The Course		
Chapter	**Chapter's Name**	**Details of The Chapter**
1	Affine Space	Affine Space Affine Lines Parameterization of a Line Affine Transformation Affine Conic Conic in $\mathbb{A}_n(\mathbb{R})$ Conic in $\mathbb{A}_n(\mathbb{C})$ Intersecting Lines and Circles Parameterizing Circles
2	Projective Space	Introduction $\mathbb{P}_n(F)$ (or $PG(n,F)$) Describe points in the real plane Describe lines in the real plane Projective Algebraic Sets in $\mathbb{P}_2(F)$ Projective Line in $\mathbb{P}_2(F)$ Projective Transformations of $\mathbb{P}_n(F)$ Projective transformation in $\mathbb{P}_1(F)$ Homogenizing Affine Equations Solving equations in $\mathbb{P}_1(F)$ Intersecting Projective Lines and Curves Conic in $\mathbb{P}_2(F)$ Projective Parameterizations Intersecting Conics with Curves

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[2] J. W. P. Hirschfeld. Projective geometries over Finite Fields:2nd edition, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998.

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Chapter One

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Affine Space

Chapter One

Affine Space

I-Affine Space

Affine spaces are the right framework for dealing with motions, trajectories, and physical forces, among other things. Thus, affine geometry is crucial to a clean presentation of kinematics, dynamics, and other parts of physics (for example, elasticity). After all, a rigid motion is an affine map, but not a linear map in general.

Definition: Let $V\ (n+1, F)$ be a vector space of rank (dimension) $n+1$ over a field F. The **points, lines, planes, . . . , hyperplanes** are the subspaces of $V\ (n+1, F)$ of rank $0,1,2,3,\dots,n$.

Remark: A **hyperplane** is a subspace of co-dimension 1, that is, of dimension one less than the whole space.

Definition: Let $K \subseteq F$(field). Then K is a **flat** if $K = u + W := \{u + w \mid w \in W\}$ for some subspace $W \subseteq F$ and some vector $u \in F$. We also call $K = \emptyset$ a flat. $\dim(K) = \dim(W)$.

Definition: Affine $\boldsymbol{n}$-space (or $AG(n,F)$**)** over a field F is the set

$$\mathbb{A}_n(F) = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n, \in F\},$$

whose points, lines, planes,..., hyperplanes, are the cosets of the subspaces of $V(n,F)$ of rank $0, 1, 2, \dots, n-1$. The dimension of a subspace of $AG(n,F)$ is the rank of a subspace of $V(n,F)$.

If $F = F_q$, then we write $AG(n,q)$ for affine n-space over a finite field F_q.

Definition: (Analytic View of $\boldsymbol{AG(2,F)}$)

The points of the affine plane $AG(2,F)$ are the ordered pairs (x,y), where $x, y \in F$. The lines are collections of points (x,y) satisfying equations of the form $ax + by = c$, where $a, b, c \in F$.

Definition: (Geometric View of $\boldsymbol{AG(2,F)}$)

An **affine plane** $AG(2,F)$ over a field F is a set, whose elements are called **points**, and a set of subsets, called **lines**, satisfying the following three axioms, **A1–A3**. We will use the terminology ”P lies on l” or ”l passes through P” to mean the point P is an element of the line l.

A1: Given two distinct points P and Q, there is one and only one line containing both P and Q.

(We say that two lines are **parallel** if they are equal or if they have no points in common).

A2: Given a line l and a point P not on l, there is one and only one line m which is parallel to l and which passes through P.

A3: There exist three non-collinear points. (A set of points $P_1, \dots, P_n$ is said to be **collinear** if there exists a line l containing them all).

Remark: Affine space over the real numbers $\mathbb{R}$ has less structure than Euclidean space; there is no inner product, and thus no way of measuring distances or angles between vectors.

Example: (i) The ordinary plane, known to us from Euclidean geometry, satisfies the axioms A1–A3, and therefore is an affine plane.

(ii) Construct $AG(2,2)$.

Let's figure out the geometric structure for the collection of all ordered pairs (a,b), where $a,b \in F_2$. We list the four points as columns in a matrix.

$$\begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array}$$
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

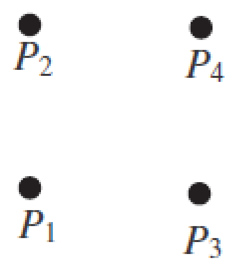


Figure 1: The four points of the affine plane $AG(2,2)$

In Figure 1, we draw the four points in a 2×2 grid, as we do in the Euclidean plane. The line is the collection of points (x,y) satisfying $ax + by = c$, where $a,b,c \in F_2$.

How many lines are there? Assume our line has equation $ax + by = c$, and let's count the number of distinct equations in a systematic way.

- Case 1: $a \neq 0$. Then we can divide both sides of $ax + by = c$ by a, yielding an equation that looks like $x + \acute{b}y = \acute{c}$. There are two choices for $\acute{b}$ and two for $\acute{c}$, so there are four lines in this case.

(i) If $b = 1 \Longrightarrow x + y = \acute{c}$, so we have two lines each line has two points as given below.

$l_1: x + y = 0$ through $P_1 = (0,0), P_4 = (1,1)$.

$l_2: x + y = 1$ through $P_3 = (0,1), P_2 = (1,0)$.

(ii) If $b = 0 \Longrightarrow x = \acute{c}$, so we have two lines each line has two points as given below.

$l_3: x = 0$ through $P_1 = (0,0), P_3 = (0,1)$.

$l_4: x = 1$ through $P_2 = (1,0), P_4 = (1,1)$.

- Case 2: $a = 0$. Then, dividing by b (which must be non-zero), we get an equation of the form $y = \acute{c}$, so we get two more lines in this case.

(i) If $\acute{c} = 0 \Longrightarrow l_5 : y = 0$, through $P_1 = (0,0), P_2 = (1,0)$.

(ii) If $\acute{c} = 1 \Longrightarrow l_6 : y = 1$, through $P_3 = (0,1), P_4 = (1,1)$.

This gives us a total of 6 lines. A picture of the entire collection of points and lines is shown in Figure 2.

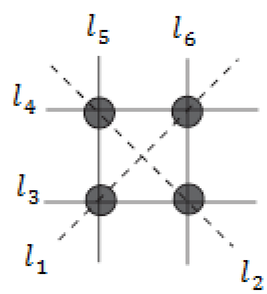


Figure 2: The affine plane $AG(2,2)$ has four points and 6 lines.

(iii) Construct $AG(2,3)$.

Let's figure out the geometric structure for the collection of all ordered pairs (a, b), where $a, b \in F_3$. We list the nine points as columns in a matrix.

$$\begin{matrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \end{matrix}$$
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}.$$

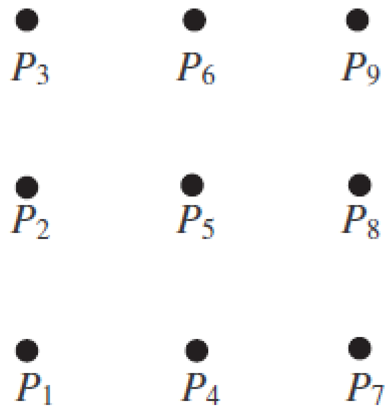


Figure 3: The nine points of the affine plane $AG(2,3)$

In Figure 3, we draw the nine points in a 3×3 grid, as we do in the Euclidean plane. The line is the collection of points (x, y) satisfying $ax + by = c$, where $a, b, c \in F_3$.

How many lines are there? Assume our line has equation $ax + by = c$, and let's count the number of distinct equations in a systematic way.

- Case 1: $a \neq 0$. Then we can divide both sides of $ax + by = c$ by a, yielding an equation that looks like $x + \acute{b}y = \acute{c}$. There are three choices for $\acute{b}$ and three for $\acute{c}$, so there are nine lines in this case.
- Case 2: $a = 0$. Then, dividing by b (which must be non-zero), we get an equation of the form $y = \acute{c}$, so we get three more lines in this case.

This gives us a total of 12 lines. A picture of the entire collection of points and lines is shown in Figure 4.

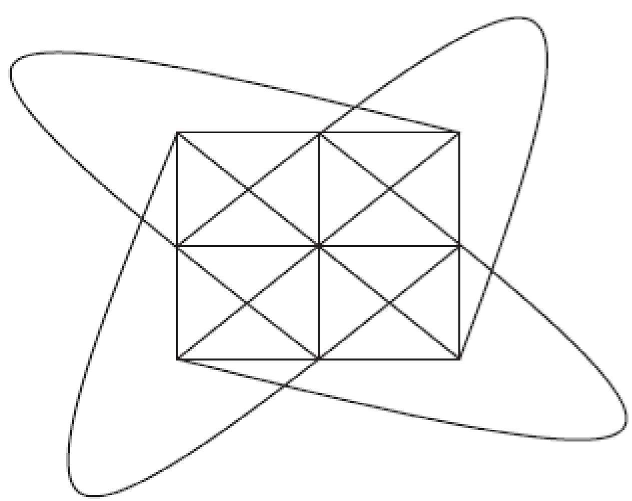


Figure 4: The affine plane $AG(2,3)$ has nine points and 12 lines.

Theorem: In $\boldsymbol{AG(2,F)}$

(i) Parallelism is an equivalence relation.

(ii) Two distinct lines in have at most one point in common.

(iii) An affine plane has at least four points. There is an affine plane with four points.

Proof:

(i) We must check the three properties:

1. Any line is parallel to itself, by definition.
2. $l \parallel m \implies m \parallel l$ by definition.
3. If $l_1 \parallel l_2$, and $l_2 \parallel l_3$, we wish to prove $l_1 \parallel l_3$.

If $l_1 = l_3$, there is nothing to prove. If $l_1 \neq l_3$, and there is a point $P \in l_1 \cap l_3$, then l_1, l_3.are both $\parallel l_2$ and pass through P, which is impossible by axiom A2. We conclude that $l_1 \cap l_3 = \emptyset$ (the empty set), and so $l_1 \parallel l_3$.

(ii) For if l, m both pass through two distinct points P, Q, then by axiom A1, $l = m$.

(iii) Without proof.

Theorem: When $F = F_q$, $AG(2,q)$ then the following holds.

1- Number of points is q^2.

2- Number of lines is $q^2 + q$ lines.

3- Any family of parallel lines consists of exactly q lines.

4- There are exactly $(q+1)$ families of parallel lines.

4- Every line has exactly q points.

5- There are exactly $(q+1)$ lines that pass through a given point.

Proof: Exercise.

II-Affine Lines

Definition: We say that $V \subseteq \mathbb{A}_n(F)$ is an algebraic set if there are polynomials $f_1, f_2, \dots, f_m \in F[X_1, X_2, \dots X_n]$ such that

$$V = \{x \in \mathbb{A}_n(F) : f_1(x) = f_2(x) = \cdots = f_m(x) = 0\}.$$

If $m = 1$, i.e. V is the solutions of a single polynomial, we call V a **hypersurface**. We will be particularly interested in the case where $n = 2$ and $m = 1$. In this case we call V **a plane algebraic curve.**

Suppose $f \in F[X_1, X_2, \dots X_n]$ is nonzero. We can write

$$f = \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} a_{i_1,\dots,i_n} {X_1}^{i_1} {X_2}^{i_2} \dots {X_n}^{i_n}.$$

Definition: The degree of f is defined by

$$\deg(f) = \{i_1 + \cdots + i_n : i_1 \leq m_1, \dots, i_n \leq m_n, a_{i_1,\dots,i_n} \neq 0\}.$$

If f is a nonzero constant, then f has degree 0.

Example: $f = X^4 + 2X^3YZ - X^2Y + YZ^3$ has degree 5.

Definition: Polynomials $f \in F[X, Y]$ of degree 1 are called **linear**. A linear polynomial is of the form

$$aX + bY + c$$

where at least one of a and b is nonzero. The zero set of a linear polynomial is called a **line**.

Remark: Of course, if $F = \mathbb{R}$ then lines have a clear geometric meaning. But if F is the field F_3 then the line $X + 2Y + 1 = 0$ is just the discrete set of points $L = \{(0,1), (1,2), (2,0)\}$. We will see that the well-know geometric properties of lines hold in arbitrary fields even when there is no obvious geometry.

III-Parameterization of a Line:

Suppose L is the line $X + bY + c = 0$. We can easily find $\varphi: F \longrightarrow L$ a parameterization of L.

(i) If $b \neq 0$, let

$$\varphi(t) = \left(t, \frac{-c - at}{b}\right).$$

(ii) $b = 0$, then $a \neq 0$ and let

$$\varphi(t) = \left(\frac{-c}{a}, t\right).$$

Theorem: $\varphi: F \longrightarrow L$ is a bijection. In particular, if F is infinite, then so is L.

Proof: Exercise.

Suppose $f_i = a_iX + b_iY - c_i$ for $i = 1,2$. Let L_i be the line $f_i = 0$. If $(x, y) \in L_1 \cap L_2$, then

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Linear algebra tells us exactly what the solutions look like.

If the matrix $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is invertible, then $A^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is the unique solution.

But A is invertible if and only if the rows are linearly independent. Thus A is not invertible if and only if there is a λ, such that $a_2 = \lambda a_1$ and $b_2 = \lambda b_1$. In this case there are two possibilities. If $c_2 = \lambda c_1$, then $f_2 = \lambda f_1$ and L_1 and L_2 are the same line. If $c_2 \neq \lambda c_1$,, then the system has no solution and $L_1 \cap L_2 = \emptyset$.

We summarize these observations in the following Theorem.

