$$1 = 1 \cdot e^{0i} = 1 \cdot e^{2k\pi i}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Then we take the $6^{\text{th}}$ root and find

$$z_k = 1^{1/6} e^{2k\pi i/6} = e^{k\pi i/3}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

The six roots are

$$z_0 = 1 \qquad z_1 = e^{\pi i/3} = \tfrac{1}{2} + \tfrac{i}{2}\sqrt{3} \qquad z_2 = e^{2\pi i/3} = -\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3}$$

$$z_3 = -1 \qquad z_4 = e^{4\pi i/3} = -\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3} \qquad z_5 = e^{5\pi i/3} = \tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}$$

2- Find all cubic roots of $w = -1 + i$.

**Solution:**

Let $z^3 = w$. First we write $-1 + i$ in polar form.

$r = |w| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$.

$\theta = \arctan\left(\frac{-1}{1}\right) = \arctan(-1)$. Since $w$ lies in the second quadrant, therefore,
$\theta = \frac{3\pi}{4} = 135^\circ$.

$$z_k = r^{1/n} e^{i\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right)} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right) + i\sin\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right)\right].$$

$n$=3, $k$ =0,1,2.