Theorem: Suppose $f_1, f_2 \in F[X]$ are linear polynomials and L_i is the line $f_i = 0$ for $i = 1, 2$.

i) $L_1 = L_2$ *if and only if* $f_2 = \lambda f_1$ *for some* $\lambda \in F$.

ii) If L_1 *and* L_2 *are distinct lines, then* $|L_1 \cap L_2| \leq 1$.

iii) If $L_1 \cap L_2 = \emptyset$ *and* $f_1 = a_1X + b_1Y + c_1$, *then for some* λ $f_2 = \lambda a_1 X + \lambda b_1 Y + d$ *where* $d \neq \lambda c_1$.

Theorem: If (x_1, y_1) and (x_2, y_2) are distinct points in $\mathbb{A}_2(F)$, there is a unique line containing both points.

Proof : We are looking for $f = aX + bY - c$ such that

$$\begin{matrix} ax_1 & + & by_1 & - & c_1 = 0 \\ ax_2 & + & by_2 & - & c_2 = 0. \end{matrix}$$

This is a system of linear equations in the variables a, b, c. Since(x_1, y_1) and (x_2, y_2) are distinct, the rows of the matrix $\begin{pmatrix} x_1 & y_1 & -1 \\ y_2 & y_2 & -1 \end{pmatrix}$ are linearly independent. Since the matrix has rank 2, linear algebra tells us that we can find an nontrivial solution (a, b, c) and every other solution is of the form $(\lambda a, \lambda b, \lambda c)$. Thus there is a unique line through (x_1, y_1) and (x_2, y_2).

IV-Affine Transformation:

In $\mathbb{R}^2$ we can transform any line to any other line by rotating the plane and then translating it. We will show that this is possible for any field.

Definition: We say that $T\colon \mathbb{A}_2(F), \longrightarrow \mathbb{A}_2(F)$, is an **affine transformation** if there are linear polynomials f and g such that $T(x, y) = (f(x, y), g(x, y))$. In this case there is a 2×2 matrix A with entries from F and a vector $\vec{b} \in F^2$ such that

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \vec{b}.$$

If $\vec{b} = 0$ we say that T is a **linear transformation**.

An affine transformation can be thought of as a linear change of variables

$$\begin{aligned} U &= a_1X + b_1Y + c_1 \\ V &= a_2X + b_2X + c_2 \end{aligned}$$

Theorem: If L is a line in $\mathbb{A}_2(F)$ there is an invertible affine transformation taking L to the line $X = 0$.

Proof: Suppose L is given by the equation $aX + bY + c = 0$.

(i) If $a \neq 0$, consider the affine transformation

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix}.$$

In other words we make the invertible change of variables $U = aX + bY + c$ and

$V = Y$. This transforms L to the line $U = 0$.

(ii) If $a = 0$, we use the transformation

$$\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix},$$

that is, the change of variables $U = bY + c$ and $V = X$, to transform L to $U = 0$.

V-Affine Conic:

The zero set of a second order polynomial is called a conic.

Theorem: Suppose F is a field and the characteristic of F is not 2. If $p(X, Y) \in F[X, Y]$ has degree 2, there is an affine transformation taking the curve $p(X, Y) = 0$ to one of the form $aX^2 + bY^2 + c = 0$ where $b \neq 0$, $aX^2 + Y = 0$ or $X^2 + c = 0$.

Proof:

We will prove this by making a sequence of affine transformations. We begin with a polynomial

$$aX^2 + bY^2 + cXY + dX + eY + f = 0.$$

Claim 1: We may assume that a $c \neq 0$.

If $a = 0$ and b $b \neq 0$, we use the transformation $T(x, y) = (y, x)$.

If $a = b = 0$, then since the polynomial has degree 2 we must have $c \neq 0$. We make the change of variables

$$\begin{aligned} X &= \quad X \\ V &= \quad Y - X \end{aligned}$$

Then $XV = XY - X^2$ and our curve is transformed to

$$cX^2 + cXV + (d + e)X + eV + f = 0.$$

Claim 2: We may assume that $c = 0$.

This is the old algebra trick of "completing the square". We make a change of variables $U = X + \alpha Y$ so that $aU^2 = aX^2 + cXY + \beta Y^2$ for some appropriate β. To get this to work we would need $2a\alpha = c$. So we make the change of variables $U = X + \frac{c}{2\alpha}Y$. Note at this point we have to divide by 2. This is one reason we had to assume that the characteristic of F is not 2.

This change of variables transforms the curve to:

$$aU^2 + \left(b - \frac{c^2}{4a^2}\right)Y^2 + dU + \left(e - \frac{dc}{2a}\right)Y + f = 0.$$

Thus, by affine transformations, we may assume that our curve is given by

$$aX^2 + bY^2 + cX + dY + e = 0.$$

There are two cases to consider.

Case 1 $b \neq 0$.

We must do two more applications of completing the square. First we make a change of variables $U = X + \alpha$ so that $aU^2 = aX^2 + cX + \beta$. We need $2a\alpha = c$ so $\alpha = \frac{c}{2a}$. Similarly we let $V = Y + \frac{d}{2a}$. The transformed curve is given by

$$aU^2 + bV^2 + e - \frac{c^2 + d^2}{4a^2} = 0.$$

Case 2 $b = 0$.

As above we complete the square by taking $U = X + \frac{c}{2a}$. That transforms the curve to

$$aU^2 + dY + e - \frac{c^2}{4a^2} = 0.$$

If $d \neq 0$, the change of variables $V = dY + e - \frac{c^2}{4a^2}$ gives us

$$aU^2 + V = 0.$$

If $d = 0$, then the curve is already given by the equation

$$X^2 + \frac{e - \frac{c^2}{4a^2}}{a} = 0.$$

VI- Conic in $\mathbb{A}_n(\mathbb{R})$.

When our field is the field $\mathbb{R}$ of real numbers, we can give even more precise information.

Theorem 2.10 *If $p \in \mathbb{R}[X, Y]$ has degree 2, then there is an affine tranformation taking the curve $p = 0$ to one of the following curves:*

i) (parabola) $Y = X^2$;
ii) (circle) $X^2 + Y^2 = 1$;
iii) (hyperbola) $X^2 - Y^2 = 1$;
iv) (point) $X^2 + Y^2 = 0$;
v) (crossed-lines) $X^2 - Y^2 = 0$;
vi) (double line) $X^2 = 0$;
vii) (parallel lines) $X^2 = 1$
viii) (empty set) $X^2 = -1$
ix) (empty set) $X^2 + Y^2 = -1$.

Proof

If we can transform p to $aX^2+Y=0$, then the tranformation $V=\frac{-Y}{a}$ gives the parabola $V=X^2$.

Suppose we can transform p to $X^2+c=0$. If $c<0$, the transformation $U=\frac{X}{\sqrt{-c}}$, transforms p to $X^2=1$. While if $c>0$, the transformation $U=\frac{X}{\sqrt{c}}$, transforms p to $X^2=-1$.

Suppose we have transformed p to $aX^2+bY^2+c=0$.

case 1 $c\neq 0$

Our curve is the same as the curve $\frac{-a}{c}X^2+\frac{-b}{c}Y^2=1$. Thus we may assume our curve is $aX^2+bY^2=1$.

case 1.1 $a>0$ and $b>0$

The transformation $U=\frac{X}{\sqrt{a}}$, $V=\frac{Y}{\sqrt{b}}$ transforms the curve to the circle $U^2+V^2=1$.

case 1.2 $a>0$ and $b<0$

The transformation $U=\frac{X}{\sqrt{a}}$, $V=\frac{Y}{\sqrt{-b}}$ transforms the curve to the hyperbola $U^2-V^2=1$.

case 1.3 $a<0$ and $b>0$

We make the transformation $T(x,y)=(y,x)$ which transforms the curve to case 1.2.

case 1.4 $a<0$ and $b<0$

The transformation $U=\frac{X}{\sqrt{-a}}$, $V=\frac{Y}{\sqrt{-b}}$ transforms the curve to $U^2+V^2=-1$, which has no solutions in $\mathbb{R}^2$.

case 2 $c=0$.

If $b=0$, then we have curve $aX^2=0$ which is the same as $X^2=0$.

Suppose $b\neq 0$. We may assume $a>0$. If $b>0$, then the change of variables $U=\frac{X}{\sqrt{a}}$, $V=\frac{Y}{\sqrt{b}}$ transforms the curve to $U^2+V^2=0$ which has a single solution $\{(0,0)\}$.

If $b<0$, then the change of variables $U=\frac{X}{\sqrt{a}}$, $V=\frac{Y}{\sqrt{-b}}$ transforms the curve to $U^2-V^2=0$. The solution set is pair of lines $U+V=0$ and $U-V=0$.

Cases i)–iii) are considered nondegenerate.

VII- Conic in $\mathbb{A}_n(\mathbb{C})$.

Over the complex field, we can simplify the classification.

Theorem 2.11 *If $p \in \mathbb{C}[X, Y]$ has degree 2, then there is an affine tranformation taking the curve $p = 0$ to one of the following curves:*
i) (parabola) $Y = X^2$;
ii) (circle) $X^2 + Y^2 = 1$;
iii) (crossed-lines) $X^2 - Y^2 = 0$;
iv) (double line) $X^2 = 0$;
v) (parallel lines) $X^2 = 1$.

Proof We have gotten rid of four cases.

The change of variable $V = iY$ transforms the hyperbola $X^2 - Y^2 = 1$ to the circle $X^2 + V^2 = 1$.

The same change of variables transforms the $X^2 + Y^2 = 0$ to the double line $X^2 - V^2 = 0$ (Note: in $\mathbb{A}_2(\mathbb{C})$ $X^2 + Y^2 = 0$ is the two lines $(X + iY) = 0$ and $(X - iY) = 0$.)

Finally the transformations $U = iX$, $V = iY$ transform $X^2 + Y^2 = -1$ to the circle $U^2 + V^2 = 1$ and transform $X^2 = -1$ to $X^2 = 1$.

VIII- Intersecting Lines and Circles

For the moment we will restrict our attention to $\mathbb{A}_n(\mathbb{R})$.What happens when we intersect the circle C with the line L given by

equation $Y = aX + b$? If $(x, y) \in C \cap L$ then

$$\begin{aligned} y &= ax + b \\ x^2 + y^2 &= 1 \end{aligned}$$

Thus $x^2 + (ax + b)^2 = 1$ and $(a^2 + 1)x^2 + 2abx + b^2 = 1$. The polynomial

$$g(X) = (a^2 + 1)X^2 + 2abX + (b^2 - 1)$$

is a nonzero polynomial of degree at most 2 (Note: if $a^2 = -1$ then g has degree at most 1), and hence has at most two solutions. Thus $|C \cap L| \leq 2$. It is easy to see that all of these possibilities occur. Let L_a be the line $Y = a$. Then $C \cap L_0 = \{(\pm 1, 0)\}$, $C \cap L_1 = \{(0,1)\}$and $C \cap L_2 = \emptyset$. A separate but similar argument is needed to show that lines $x = a$ will also intersect C in at most two places. The same ideas work just as well for nondegenerate conics.

Theorem: If C is any parabola, circle or hyperbola in $\mathbb{A}_{\boldsymbol{n}}(\mathbb{R})$ and L is a line, then $|C \cap L| \leq 2$.

Note that the proposition is not true for degenerate conics in $\mathbb{A}_2(\mathbb{R})$. For example, the conic $X^2 = 1$ has infinite intersection with the line $X = 1$.

Let's work in $\mathbb{A}_2(\mathbb{C})$ instead of $\mathbb{A}_2(\mathbb{R})$. There are several cases to consider.

Case 1 $a = \pm i$ and $b = 0$.

In this case g is the constant polynomial -1 and there are no solutions.

Case 2 $a = \pm i$ and $b \neq 0$.

In this case g is a linear polynomial and there is a single solution.

Case 3 $a \neq \pm i$ and $b^2 - a^2 = 1$.

In this case

$$g = \frac{1}{a^2+1}((a^2+1)X + ab)^2$$

and the unique point of intersection is $(\frac{-ab}{a^2+1}, \frac{b}{a^2+1})$.

If (c, d) is a point on the circle, then a little calculus shows that the tangent line has slope $-\frac{c}{d}$ and equation

$$Y = -\frac{c}{d}X + \frac{c^2}{d} + d.$$

Thus the tangent line at $(\frac{-ab}{a^2+1}, \frac{b}{a^2+1})$ is $Y = aX + b$. In other words, case 3 arises when the line $Y = aX + b$ is tangent to the curve.

Case 4 $a \neq \pm i$ and $b^2 - a^2 \neq 1$.

In this case g has two distinct zeros in $\mathbb{C}$ and $|C \cap L| = 2$.

Case 4 is the general case. We understand why there is only one solution in case 3 as the line is tangent. Cases 1 and 2 are annoying.

IX- Parameterizing Circles

Consider the circle $C \subset \mathbb{A}_{\boldsymbol{n}}(\mathbb{R})$ given by the equation $X^2 + Y^2 = 1$. It is well known that we can parameterize the curve by taking $f(t) = (\cos t, \sin t)$. There are two problems with this parameterization. First, we are using transcendental functions. Second, the parameterization is not one-to-one. For each $(x, y) \in C$ if $f(\theta) = (x, y)$, then $f(\theta + 2\pi) = (x, y)$. We will show how to construct a better parameterization.

Note that the point $(0, 1) \in C$. Let L_λ be the line $\lambda X + Y = 1$. Then L_λ is the line throught the point $(0, 1)$ with slope $-\lambda$. Consider $L_\lambda \cap C$. If $(x, y) \in L_\lambda \cap C$, then

$$\begin{aligned} x^2 + (1 - \lambda x)^2 &= 1 \\ (\lambda^2 + 1)x^2 - 2\lambda x &= 0 \\ x((\lambda^2 + 1)x - 2\lambda x) &= 0 \end{aligned}$$

Thus there are two solutions $x = 0$ and $x = \frac{2\lambda}{\lambda^2+1}$. The solution $x = 0$ corresponds to the point we already know $(0, 1)$. So we have one additional point

$$x = \frac{2\lambda}{\lambda^2 + 1}, y = \frac{1 - \lambda^2}{\lambda^2 + 1}.$$

We consider the parameterization $\rho : \mathbb{R} \to C$ given by

$$\rho(\lambda) = \left(\frac{2\lambda}{\lambda^2 + 1}, \frac{1 - \lambda^2}{\lambda^2 + 1} \right).$$

Suppose $(x, y) \in C$ and $x \neq 0$. Let $\lambda = \frac{1-y}{x}$. Then

$$\begin{aligned} \rho(\lambda) &= \left(\frac{2\frac{1-y}{x}}{\frac{1-y}{x}^2 + 1}, \frac{1 - \frac{1-y}{x}^2}{\frac{1-y}{x}^2 + 1} \right) && (1) \\ &= \left(\frac{2(1-y)x}{1 - 2y + y^2 + x^2}, \frac{x^2 - 1 - 2y + y^2}{(1 - 2y + y^2) + x^2} \right) && (2) \\ &= \left(\frac{2(1-y)x}{2 - 2y}, \frac{2y - 2y^2}{2 - 2y} \right) && (3) \\ &= (x, y) && (4) \end{aligned}$$

Where we get (2) from (3) since $x^2 + y^2 = 1$ and get (4) since $y \neq 1$.

The following proposition is now clear.
Theorem: The parameterization $\rho: \mathbb{R} \longrightarrow C$ is one-to-one. The image is $C \setminus (0, -1)$.

It is annoying that the image misses one point on the circle. If we wanted to get the point $(0, -1)$ we would need to use the line $x = 0$ with infinite slope. This construction is very general and could be used for any nondegenerate conic once we know one point.

Chapter Two

Dr. Emad Bakr Al-Zangana
Projective Space

Projective Space

I-Introduction

Projective geometry (space) is the study of the properties of geometric figures that are not altered by projections.

One important problem in algebraic geometry is understanding $|L \cap C|$ where L is a line and C is a curve of degree d. The basic idea is that if L is given by $Y = aX + b$ and C is given by $f(X,Y) = 0$, we substitute and must solve the equation $f(X, aX + b) = 0$. Usually this is a polynomial of degree d and there are at most d solutions. We would like to say that there are exactly d solutions, but below examples showing that this is not true.

(1) In $\mathbb{A}_{\boldsymbol{n}}(\mathbb{R})$ the line $Y = X$ is a subset of the solution set of $X^2 - Y^2 = 0$. In this case $L \cap C$ is infinite.

(2) In $\mathbb{A}_{\boldsymbol{n}}(\mathbb{R})$ the line $Y = 1$ is a tangent to the circle of $X^2 + Y^2 = 1$. In this case $|L \cap C| = 1$.

(3) In $\mathbb{A}_{\boldsymbol{n}}(\mathbb{R})$ the line $Y = 2$ does not intersect the circle $X^2 + Y^2 = 1$, because there are no real solutions to the equation $X^2 + 3 = 0$.

(4) In $\mathbb{A}_{\boldsymbol{n}}(\mathbb{C})$ the point $(0,1)$ is the only point of intersection of the line $Y = iX + 1$ and the circle $X^2 + Y^2 = 1$, because $X^2 + (iX + 1)^2 = 1$ if and only if $X = 0$.

(5) Even if we only consider the intersection of two lines L_1 and L_2 we might have no intersection points if the lines are parallel.

We can avoid problem **(1)** by only looking at the cases where $L \nsubseteq C$. For example, we will later see that this holds if C is irreducible and $d > 1$. Problem **(2)** is unavoidable. We will eventually get around this by carefully assigning multiplicities to points of intersection. Tangent lines will intersect with multiplicity at least 2. This will allow us to prove results saying that "counted correctly" there are d points of intersection. In arbitrary fields F we will always run into problems like **(3)** where there are fewer than d points of intersection because there are polynomials with no zeros. We can avoid this by restricting our attention to algebraically closed fields. In this section we will try to avoid problems **(4)** and **(5)** by working in projective space rather than affine space.

(6) The parameterization we gave of the circle missed the point $(0, -1)$ because we needed to use the line $X = 0$ with infinite slope.

II- $\mathbb{P}_{\boldsymbol{n}}(\boldsymbol{F})$ (or $\boldsymbol{PG}(\boldsymbol{n}, \boldsymbol{F})$)

Let F be a field. We define an equivalence relation $\sim$ on $F^{n+1} \backslash \{(0, \dots, 0)\} = V(n+1, F) \backslash \{(0, \dots, 0)\}$ by $(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$ if and only if there is¸ $\lambda \in F$ such that

$$(x_1, \dots, x_{n+1}) = (\lambda y_1, \dots, \lambda y_{n+1}).$$

We let

$$[x_1, \dots, x_{n+1}] = \{(y_1, \dots, y_{n+1}) \in F^{n+1} \backslash \{\mathbf{0}\} | \, (x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1}) \}$$

denote the equivalence class of $(x_1, \dots, x_{n+1})$.

Projective Space

Definition: Projective $\boldsymbol{n}$-space over a field F is

$$\mathbb{P}_{\boldsymbol{n}}(F) = \{[x_1, \dots, x_{n+1}]: (x_1, \dots, x_{n+1}) \in F^{n+1} \setminus \{\mathbf{0}\}\}.$$

We say that $(x_1, \dots, x_{n+1})$ is a homogeneous coordinates for the $\sim -$equivalence class $[x_1, \dots, x_{n+1}]$.

Remark: (i) The projective space $PG(n, F)$ is the geometry whose points, lines, planes, . . ., hyperplanes are the subspaces of $V(n + 1, F)$ of rank $1, 2, 3, \dots, n$. The dimension of a subspace of $PG(n, F)$ is one less than the rank of a subspace of $V(n + 1, F)$.

(ii) For any point $p \in \mathbb{P}_{\boldsymbol{n}}(F)$**,** we have a number of choices for homogeneous coordinates. If $(x_1, \dots, x_{n+1})$ is one choice of homogeneous coordinates for p. Then $[x_1, \dots, x_{n+1}]$ is exactly the line with parametric equation

$$f(t) = \begin{pmatrix} tx_1 \\ \vdots \\ tx_{n+1} \end{pmatrix}.$$

Thus the $\sim -$equivalence classes are exactly the lines through $\mathbf{0} = (0, \dots, 0)$ in F^{n+1}. This gives alternative characterization of $\mathbb{P}_{\boldsymbol{n}}(F)$. So, the following theorem is hold.

Theorem: There is a bijection between $\mathbb{P}_{\boldsymbol{n}}(F)$ and the set of lines through $\mathbf{0}$ in F^{n+1}.

Let $U = \{p \in \mathbb{P}_{\boldsymbol{n}}(F): p \text{ has homogenous coordinates } (x_1, \dots, x_{n+1}) \text{ where } x_{n+1} \neq 0\}$**.** If $[x_1, \dots, x_{n+1}] \in U$, then

$$(x_1, \dots, x_{n+1}) \sim \left(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, 1\right)$$

and if $(y_1, \dots, y_{n+1})$ are also homogenous coordinate for p, then

$$y_1 = \frac{x_1}{x_{n+1}}, y_2 = \frac{x_2}{x_{n+1}}, \dots, y_n = \frac{x_n}{x_{n+1}}.$$

Therefore the following theorem is hold.

Theorem: The function $(x_1, \dots, x_n) \longrightarrow [x_1, \dots, x_n, 1]$ is bijection between $\mathbb{A}_{\boldsymbol{n}}(F)$ and U.

Remark:

(1) The homogeneous coordinates is a mapping from $V(n, F)$ to $V(n + 1, F)$.

(2) In this way we view $\mathbb{A}_{\boldsymbol{n}}(F)$as a subset of $\mathbb{P}_{\boldsymbol{n}}(F)$.

(3) Note that we had a great deal of freedom in this choice. If

$$U_i = \{p \in \mathbb{P}_{\boldsymbol{n}}(F) | p \text{ has homogeneous coordinates } [x_1, \dots, x_{n+1}] \text{ where } x_i \neq 0\}.$$

Then we could also identify $\mathbb{A}_{\boldsymbol{n}}(F)$ with U_i.

(4) We think of the points of $\mathbb{P}_{\boldsymbol{n}}(F) \setminus U$ as being "points at infinity" called **ideal pints**. The points in $\mathbb{P}_{\boldsymbol{n}}(F) \setminus U$ are those with homogeneous coordinates $[x_1, \dots, x_n, 0]$ where not all $x_i = 0$.

(5) $\mathbb{P}_{\boldsymbol{n}}(F) = \mathbb{A}_{\boldsymbol{n}}(F) \cup \{\text{ideal points which are of the form} [x_1, \dots, x_n, 0]\}$

(6) The set of homogeneous points of the set U is a kind of viewplane in F^{n+1} when $F = \mathbb{R}$.

(7) Note that $(x_1, \ldots, x_n, 0) \sim (y_1, \ldots, y_n, 0)$ if and only if $(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. This proves the following theorem.

Theorem: The function $[x_1, \ldots, x_n, 0] \longrightarrow [x_1, \ldots, x_n]$ is bijection between $\mathbb{P}_{\boldsymbol{n}}(F)\backslash U$ and $\mathbb{P}_{\boldsymbol{n-1}}(F)$.

For $n = 0$, if $x, y \in F\backslash\{0\}$, then $(y) = \frac{y}{x}(x)$. Thus $(y) \sim (x)$ and $\mathbb{P}_{\boldsymbol{0}}(F)$ is a single point.

We call $\mathbb{P}_{\boldsymbol{0}}(F)$ **the projective point over $\boldsymbol{F}$.**

For $n = 1$, we have $U = \{p \in \mathbb{P}_{\boldsymbol{1}}(F) : p$ has homogenous coordinates $(x, 1)\}$ that we identify with $\mathbb{A}_{\boldsymbol{1}}(F)$. There is a unique point $p \in \mathbb{P}_{\boldsymbol{1}}(F)\backslash U$ and p has homogeneous coordinates $(1,0)$.

We call $\mathbb{P}_{\boldsymbol{1}}(F)$ **the projective line over $\boldsymbol{F}$.**

For $n = 2$, we have $U = \{p \in \mathbb{P}_{\boldsymbol{2}}(F) : p$ has homogenous coordinates $(x, y, 1)\}$ that we identify with $\mathbb{A}_{\boldsymbol{2}}(F)$. The points in $\mathbb{P}_{\boldsymbol{2}}(F)\backslash U$ have homogeneous coordinates $(x, y, 0)$.

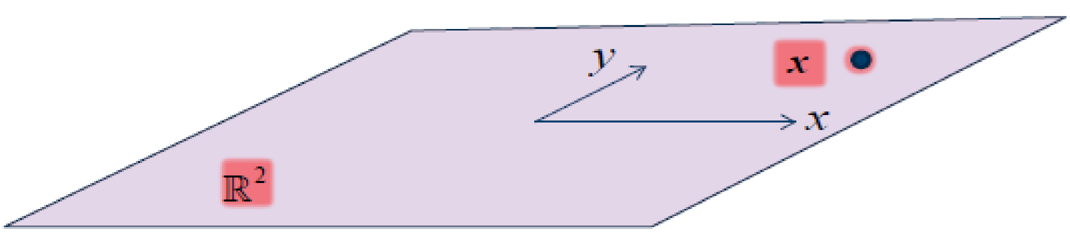
We call $\mathbb{P}_{\boldsymbol{2}}(F)$ **the projective plane over $\boldsymbol{F}$.**

III- Describe points in the real plane

Euclidean plane $\mathbb{R}^2$

(1) Choose a 2D coordinate frame.

(2) Each point corresponds to a unique pair of Cartesian coordinates $\boldsymbol{x} = (x, y) \in \mathbb{R}^{\boldsymbol{2}}$**.**



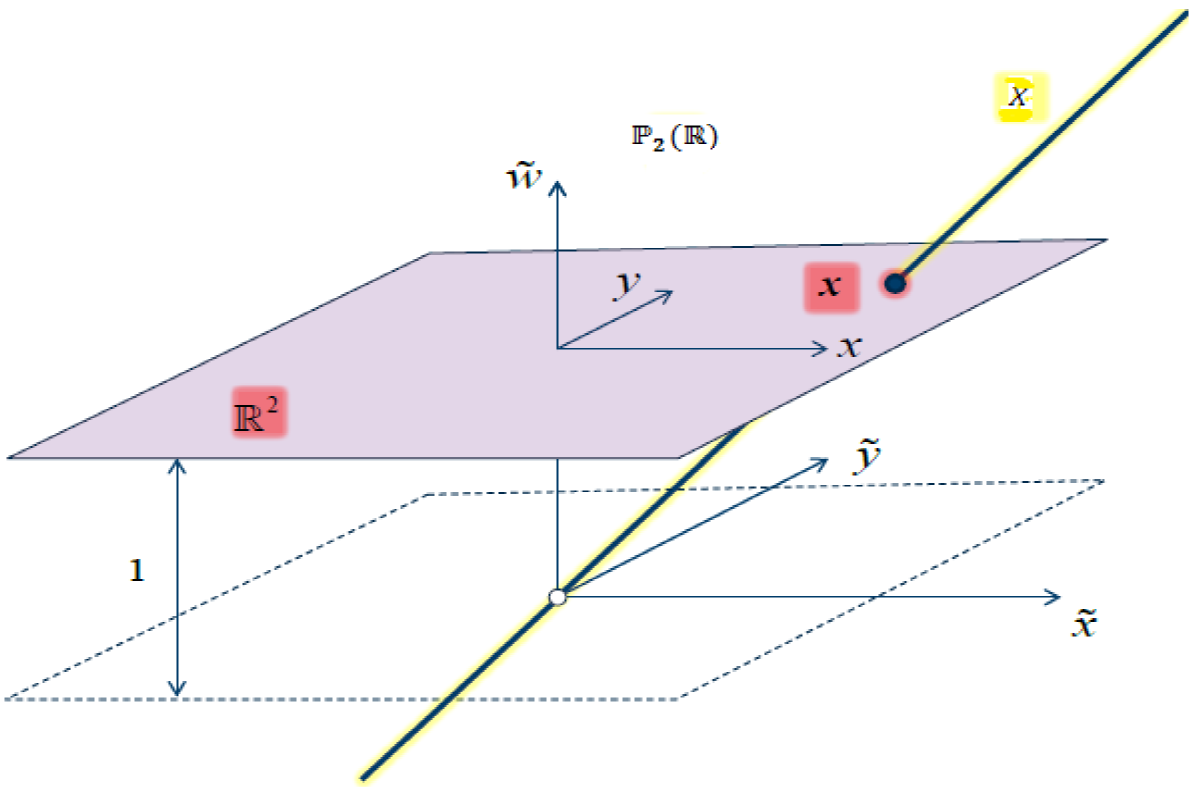
Euclidean projective plane $\mathbb{P}_{\boldsymbol{2}}(\mathbb{R})$

(1) Expand coordinate frame to 3D.

(2) Each point corresponds to a triple of homogenous coordinates $X = [\tilde{x}, \tilde{y}, \widetilde{w}]$.

Therefore, any point $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$ has a corresponding point X in $\mathbb{P}_2(\mathbb{R})$ with homogenous point $(\tilde{x}, \tilde{y}, 1)$ and point X in $\mathbb{P}_2(\mathbb{R})$ with homogenous point $(\tilde{x}, \tilde{y}, 0)$ has no counterpart in $\mathbb{R}^2$.

These points are corresponding to points at infinity.



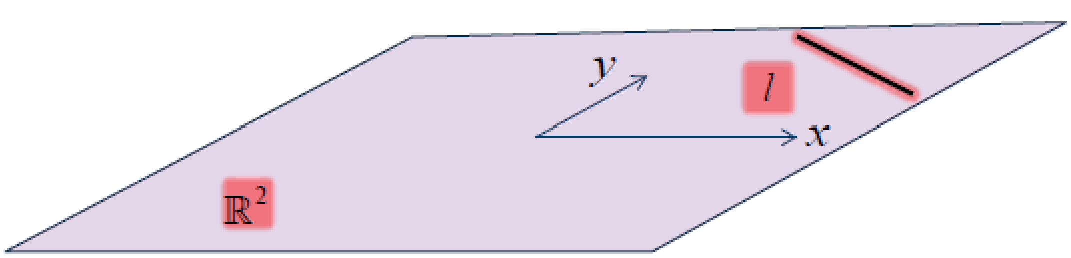
According to this, when we work in the projective plane, we can swap between Euclidean representation and the projective representation as follows:

$$\mathbb{R}^2 \ni \boldsymbol{x} = [x, y] \quad \longmapsto \quad X = [x, y, 1] \in \mathbb{P}_2(\mathbb{R})$$
$$\mathbb{P}_2(\mathbb{R}) \ni X = [\tilde{x}, \tilde{y}, \tilde{w}], \tilde{w} \neq 0 \quad \longmapsto \quad \boldsymbol{x} = [\tilde{x}/\tilde{w}, \tilde{y}/\tilde{w}] \in \mathbb{R}^2$$

IV- Describe lines in the real plane

Euclidean plane $\mathbb{R}^2$

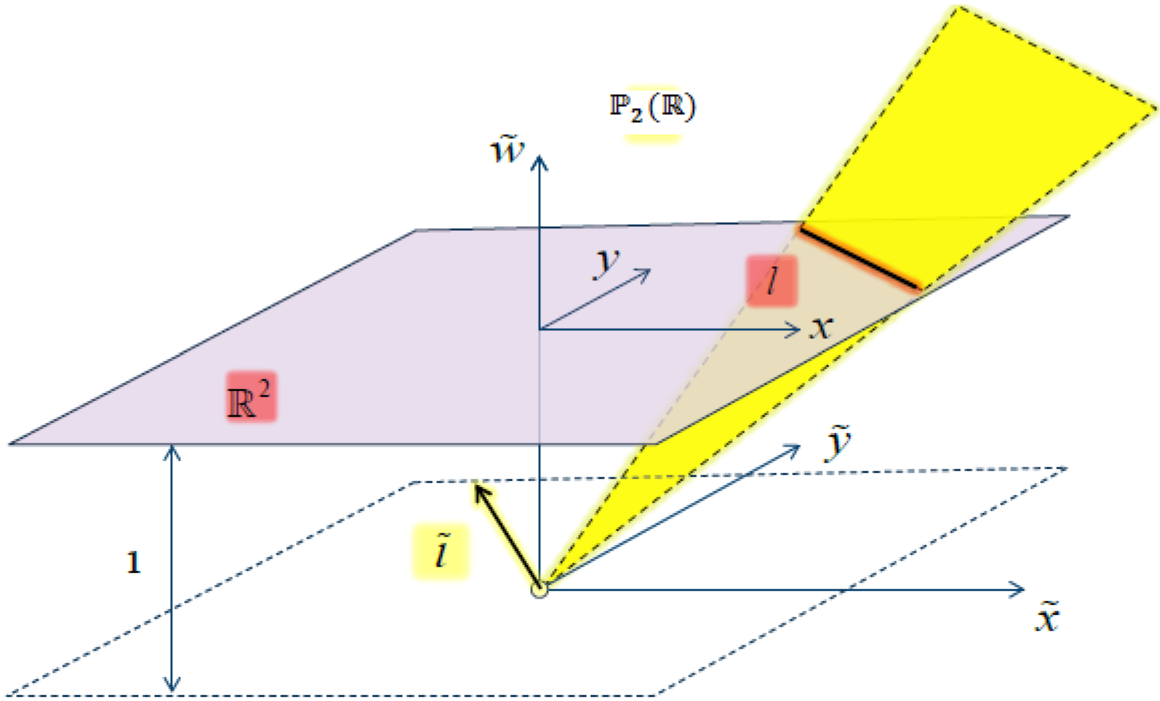
The triple$(a, b, c) \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$, determine a line $l = \{(x, y) \in \mathbb{R}^2 | ax + by + c = 0\}$ in $\mathbb{R}^2$.



Projective Space

Euclidean projective plane $\mathbb{P}_2(\mathbb{R})$

Homogenous vector $\tilde{l} = [a, b, c]$, determine a line $l = \{X = [\tilde{x}, \tilde{y}, \tilde{w}] \in \mathbb{P}_2(\mathbb{R}) | X \cdot \tilde{l}^T = 0\}$.

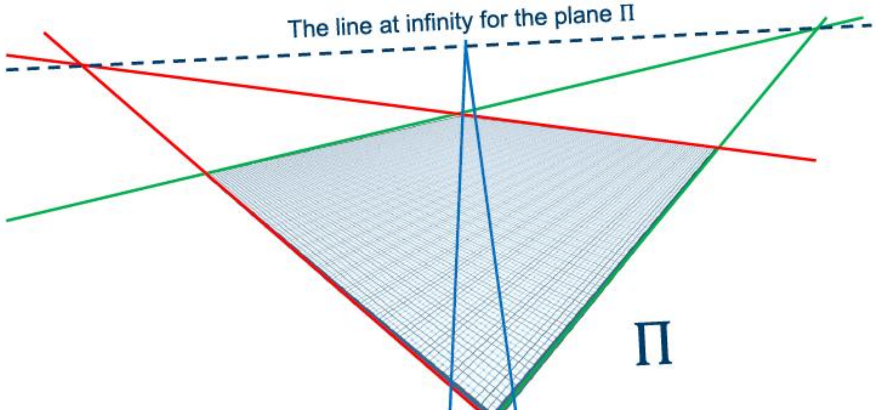


Remark:

(1) All lines in the Euclidean plane have a corresponding line in the projective plane.

(2) The line $\tilde{l} = [a, b, 0]$ in the projective plane does not has an Euclidean counterpart. This line consists entirely of ideal points and is known as a line at infinity called **ideal line**.

Projective Space



A point at infinity

V- Projective Algebraic Sets in $\mathbb{P}_2(F)$

How do we talk about solutions to polynomial equations in $\mathbb{P}_2(F)$? Some care is needed. For example, let $f(X,Y,Z) = X^2 + Y^2 + Z$. If $p \in \mathbb{P}_2(F)$ has homogeneous coordinates $(1,1-2)$ then $f(1,1-2) = 0$. But p also has homogeneous coordinates $(3,3-6)$ and $f(3,3-6) = 12$.

Definition: A field F is **algebraically closed** if every non constant polynomial has a zero in F.

Definition: A **monomial of degree d** in $F[X,Y,Z]$ is a polynomial $aX^iY^jZ^k$ where $a \in F$ and $i + j + k = d$.

We say that a polynomial $f \in F[X,Y,Z]$ is **homogeneous of degree d** if it is a sum of monomials of degree d.

Example: $f(X,Y,Z) = X^2 + Y^2 - Z^2$ is homogeneous of degree 2.

Theorem: If f is homogeneous of degree d, then

(1)

$$f(tx,ty,tz) = t^d f(x,y,z)$$

for all $t,x,y,z \in F$.

(2) If $f(X,Y,Z) = 0$, then $f(tx,ty,tz) = 0$ for all $t \in F$.

Proof: Exercise.

Definition: An **algebraic set** in $\mathbb{P}_2(F)$ is the set of

$$\{[x,y,z] \in \mathbb{P}_2(F) : f_1(x,y,z) = \cdots = f_m(x,y,z) = 0\}$$

where $f_1, \dots, f_m$ are homogeneous polynomials.

Projective Space

A **projective curve** in $\mathbb{P}_2(F)$ is

$$C = V_{\mathbb{P}}(f) = \{[x, y, z] \in \mathbb{P}_2(F) : f(x, y, z) = 0\}$$

where f is a nonzero homogeneous polynomial.

VI- Projective Line in $\mathbb{P}_2(F)$

A homogeneous polynomial of degree 1 in $F[X, Y, Z]$ is of the form $aX + bY + cZ$ where at least one of $a, b, c \neq 0$. The zero set of such a polynomial is a **projective line.**

Theorem: If L_1 and L_2 are distinct projective lines, then $|L_1 \cap L_2| = 1$.

Proof: Suppose L_i is the line $a_iX + b_iY + c_iZ = 0$, $i = 1,2$. Points $p \in L_1 \cap L_2$ have homogeneous coordinates (x, y, z) such that

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

a homogeneous system of 2 linear equations in 3 unknowns.

If $(a_1, b_1, c_1) = \lambda(a_2, b_2, c_2)$ for some $\lambda \in F\backslash\{0\}$, then L_1 and L_2 are the same line. Thus we may assume that (a_1, b_1, c_1) and (a_2, b_2, c_2) are linearly independent.

Thus the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

has rank 2. It follows that the homogeneous system has a one dimensional solution space. Let (x, y, z) be a nonzero solution. Then all solutions are of the form $(\lambda x, \lambda y, \lambda z)$ for some $\lambda \in F\backslash\{0\}$. In other words, $L_1 \cap L_2 = \{[x, y, z]\}$.

Remark: The above theorem means that, all lines in the projective plane are intersected.

Suppose L is the line in $\mathbb{P}_2(F)$ given by the equation $aX + bY + cZ = 0$. A point p with homogeneous coordinates $(x, y, 1)$ is on L if and only if $aX + bY + c = 0$. Thus when we identify affine space $\mathbb{A}_2(F)$ with

$$U = \{p \in \mathbb{P}_2(F): p \text{ has homogenous coordinates } (x, y, z) \text{ with } z \neq 0\}.$$

Then the points on $L \cap \mathbb{A}_2(F)$ are exactly the points on the affine line $aX + bY + c = 0$.

Start with an affine line $aX + bY + c = 0$ where at least one of $a, b \neq 0$ and let L be the projective line $aX + bY + cZ = 0$. If $ax + by = 0$, then $(x, y, 0)$ is also a point on L. It is easy to see that any such point is of the form $(\lambda b, -\lambda a, 0)$. Thus L contains a unique point in $\mathbb{P}_2(F)\backslash \mathbb{A}_2(F)$. We consider this **the point at infinity on L**.

Note that $[b, -a, 0]$ is also a point on the line $aX + bY + dZ = 0$ for any d. Thus we have shown that "parallel" affine line intersects at the point at infinity.

When $a = b = 0$, we just have the line $Z = 0$. As this line contains no points of U, we think of it as the line at infinity.

Theorem:

(1) If $p_1, p_1 \in \mathbb{P}_2(F)$ are distinct points, there is a unique line L with $p_1, p_1 \in L$.

(2) Let L_i be the line $a_i X + b_i Y + c_i Z = 0,\ i = 1,2$. Then

$$L_1 = L_2 \text{ if and only if } (a_1, b_1, c_1) \sim (a_2, b_2, c_2).$$

(3) Let $L_{a,b,c}$ be the line with equation $aX + bY + cZ = 0$. The map

$$[a, b, c] \longmapsto L_{a,b,c}$$

is a bijection between the points and lines of $\mathbb{P}_2(F)$.

Proof: Exercise.

Remark:

(1) Points (3) in the above theorem means that, points and lines in the projective plane have the same representations. That is, points and lines are dual objects in the projective plane.

(2) The line $\tilde{l}$ passing through points $p_1, p_1 \in \mathbb{P}_2(F)$ can be given by cross product of two vectors as follows: $\tilde{l} = p_1 \times p_1$. Therefore, using duality, two lines $\tilde{l}_1,\ \tilde{l}_2$ intersect at the point $X = \tilde{l}_1 \times \tilde{l}_2$.

(3) The cross product of two vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ can be represented by a 3×3 matrix,

$$u \times v \longmapsto [u]_\times v^T$$

where

$$[u]_\times \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}.$$

Example: Determine the line passing through the point $p_1 = (2,4)$ and $p_2 = (5,13)$ in $\mathbb{A}_2(\mathbb{R})$.

Solution: Homogeneous representation of the points.

$$\tilde{p}_1 = [2,4,1], \qquad \tilde{p}_2 = [5,13,1].$$

Homogeneous representation of the line.

$$\tilde{l} = \tilde{p}_1 \times \tilde{p}_2 = [\tilde{p}_1]_\times \tilde{p}_2^{\,T} = \begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & -2 \\ -4 & 2 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 13 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}.$$

The line equation is $-3X + Y + 2 = 0$.

VII- Projective Transformations of $\mathbb{P}_n(F)$

Recall that $T: F^n \longrightarrow F^n$ is a **linear transformation** if $T(ax + by) = aT(x) + bT(y)$ for all $a, b \in F$ and $x, y \in F^n$. Let $GL_n(F)$ be the set of invertible $n \times n$ matrices with entries from F. If T is a linear transformation of F^n, then there is an $n \times n$ matrix A such that

$T(x) = Ax^T$ for all $x \in F^n$. If T is invertible, then $A \in GL_n(F)$.

If $T: F^{n+1} \longrightarrow F^{n+1}$ is a linear transformation, then $T(0) = 0$ and $T(\lambda x) = \lambda T(x)$. Thus if $x \sim y$, then $T(x) \sim T(y)$. Moreover, if T is invertible and $T(y) = \lambda T(x)$, then $y = \lambda x$. Thus

$$x \sim y \text{ if and only if } T(x) \sim T(y)$$

for invertible linear T. In particular if $T: F^{n+1} \longrightarrow F^{n+1}$ is an invertible linear transformation, then there map

$$[x] \longmapsto [T(x)]$$

is a well-defined function from $\mathbb{P}_n(F)$ to $\mathbb{P}_n(F)$. We call such functions **projective transformations.**

VIII- Projective transformation in $\mathbb{P}_1(F)$

Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible, then

$$T([x, y]) = [ax + by, cx + dy]$$

is a projective transformation of $\mathbb{P}_1(F)$.

Put $[1,0] = \infty$, so $\mathbb{P}_1(F)$=$\mathbb{A}_1(F) \cup \{\infty\}$. Now to see how does T act on $\mathbb{P}_1(F)$.

$$T([x, 1]) = [ax + b, cx + d] = \left[\frac{ax + b}{cx + d}, 1\right]$$

if $cx + d \neq 0$. Thus we can view T as extending the function $x \longmapsto \frac{ax + b}{cx + d}$ on $F \backslash \{\frac{-d}{c}\}$.

Since A is invertible, $a\frac{-d}{c} + b \neq 0$. Thus

$$T\left(\left[\frac{-d}{c}, 1\right]\right) = \left[a\frac{-d}{c} + b, 0\right] = [1,0].$$

Also,

$$T([1,0]) = [a,c].$$

There are two cases to consider.

Case 1: $c = 0$

In this case

$$T(x) = \frac{ax+b}{d}$$

for $x \in F$ and $T(\infty) = \infty$.

Case 2: $c \neq 0$

In this case

$$T(x) = \frac{ax + b}{cx + d}$$

for $x \in F\backslash\left\{\frac{-d}{c}\right\}$, $T\left(\frac{-d}{c}\right) = \infty$ and

$$T(\infty) = \frac{a}{c}.$$

.

In case 1, T extends an affine transformation of $\mathbb{A}_{\mathbf{1}}(F)$. This is not true in case 2.

Theorem: If $L \subset \mathbb{P}_2(F)$ is a line, then $T(L)$ is a line.

Proof: Let L be the line $aX + bY + cZ = 0$. Let $A \in GL_3(F)$ be a matrix such that $T([x]) = [Ax^T]$. Then L is the set of $[x,y,z]$ where (x,y,z) is a solution to

$$(a \quad b \quad c)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Let

$$(\alpha \quad \beta \quad \gamma) = (a \quad b \quad c)A^{-1}.$$

Then

$$(\alpha \quad \beta \quad \gamma)A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Thus $T([x,y,z])$ is a point on the line L_1 with equation $\alpha X + \beta Y + \gamma Z = 0$.

On the other hand if $\alpha x + \beta y + \gamma z = 0$, then

$$(a \quad b \quad c)A^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\alpha \quad \beta \quad \gamma)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Thus, $T^{-1}([x,y,z])$ is on the line L. Thus $T(L) = L_1$ and $T^{-1}(L_1) = L$.

Theorem: If $L \subset \mathbb{P}_2(F)$ is projective line, there is a projective transformation taking L to the line $X = 0$.

Proof: Exercise.

IX-Homogenizing Affine Equations

In Section **II,** we identify $\mathbb{A}_2(F)$ with the subset $U = \{[x, y, 1] \in \mathbb{P}_2(F): x, y \in F\}$ of $\mathbb{P}_2(F)$.

If C is a projective curve given by $K(X, Y, Z) = 0$, then

$$C \cap \mathbb{A}_2(F) = \{(x, y): f(x, y) = 0\}$$

where $f(X, Y) = K(X, Y, 1)$. Thus $C \cap \mathbb{A}_2(F)$ is affine curve.

On the other hand, suppose $f(X, Y) \in F[X, Y]$ has degree d. For $i \leq d$, let f_i be the sum of all monomials in f of degree i. Then f_i is homogeneous of degree i and $f = \sum_{i=0}^{d} f_i$.

Let

$$K(X, Y, Z) = \sum_{i=0}^{d} f_i(X, Y) Z^{d-i}.$$

Then $f(X, Y) = K(X, Y, 1)$, K is homogeneous of degree d, and the affine curve $f(X, Y) = 0$ is the affine part of the projective curve $K(X, Y, Z) = 0$.

For example if $f(X, Y) = Y - X^2$, then $K(X, Y, Z) = YZ - X^2$.
This trick will allow us to use projective methods to study affine equations.

X- Solving equations in $\mathbb{P}_1(F)$

In this section, we will study the zero set of homogeneous polynomial in $\mathbb{P}_1(F)$.

Suppose $K(X, Y) \in F[X, Y]$ is a homogeneous polynomial of degree d.
Recall that we can write $\mathbb{P}_1(F)=\mathbb{A}_1(F) \cup \{\infty\}$, where $[1,0] = \infty$.

Let

$$K(X, Y) = \sum_{i=0}^{d} a_i X^i Y^{d-i} = a_0 Y^d + a_1 X Y^{d-1} + \cdots + a_d X^d$$

and let

$$f(X) = K(X, 1) = \sum_{i=0}^{d} a_i X^i.$$

The solutions to $K(X, Y) = 0$ in $\mathbb{A}_1(F)$ are points $p = [x, 1]$ where x is zero of f. We let $\boldsymbol{m_p}$ **be the multiplicity of $\boldsymbol{f}$ at $\boldsymbol{x}$.**

The point $\infty = [1,0]$ is a solution if and only if $a_d = 0$ and Y divides $K(X, Y)$. Let k be the maximal such that Y^k divides $K(X, Y)$. We call $\boldsymbol{k = m_\infty}$ **the multiplicity at ∞.**

Theorem: If $K(X,Y)$ is homogeneous of degree d, $p_1,\dots,p_k$ are the distinct zeros of K in $\mathbb{P}_1(F)$ and m_{p_i} is the multiplicity at p_i, then

$$\sum m_{p_i} \le d.$$

.

If F is algebraically closed, then

$$\sum m_{p_i} = d.$$

Proof:

We can write $K(X,Y) = Y^m G(\mathrm{X},\mathrm{Y})$, where G is homogeneous of degree $d-m$ and Y does not divide G. If $m > 0$, then ∞ is a zero of K of multiplicity m. The affine zeros of $K$are the zeros of $g(X) = G(\mathrm{X},1)$, a polynomial of degree $d-m$. We know that the sum of the multiplicities of zeros of g is at most $d-m$, with equality holding if F is algebraically closed.

XI- Intersecting Projective Lines and Curves

In this section we will show that projective lines intersect curves of degree d is exactly d points.

Theorem: Suppose $K(X,Y,Z)$ is a homogeneous polynomial of degree d and C is the projective curve $K=0$. Let $L \subset \mathbb{P}_2(F)$ be a projective line such that $L \nsubseteq C$. Then $|L \cap C| \le d$.

Proof:

Let L be the line $aX + bY + cZ = 0$. We will assume that $c \neq 0$ (the other cases are similar). If (x,y,z) are the homogeneous coordinates for a point of $L \cap C$, then $K\left(x,y,\frac{-ax-by}{c}\right) = 0$. Let $G(X,Y)$ be the polynomial obtained when we substitute $Z = \frac{-ax-by}{c}$ into $K(X,Y,Z)$. If $K(x,y,z) = 0$ and $ax + by + cz = 0$, then $G(x,y) = 0$. Moreover, if $G(x,y) = 0$, then $K\left(x,y,\frac{-ax-by}{c}\right) = 0$.

Claim Either $G(X,Y) = 0$ or $G(X,Y)$ is homogeneous of degree d.

Suppose

$$K(X,Y,Z) = \sum_{n=0}^{m} a_n X^{i_n} Y^{j_n} Z^{k_n}$$

where in $i_n + j_n + k_n = d$ for all n.

When we expand $(-aX - bY)^{k_n}$ we get a homogeneous polynomial of degree k_n. Thus each monomial occurring in $X^{i_n} Y^{j_n} (-aX - bY)^{k_n}$ has degree d.

When we add up all of these terms, either they all cancel out and we get $K=0$ or they don't and K is homogeneous of degree d.

(i) Suppose $G(X,Y) = 0$. If $ax + by + cz = 0$, $K(x,y,z) = 0$. Thus the line L is contained in C.

(ii) Suppose $G(X,Y)$ is homogeneous of degree d. If $[x,y,z] \in L \cap C$, then $[x,y]$ is a zero of G in $\mathbb{P}_1(F)$. On the other hand, if $[x,y]$ is a zero of G in $\mathbb{P}_1(F)$, then $\left[x,y,\frac{-ax-by}{c}\right]$ is a point on $L \cap C$. The map

$$[x,y] \mapsto \left[x,y,\frac{-ax-by}{c}\right]$$

is well-defined and one-to-one. Thus there is a bijection between points on $L \cap C$ and zeros of G. We know that there are at most d distinct zeros to G in $\mathbb{P}_1(F)$. Thus $|L \cap C| \leq d$.

Example:

(i) Let C be the curve $X^2 + Y^2 - Z^2 = 0$ in $\mathbb{P}_2(\mathbb{C})$. Then $C \cap \mathbb{A}_2(\mathbb{C})$ is the affine circle $X^2 + Y^2 = 1$. Let L_1 be the line $iX - Y + Z = 0$. This is the projective version of the line $Y = iX + 1$. $L_1 \cap C \cap \mathbb{A}_2(\mathbb{C}) = \{(0,1)\}$.

Let

$$G(X, Y) = X^2 + Y^2 - (Y - iX^2) = 2X^2 + 2iXY = 2X(X + iY).$$

Then G has two zeros in $\mathbb{P}_1(\mathbb{C})$, $[0,1]$ and $[-i, 1]$, each with multiplicity 1. Thus $L_1 \cap C$ has two points of intersection $[0,1,1]$ and $[-i, 1,0]$. Each point of intersection has multiplicity one. Note that $[0,1,1]$ is the one point in $\mathbb{A}_2(\mathbb{C})$, while $[-i, 1,0]$ is a point at infinity.

(ii) Let C as in **(i)** and L_1 given by $iX - Y = 0$. This corresponds to intersecting the affine circle $X^2 + Y^2 = 1$ with the line $Y = iX$.

Since the Z-coordinate is zero, we consider

$$G_1(X, Z) = K(X, iX, Z) = -Z^2.$$

The equation $G_1(X, Z) = 0$ has a unique zero $[1,0]$ in $\mathbb{P}_1(\mathbb{C})$ of multiplicity 2. Thus $C \cap L_2$ has a unique point $[1, i, 0]$ of multiplicity 2.

Therefore, there are no points of intersection in $\mathbb{A}_2(\mathbb{C})$.

XII- Conic in $\mathbb{P}_2(F)$

Suppose $K(X, Y, Z)$ has degree 2. Then

$$K(X, Y, Z) = aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ.$$

Let C be the curve $V_{\mathbb{P}}(K)$.

We can think of $K(X, Y, Z) = 0$ as a metric equation as follows:

$$K(X, Y, Z) = (X \quad Y \quad Z)\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

Notice the matrix $\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$ is symmetric matrix and from linear algebra there is an invertible matrix A such that $A^T\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}A$ is diagonal matrix.

Theorem: Suppose F is a field of characteristic different from 2. There is a projective transformation that transforms a conic C to one of the following curves:

(1) $X^2 + bY^2 + cZ^2 = 0$ where $a, b, c \neq 0$.

(2) $X^2 + bY^2 = 0$ where $b \neq 0$.

(3) $X^2 = 0$.

Proof: Exercise.

Definition: The conic of type **(i)** in the above theorem called **nondegenerate conic**.

Projective Space

The following theorem give a very simple classification of conics in $\mathbb{P}_2(F)$ for algebraically closed F.

Theorem: Suppose F is a algebraic closed field of characteristic different from 2. There is a projective transformation that transforms a conic C to one of the following curves:
(1) $X^2 + Y^2 - Z^2 = 0$, Circle.
(2) $X^2 - Y^2 = 0$, Crossing lines.
(3) $X^2 = 0$, Double lines.
Proof:
(1) If we can transform C to $X^2 + bY^2 + cZ^2 = 0$ where $a, b, c \neq 0$, we let $V = \sqrt{b}Y, W = i\sqrt{c}Z$ to transform the curve to $X^2 + V^2 - W^2 = 0$.

Using this classification we can better understand what happens when a conic C intersects a line L. If C is degenerate, we can of course have $C \cap L$. This is impossible if C is nondegenerate.

Corollary: Let F be algebraically closed if $C \subset \mathbb{P}_2(F)$ is a nondegenerate conic and $L \subset \mathbb{P}_2(F)$ is a line, then $L \nsubseteq C = \phi$.
Proof:
Let C is given by the equation $X^2 + Y^2 + Z^2 = 0$. Let L be the line $aX + bY + cZ = 0$. Without loss of generality suppose $c \neq 0$. If (x, y, z) are the homogeneous coordinates for a point on $L \cap C$, then

$$z = \frac{-ax - by}{c}$$

$$x^2 + y^2 + \left(\frac{-ax - by}{c}\right)^2 = 0$$

and (x, y) is a zero of the equation

$$g(X, Y, Z) = (a^2 + c^2)X^2 + (b^2 + c^2)Y^2 + 2abXY = 0.$$

This polynomial is not identically zero, since in order to have $2ab = 0$ we must have $a = 0$ or $b = 0$, but $c \neq 0$, so we will either get $a^2 + c^2 \neq 0$ or $b^2 + c^2 \neq 0$. Thus the polynomial has at most two solutions in $\mathbb{P}_2(F)$, that is, $|L \cap C| \leq 2$.

XIV-Projective Parameterizations

We assume that F is a field of characteristic different from 2. In chapter one, we showed that

$$\rho(t) = \left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2}\right)$$

is a rational parameterization of the circle in $\mathbb{A}_{\mathbf{2}}(F)$. There were a couple of problems with this parameterization:
(i) it misses the point $(0, -1)$;
(ii) it is undefined when $t^2 = -1$.
Both of these problems disappear in $\mathbb{P}_2(F)$.
Let C be the circle in $\mathbb{P}_2(F)$ with equation $X^2 + Y^2 - Z^2 = 0$. Let think of t as being $\frac{u}{v}$. Then

$$\rho(t) = \left(\frac{2uv}{u^2 + v^2}, \frac{v^2 - u^2}{u^2 + v^2}\right).$$

This gives us an idea of how to define a projective transformation. We define $f: \mathbb{P}_1(F) \longrightarrow \mathbb{P}_2(F)$ by

$$[x,y] \longmapsto [2xy, y^2 - x^2, x^2 + y^2].$$

(1) To show that f is well-defined. First we see that f preserves $\sim -$ equivalence classes since

$$[\lambda x, \lambda y] \longmapsto [2\lambda^2 xy, \lambda^2(y^2 - x^2), \lambda^2(x^2 + y^2)] = [2xy, y^2 - x^2, x^2 + y^2].$$

Moreover, if $(2xy, y^2 - x^2, x^2 + y^2) = (0,0,0)$, then $x = 0, y = 0$. Thus f is a well-defined function. Since

$$(2xy)^2 + (y^2 - x^2)^2 - (x^2 + y^2)^2 = 0$$

then the image of f is the circle C.

(2) To show that f is one to one, it is enough to fine an inverse to f. Let define $g: C \longrightarrow \mathbb{P}_1(F)$ by

$$g([a,b,c]) = \begin{cases} [c-b, a] & if\ a \neq 0 \text{ and } b \neq c \\ [1,0] & if\ a = 0 \text{ and } b = c \neq 0 \end{cases}.$$

It is not difficult to prove that g is well defined function and $gof = fog = I$.
Thus f is bijection between C and $\mathbb{P}_1(F)$.

Remark: From above, the two given problems at the beginning of this section haven been fixed.

XV-Intersecting Conics with Curves

Theorem: Suppose F is an algebraically closed field of characteristic different from 2, $C \subseteq \mathbb{P}_2(F)$ is a nondegenerate conic and $D \subseteq \mathbb{P}_2(F)$ is a curve of degree d. Either $C \subseteq D$or $|C \cap D| \leq 2d$

Proof:
We have a projective parameterization of C. From Section XIV, there are homogeneous polynomials a, b, c of degree 2 such that

$$[x,y] \longmapsto [a(x,y), b(x,y), c(x,y)]$$

is a parameterization of C.
Suppose D is the set of solutions to the degree d polynomial $K(X,Y,Z) = 0$. Then points of $C \cap D$ are in bijective correspondence with points $[x,y] \in \mathbb{P}_1(F)$ such that

$$K\big(a(x,y), b(x,y), c(x,y)\big) = 0.$$

Let $g(X,Y) = K\big(a(X,Y), b(X,Y), c(X,Y)\big)$. Since K is homogeneous of degree d and a, b, c are homogeneous of degree 2, g is either 0 or of degree $2d$.
If $g = 0$, then $[a(x,y), b(x,y), c(x,y)] \in D$ for all $[x,y] \in \mathbb{P}_1(F)$ and $C \subseteq D$. If g has degree $2d$, then there are at most $2d$ points $p \in \mathbb{P}_1(F)$ such that $g(p) = 0$ and $[a(x,y), b(x,y), c(x,y)] \in D$. Thus $|C \cap D| \leq 2d$.

While two points determine a line, 5 points (in general position) determine a conic.

Theorem: If $p_1, p_2, p_3, p_4, p_5 \in \mathbb{P}_2(F)$ there is a conic C with $p_1, p_2, p_3, p_4, p_5 \in D$.
If no four of p_1, p_2, p_3, p_4, p_5 are colinear, then C is unique. If no three are colinear C is nondegenerate.

Proof: We first show that there is at least one conic. Let $(x_{i,1}, \dots, x_{i,5})$ be homogeneous coordinates for p_i. We are looking for $a_1, \dots, a_6$ such that

$$\begin{pmatrix} {x^2}_{1,1} & {x^2}_{1,2} & {x^2}_{1,3} & x_{1,1}x_{1,2} & x_{1,1}x_{1,3} & x_{1,2}x_{1,3} \\ & & & \vdots & & \\ {x^2}_{5,1} & {x^2}_{5,2} & {x^2}_{5,3} & x_{5,1}x_{5,2} & x_{5,1}x_{5,3} & x_{5,2}x_{5,3} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since this is a homogeneous system of 5 linear equations in 6 unknowns it has a nonzero solution $(a_1, \ldots, a_6)$. Let $K(X, Y, Z) = a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5XZ + a_6YZ$. Then $K(p_i) = 0$ for $i = 1, \ldots, 6$.

Suppose no four of p_1, p_2, p_3, p_4, p_5 are collinear. We next must show uniqueness.

Suppose $F \subseteq \bar{F}$ is algebraically closed. It is enough to show that there is a unique conic in F containing of p_1, p_2, p_3, p_4, p_5. Suppose C_1 and C_2 are conics such that $p_1, p_2, p_3, p_4, p_5 \in C_1 \cap C_2$.

Case 1: C_1 and C_2 are nondegenerate.

Then by the previous theorem $C_1 \subseteq C_2$ and $C_1 \subseteq C_2$.

Case 2: One C_i is degenerate and the other is not.

Suppose C_1 in nondegenerate and C_2 is degenerate. We know that C_1 does not contain a line and that C_2 is either two crossing lines or a single line. Since p_1, p_2, p_3, p_4, p_5 are non collinear, C_2 must be two crossing lines.

Case 3: C_1 and C_2 are degenerate.

Since the points p_1, p_2, p_3, p_4, p_5 are not collinear, $C_1 = L_0 \cup L_1$ and $C_1 = L_2 \cup L_3$ where each L_i is a line $L_0 \neq L_1$ and $L_2 \neq L_3$.

Since $\{p_1, p_2, p_3, p_4, p_5\} \subseteq L_0 \cup L_1$. At least one of the lines must contain three of the points.

Suppose $p_1, p_2, p_3 \in L_0$. Since $p_1, p_2, p_3 \in L_2 \cup L_3$ at least one of the lines must contain at least two of those points. Say $p_1, p_2 \in L_3$. But then $p_1, p_2 \in L_0 \cap L_3$. Since two points determine a line we must have $L_0 = L_1$. Thus we may assume that $C_1 = L_0 \cup L_1$ and $C_1 = L_0 \cup L_3$ and $p_1, p_2, p_3 \in L_0$.

Since no four points are collinear we must have $p_4, p_5 \in L_1$ and $p_4, p_5 \in L_2$. But then $L_1 = L_2$ and $C_1 = C_2$. Thus there is a unique conic through p_1, p_2, p_3, p_4, p_5.

If C is degenerate then C is either a line or the union of two lines. In either case at least three of the points are on a line. Thus if no three points are collinear C is nondegenerate.

Appendix

Dr. Emad Bakr Al-Zangana
Appendix

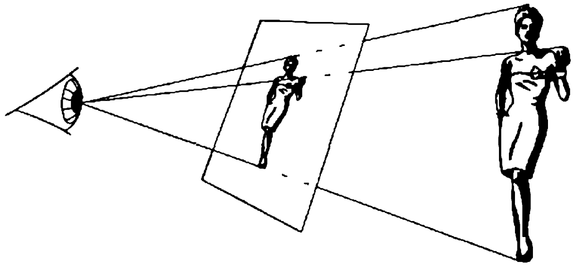


FIGURE 4.1. Projecting a scene into a plane.

1. **Central projection.** Given a point P and a plane H with $P \notin H$, define a projection function f by the formula

$$f(Q) = \overleftrightarrow{PQ} \cap H$$

for every point Q such that $\overleftrightarrow{PQ}$ is not parallel to H. f is called *projection from* P *into* H; P is the *center* of the projection f.

2. **Parallel projection.** Let $\vec{v}$ be a nonzero vector and H a plane that is not parallel to $\vec{v}$. For each point Q let L_Q be the line through Q that is parallel to $\vec{v}$. Define a projection function g by the formula

$$g(Q) = L_Q \cap H.$$

g is called *parallel projection into* H *along the direction* $\vec{v}$.

Parallel projection acts like a central projection whose center is infinitely far away.

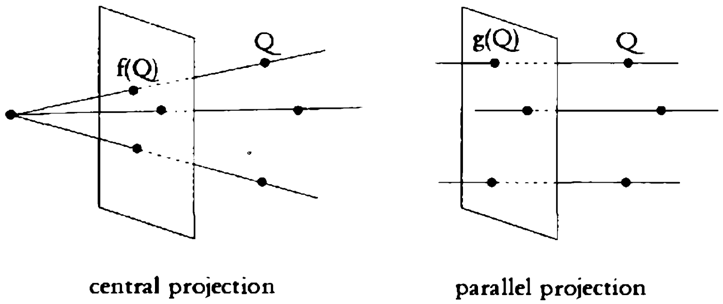


FIGURE 4.2.

Some examples of properties that are preserved by projections are: the property of being a point, the property of being a line, and the property of being a conic section. (Here and in the rest of this chapter we consider only objects that do not contain the artist's eye).

Properties that are not preserved include length, the size of angles, area, and the property of being a circle.

Projectivization. From now on we will imagine that the artist has only one eye, and that it is located at the origin, O. A *radial* line or plane is one that passes through O. The *projectivization* of a scene is the set of all radial lines that pass through points in the scene, together with all radial lines that are infinitesimally close to lines passing through points in the scene (nobody's eye is sharp enough to distinguish between lines that are infinitesimally close to each other).

As it views a scene your eye does not respond directly to the objects in the scene. Instead it responds to a projectivization of the scene, namely the projectivization that consists of all the light rays that travel along lines from points in the scene to your eye. This fact has important consequences:

1. **Radial lines look like points, and radial planes look like lines** because they are being viewed "edge on" by the eye at the origin.

2. **Radial dimensions are lost** because radial lines look like points.

3. **Non-radial lines acquire an extra "point at infinity".** The projectivization of a non-radial line L is the set of radial lines in the plane $\overline{OL}$ that connects L with the eye. Only one radial line in the plane $\overline{OL}$ does not connect a point on L to the eye. That one exception is the radial line that is parallel to L. We shall call this exceptional line P_∞, the "point at infinity" on L. To the eye P_∞ appears to be a point at infinity on L, because it is the limit of lines $\overleftrightarrow{OP}$ connecting the eye to points $P \in L$ as P approaches infinity (Fig. 4.3),

$$P_\infty = \lim_{P \to \infty} \overleftrightarrow{OP}.$$

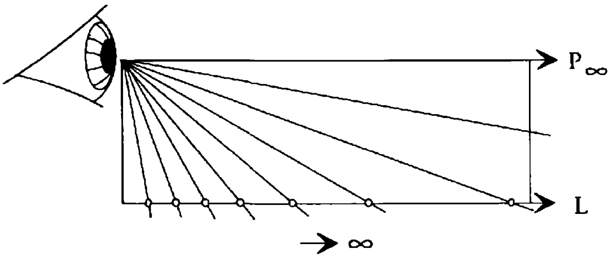


FIGURE 4.3. A point at infinity.

4. **Non-radial planes acquire an extra "line at infinity".** Let H be a non-radial plane. As $P \in H$ goes to infinity the line OP tends towards the radial plane that is parallel to H. We call this plane L_∞, the *line at infinity* of H.

$$L_\infty = \left\{ \lim_{P \to \infty} \overleftrightarrow{OP} \,\middle|\, P \in H \right\}.$$

To the eye, points in L_∞ look like they lie "at infinity" on the horizon of H (Fig. 4.4).

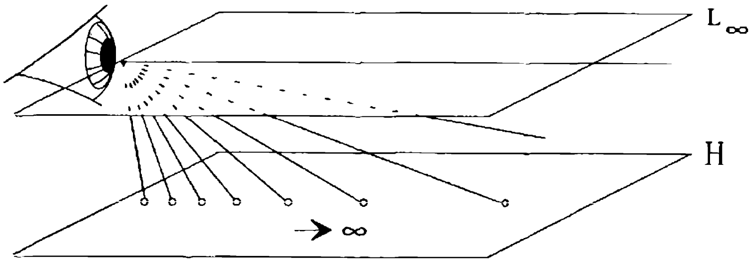
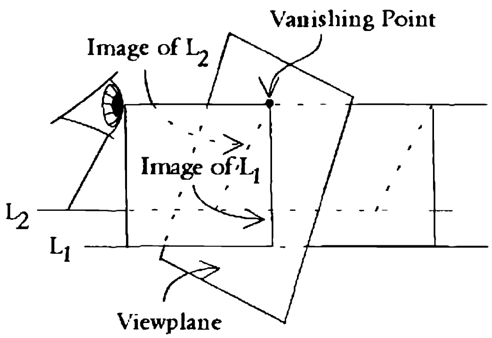


FIGURE 4.4. L_∞ is a "line at infinity".

As a general rule any figure that extends off to infinity will acquire extra "points at infinity" when it is projectivized.

Vanishing Points.

A perspective drawing is created by intersecting the projectivization of a scene with a plane. The plane, which we shall call the *viewplane*, is the artist's canvas. The *image* of the scene is the intersection of the projectivization of the scene with the viewplane. A *vanishing point* is an image of a point at infinity.



Parallel Lines have the same Vanishing Point.

FIGURE 4.5.

Parallel lines all are parallel to the same radial line, so they have the same point at infinity. Therefore the images of parallel lines all pass through the same vanishing point (Fig. 4.5). The only exception to this rule occurs when the lines are all parallel to the viewplane. In that case their point at infinity does not intersect the viewplane, so the lines have no vanishing points and their images are parallel. (Of course their common point at infinity is still visible to the eye, but it does not appear in the picture).

A plane's *horizon* is the image of its line at infinity. If the plane contains some parallel lines then their common vanishing point is on the plane's horizon.

at a point P such that the lines $\overleftrightarrow{PV_1}$, $\overleftrightarrow{PV_2}$, and $\overleftrightarrow{PV_3}$ are all perpendicular to each other. Let S_1 be the sphere with diameter $\overline{V_2V_3}$, S_2 the sphere with diameter $\overline{V_1V_3}$, and S_3 the sphere with diameter $\overline{V_1V_2}$. Show that $P \in S_1 \cap S_2 \cap S_3$. There are two points in this intersection, one on each side of the viewplane.

Where should you put your eye if the box is drawn in two point perspective? In one point perspective?

4.2 Projective Space

Definition 4.2.1 A *projective point* (notation: $\mathbf{P}^0$) is a radial line.

A *projective line* (notation: $\mathbf{P}^1$) is the set of radial lines in a radial plane.

A *projective plane* (notation: $\mathbf{P}^2$) is the set of radial lines in a radial three dimensional space.

Generalizing the above, there are two equivalent definitions for projective space.

Definition 4.2.2 An *n dimensional projective space* $\mathbf{P}^n$ is the set of radial lines in $\mathbf{R}^{n+1}$.

Definition 4.2.3 *n dimensional projective space* $\mathbf{P}^n$ is obtained by starting with $\mathbf{R}^n$ and completing it by adding on its "points at infinity".

To see why the two definitions are equivalent, recall that

$$\mathbf{R}^{n+1} = \{(x_0, x_1, \cdots, x_n) \mid x_i \in \mathbf{R},\ i = 0, \cdots, n\}.$$

Regard $\mathbf{R}^n$ as the set of points in $\mathbf{R}^{n+1}$ with x_0 coordinate equal to one:

$$\mathbf{R}^n = \{(1, x_1, \cdots, x_n) \mid x_i \in \mathbf{R},\ i = 1, \cdots, n\},$$

a kind of "viewplane" in $\mathbf{R}^{n+1}$.

Set $x = (x_1, \cdots, x_n)$. Every point $(1, x) = (1, x_1, \cdots, x_n)$ in $\mathbf{R}^n$ lies on exactly one radial line, namely, the line $L_{(1,x)}$ consisting of all scalar multiples of the vector $(1, x)$:

$$L_{(1,x)} = \{(t, tx_1, \cdots, tx_n) \mid t \in \mathbf{R}\}.$$

Radial lines

$$L_{(0,x)} = \{(0, tx_1, \cdots, tx_n) | t \in \mathbf{R}\},$$

whose points have x_0 components equal to zero, are parallel to $\mathbf{R}^n$; they represent "points at infinity" on $\mathbf{R}^n$. Thus every radial line in $\mathbf{R}^{n+1}$ can be matched up either with a point $(1, x_1, \cdots, x_n)$ in $\mathbf{R}^n$ or with a point at infinity on $\mathbf{R}^n$. This produces a one to one correspondence

$$\{\text{radial lines in } \mathbf{R}^{n+1}\} \overset{1:1}{\leftrightarrow} \mathbf{R}^n \cup \{\text{points at infinity}\},$$

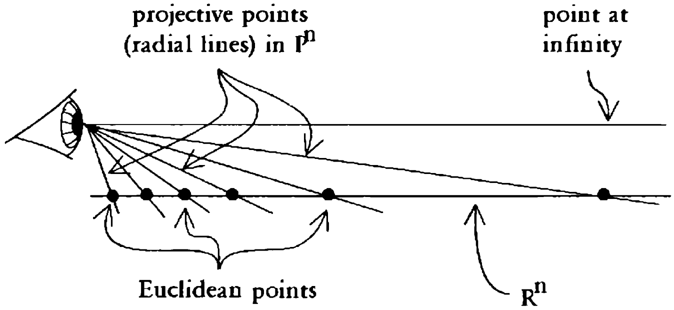


FIGURE 4.7. $\mathbf{P}^n = \mathbf{R}^n \cup \{\text{points at infinity}\}$

which shows that Definitions 4.2.2 and 4.2.3 are equivalent (see Fig. 4.7).

The fact that projective spaces contain "points at infinity" is an important difference between projective spaces and Euclidean spaces. Nevertheless the two kinds of space look the same to the eye, so they usually are depicted in the same way in diagrams.

A theorem about projective space can be interpreted as a theorem about a Euclidean space of the same dimension, provided that lines meeting at a point at infinity are interpreted as parallel lines, planes meeting along a line at infinity are interpreted as parallel planes, and so on.

Let us emphasize that *in projective space all points look exactly the same, all lines look exactly the same, and all planes look exactly the same.* In other words there is nothing special about points, lines, or planes at infinity. This means that *any* point, line, or plane in $\mathbf{P}^n$ can be regarded as a point, line, or plane at infinity, provided that this is done in a consistent way: the line at infinity in $\mathbf{P}^2$ must contain all the points at infinity, the plane at infinity in $\mathbf{P}^3$ must contain all the lines at infinity, and so on.

From now on we will drop the word "projective" wherever possible, and call projective points, projective lines and projective planes simply "points", "lines" and "planes".

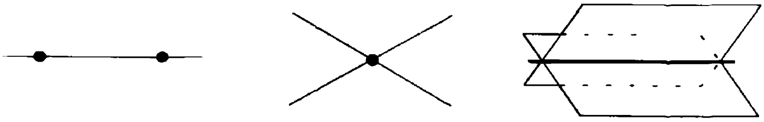
Proposition 4.2.1 *(See Fig. 4.8).*

1. *Every pair of points in* $\mathbf{P}^n$ *lies on exactly one line.*
2. *Every pair of lines in* $\mathbf{P}^2$ *intersects in exactly one point.*
3. *Every pair of planes in* $\mathbf{P}^3$ *intersects in exactly one line.*

Proof of 1. According to Definition 4.2.2, part 1) of the proposition says that a pair of radial lines in $\mathbf{R}^{n+1}$ lies in exactly one radial plane. This is clear because any pair of intersecting lines lies in a unique plane.

Proof of 2. Again using Definition 4.2.2, part 2) says that two radial planes in $\mathbf{R}^3$ must intersect in exactly one radial line. This is true because two radial planes cannot be parallel as both contain the origin.

Proof of 3 (sketch). A rigorous proof of part 3) requires one to think about intersecting three-dimensional radial subspaces of $\mathbf{R}^4$. Here is a more



two points determine a line

two coplanar lines determine a point

two planes in P^3 determine a line

FIGURE 4.8.

intuitive argument. Let H and H' be planes in $\mathbf{P}^3$. Let H_∞ be a third plane in $\mathbf{P}^3$, different from the other two. Regard H_∞ as a plane at infinity, and $\mathbf{P}^3 - H_\infty$ as a copy of $\mathbf{R}^3$. The set $H - H_\infty$ of all "finite points" on H is a plane in $\mathbf{R}^3$. Likewise the set $H' - H'_\infty$ of all finite points on H' is a plane in $\mathbf{R}^3$. These two Euclidean planes either intersect in a Euclidean line or else they are parallel and have the same line at infinity. In either case it follows that H and H' intersect in a line when points at infinity are included.

Similar arguments establish the next proposition.

Proposition 4.2.2 *Any three noncollinear points in* $\mathbf{P}^3$ *lie in exactly one plane. Three planes in* $\mathbf{P}^3$ *must either intersect in exactly one point or else they contain a common line. A line and a plane in* $\mathbf{P}^3$ *intersect in exactly one point unless the line lies in the plane. If two lines in* $\mathbf{P}^3$ *do not lie in a common plane then they are skew (and do not intersect).*

4.3 Desargues' Theorem

A set of lines is *coincident* if all the lines intersect at the same point. A *triangle* $\triangle ABC$ is the union of three intersecting but noncoincident lines

$$\triangle ABC = \overleftrightarrow{AB} \cup \overleftrightarrow{AC} \cup \overleftrightarrow{BC}.$$

Triangles $\triangle ABC$ and $\triangle A'B'C'$ are *in perspective* if the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ that join corresponding vertices on the two triangles, are coincident (Fig. 4.9).

Theorem 4.3.1 (Girard Desargues, (1591-1661)).
If the triangles $\triangle ABC$ *and* $\triangle A'B'C'$ *are in perspective then the points*

$$P = \overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}, \quad Q = \overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}, \quad R = \overleftrightarrow{BC} \cap \overleftrightarrow{B'C'},$$

where their corresponding sides intersect, are collinear (Fig. 4.9).

Proof. Let

$$X = \overleftrightarrow{AA'} \cap \overleftrightarrow{BB'} \cap \overleftrightarrow{CC'}.$$

4.4 Cross Ratios

It is impossible to calculate the exact distances between objects in a scene from data obtained by measuring a perspective drawing of the scene because the drawing does not depict radial distances. However, using something called the *cross ratio*, one can find the *relative* distances between three or more collinear points in the scene, provided that the points are not on the same radial line and one knows the location of the vanishing point of the line that contains them.

Notation: the *ratio of two parallel vectors* in Euclidean space is

$$\frac{\overrightarrow{v}}{\overrightarrow{w}} = t \text{ if } \overrightarrow{v} = t\overrightarrow{w} \text{ and } \overrightarrow{w} \neq 0.$$

Definition 4.4.1 The Cross Ratio of Four Points on a Euclidean Line.

The *cross ratio* $[A, B, C, D]$ of four distinct points A, B, C, D on a Euclidean line is

$$[A, B, C, D] = \frac{\overrightarrow{AC}}{\overrightarrow{AD}} \frac{\overrightarrow{BD}}{\overrightarrow{BC}}.$$

Example 4.4.1 The cross ratio of four numbers a, b, c, d on a number line is

$$[a, b, c, d] = \frac{c-a}{d-a} \frac{d-b}{c-b}.$$

If you rearrange the points their cross ratio may change. For example $[B, A, C, D] = 1/[A, B, C, D]$.

The cross ratio is significant in projective geometry because it is not changed by projections (see part 2 of the next proposition).

Proposition 4.4.1 *Let A, B, C, and D be four points on a Euclidean line, and P a point that is not on that line. Then*

1.
$$[A, B, C, D] = \frac{\sin \angle APC}{\sin \angle APD} \frac{\sin \angle BPD}{\sin \angle BPC}. \tag{4.1}$$

2. *Let A', B', C', D' be the projections, respectively, of A, B, C, D from P onto another line. Then*

$$[A, B, C, D] = [A', B', C', D'].$$

(See Fig. 4.12).

It is important to keep track of the signs. The angles in part 1 of the above proposition are *oriented* angles, with a direction of rotation chosen so that an angle $\angle XYZ$ indicates a rotation between 0^o and 180^o carrying

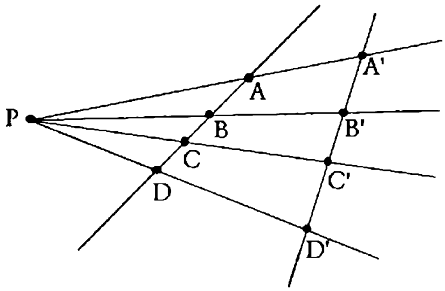


FIGURE 4.12. [A,B,C,D]=[A',B',C',D'].

$\overrightarrow{YX}$ to $\overrightarrow{YZ}$. The sines of two of these angles have the same sign if and only if the angles rotate in the same direction.

Proof. Taking into account the orientations of the angles, it is easy to check that the left and right sides of Equation 4.1 have the same signs. It remains to show that their magnitudes are equal.

The magnitude of the cross ratio is

$$\begin{aligned}
|[A,B,C,D]| &= \frac{AC}{AD}\frac{BC}{BD} \\
&= \frac{\text{area}\,\triangle\,APC}{\text{area}\,\triangle\,APD}\frac{\text{area}\,\triangle\,BPD}{\text{area}\,\triangle\,BPC} \\
&= \left|\frac{(AP)(CP)\sin\angle APC}{(AP)(DP)\sin\angle APD}\right|\left|\frac{(BP)(DP)\sin\angle BPD}{(BP)(CP)\sin\angle BPC}\right| \\
&= \left|\frac{\sin\angle APC}{\sin\angle APD}\frac{\sin\angle BPD}{\sin\angle BPC}\right|
\end{aligned}$$

This proves part 1 of the proposition.

Part 2 is an immediate consequence of part 1 since $\angle APB = \angle A'PB'$ and so on.

This completes the proof.

If one of the four points A, B, C, D in Definition 4.4.1 is a point at infinity we can still compute the cross ratio by taking a limit. For example if $A = \infty$ then

$$[A,B,C,D] = \lim_{P\to\infty}[P,B,C,D]$$

and so on.[1] The same result is obtained simply by defining $\frac{\infty}{\infty} = 1$ and $-\frac{\infty}{\infty} = -1$.

Example 4.4.2 If b, c, d are numbers on a number line,

$$[\infty,b,c,d] = \lim_{p\to\infty}\frac{c-p}{d-p}\frac{d-b}{c-b}$$

[1]Of course P, B, C, and D must be collinear.

$$= \frac{d-b}{c-b}.$$

The conclusion of part 2 of Proposition 4.4.1 remains true if A, B, C, D or P is a point at infinity. In particular if P is a point at infinity then "projection from P" is parallel projection, and the proof of part 2 of the proposition is a simple application of similar triangles.

The next proposition shows how to compute the ratio of distances between points in a scene by using the cross ratio of their images in a perspective drawing.

Corollary 4.4.1 *Let B, C, D be three points on a Euclidean line, let B', C', D' be their images in a perspective drawing and let V' be the vanishing point of the line. Then*

$$\frac{BD}{BC} = [V', B', C', D'].$$

Proof. By Proposition 4.4.1,

$$\begin{aligned} [V', B', C', D'] &= [\infty, B, C, D] \\ &= \frac{BD}{BC}. \end{aligned}$$

Example 4.4.3 (See [9, Chap. 3, page 47.]). The following procedure will produce a perspective drawing of a Euclidean line segment that has been subdivided into n equal parts.

On the viewplane, let $\overline{AB}$ be the image of a Euclidean line segment and V be the vanishing point of the corresponding Euclidean line. Let $P \notin \overleftrightarrow{AB}$ be a point in the viewplane and let $L \neq \overleftrightarrow{PV}$ be a line parallel to $\overleftrightarrow{PV}$ in the viewplane. Set

$$A' = \overleftrightarrow{AP} \cap L \qquad \text{and} \qquad B' = \overleftrightarrow{BP}.$$

Choose points $A'_1, \ldots, A'_{n-1} \in L$ such that

$$A'A'_1 = A'_1A'_2 = \ldots = A'_{n-1}B'.$$

For each $i = 1, \ldots, n$ set

$$A_i = \overleftrightarrow{A'_iP} \cap \overline{AB}.$$

Then $A_1, A_2, \ldots, A_{n-1}$ are the images of points that subdivide the original Euclidean line segment into n segments of equal length (see Fig. 4.13).

Exercise 4.4.1 Prove that the construction in Example 4.4.3 really does produce a perspective drawing of a line segment that has been divided into pieces of equal length. (Hint: Use part 2 of Proposition 4.4.1 and Corollary 4.4.1.

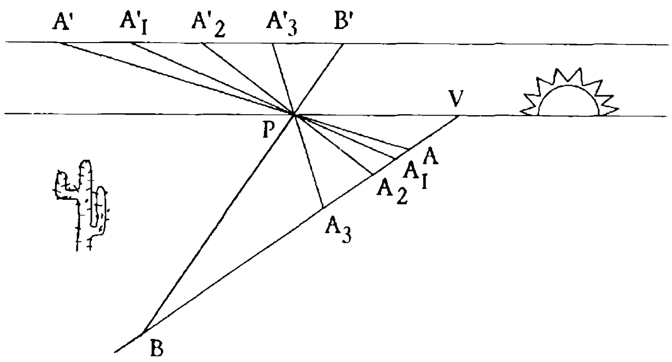


FIGURE 4.13. Equal lengths.

More Cross Ratios

Definition 4.4.1 and Proposition 4.4.1 can be extended in many ways to cover produce additional kinds of cross ratios.

Definition 4.4.2 The Cross Ratio of Four Points on a Projective Line. Let $P_1, \ldots, P_4$ be four points on a projective line, let H be a Euclidean viewplane, and let $P'_1 = P_{\cap}H, \ldots, P'_4 = P_4 \cap H$ be the images of $P_1, \ldots P_4$ in H. Then

$$[P_1, P_2, P_3, P_4] = [P'_1, P'_2, P'_3, P'_4].$$

Definition 4.4.2 would be useless if intersecting $P_1, \ldots, P_4$ with different planes H gave different cross ratios, but part 2 of Proposition 4.4.1 guarantees that this never happens.

Corollary 4.4.2 *In projective space, if $P_1, \ldots, P_4$ are four points on a line L, and $P'_1, \ldots, P'_4$ are their images under a central projection mapping L to another line L', then*

$$[P_1, P_2, P_3, P_4] = [P'_1, P'_2, P'_3, P'_4].$$

Proof. If you intersect everything with a viewplane then 4.4.2 becomes part 2 of Proposition 4.4.1.

Definition 4.4.3 The Cross Ratio of Four Coincident Lines in a Plane. Let $L_1, \ldots, L_4$ be four coincident lines in a plane. If $P_1 \in L_1$, $P_2 \in L_2$, $P_3 \in L_3$, and $P_4 \in L_4$ are any four *collinear* points, define

$$[L_1, L_2, L_3, L_4] = [P_1, P_2, P_3, P_4].$$

Definition 4.4.3 applies in both projective and Euclidean spaces. In either case, part 2 of Proposition 4.4.1 guarantees that the cross ratio of the four lines is the same regardless of the choice of the points $P_1, \ldots, P_4$.

Proposition 4.4.2 Cross ratios of lines are not changed by projections. *In* $\mathbf{P}^3$ *or* $\mathbf{E}^3$ *let* L_1, L_2, L_3, L_4 *be four coincident lines in a plane* H. *If* H' *is another plane and* f *is a central projection from* H *into* H' *then*

$$[f(L_1), f(L_2), f(L_3), f(L_4)] = [L_1, L_2, L_3, L_4].$$

Proof. (See Fig. 4.14). Let M be a line in H. For each $i = 1, \dots, 4$ set $P_i \in M \cap L_i$. Then

$$\begin{aligned} [L_1, L_2, L_3, L_4] &= [P_1, P_2, P_3, P_4] \\ &= [f(P_1), f(P_2), f(P_3), f(P_4)] \\ &= [f(L_1), f(L_2), f(L_3), f(L_4)]. \end{aligned}$$

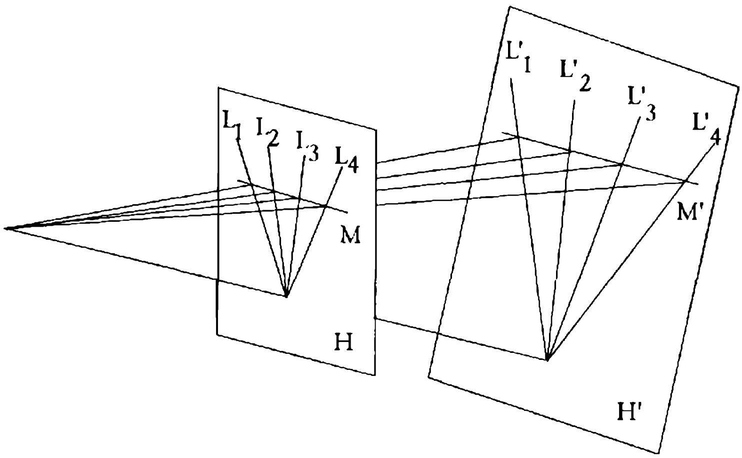


FIGURE 4.14. Cross ratios are not changed by projections.

A *conic* in the projective plane is the projectivization of a conic in the Euclidean plane. The projective conic is "smooth" if the corresponding Euclidean conic is smooth.

Proposition 4.4.3 *Let* A, B, C, D *be four points on a smooth conic* K. *Then for all* $P, Q \in K$,

$$[\overleftrightarrow{PA}, \overleftrightarrow{PB}, \overleftrightarrow{PC}, \overleftrightarrow{PD}] = [\overleftrightarrow{QA}, \overleftrightarrow{QB}, \overleftrightarrow{QC}, \overleftrightarrow{QD}].$$

Proof. (See Fig. 4.15). Clearly it is enough to prove the proposition for a Euclidean cone, for every projective cone can be made into a Euclidean cone by intersecting it with a viewplane.

Every smooth Euclidean conic is a section of a right circular cone. Let V be the vertex of the cone. If you project K from V into a plane that is perpendicular to the axis of the cone the image of K will be a circle, and track at the point where the second diagonal crosses the track. This step is repeated until all the ties are drawn.

In the actual scene the rectangles formed by the tracks and the ties all are parallel and congruent. Thus their diagonals also are parallel, so corresponding sides and diagonals meet at the same points on the horizon.

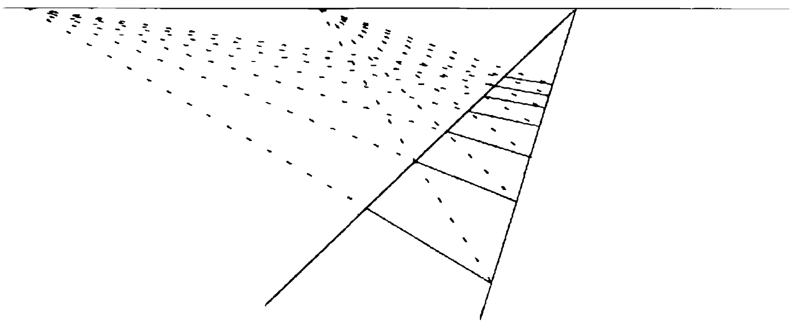


FIGURE 4.17. Railroad tracks.

Exercise 4.4.3 a) Draw a checkerboard that extends toward the horizon in all directions.

b) Draw a scene showing a row of equally spaced houses along a street that runs toward the horizon.

4.5 Projections in Coordinates

In this section (x, y, z) are the standard coordinates in $\mathbf{R}^3$.

Example 4.5.1 Given two parallel lines

$$\begin{aligned} L_1 &= \{x = 1 \text{ and } z = -1\} \\ L_2 &= \{x = -1 \text{ and } z = -1\} \end{aligned}$$

contained in the plane

$$G = \{z = -1\},$$

project L_1 and L_2 from the origin into the viewplane

$$H = \{y = 1\}.$$

Solution. (See Fig. 4.18). Points $(x, 1, z) \in H$ and $(x', y', -1) \in G$ lie on the same radial line if and only if

$$(x', y', -1) = t(x, 1, z)$$

for some scalar t, that is, if and only if

$$x' = tx, \quad y' = t, \quad \text{and} \quad -1 = tz.$$

The third equation says that $t = -1/z$; using this the first two become

$$x' = -\frac{x}{z} \quad \text{and} \quad y' = -\frac{1}{z}. \tag{4.2}$$

$(x', y', -1)$ lies on L_1 if and only if $x' = 1$. Hence $(x, 1, z)$ lies on the projection of L_1 if and only if $-x/z = 1$. Multiplying through by z to clear the fractions, we get the equation for the projection of L_1:

$$-x = z.$$

Similarly, the equation for the projection of L_2 is $x' = -x/z = -1$, i.e.

$$x = z.$$

The horizon of G is the intersection of the viewplane with the plane $z = 0$, which is parallel to G. Thus the horizon is the line $z = 0$ in H. The projections of L_1 and L_2 meet at the common vanishing point, $(x, y, z) = (0, 1, 0)$, of L_1 and L_2 in H, which lies on the horizon of G.

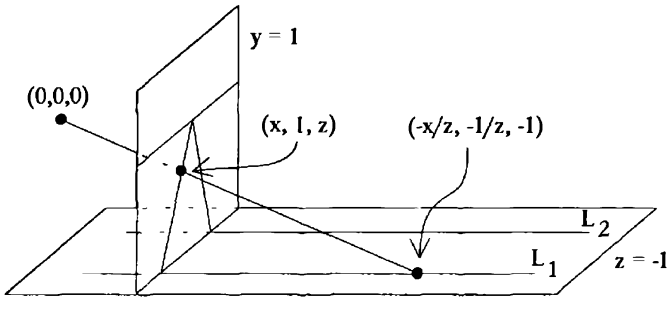


FIGURE 4.18. Images of parallel lines meeting at the horizon.

Example 4.5.2 Project the parabolas

$$Q_1 = \left\{ y = 1 + \frac{x^2}{4}, z = -1 \right\} \quad \text{and}$$
$$Q_2 = \left\{ y = \frac{x^2}{4}, z = -1 \right\}$$

from the plane $G = \{z = -1\}$ into the $y = 1$ plane.

Solution. (See Fig. 4.19). By plugging Formulas 4.2 from the previous example into the formula for Q_1, one obtains a formula $-1/z = 1 +$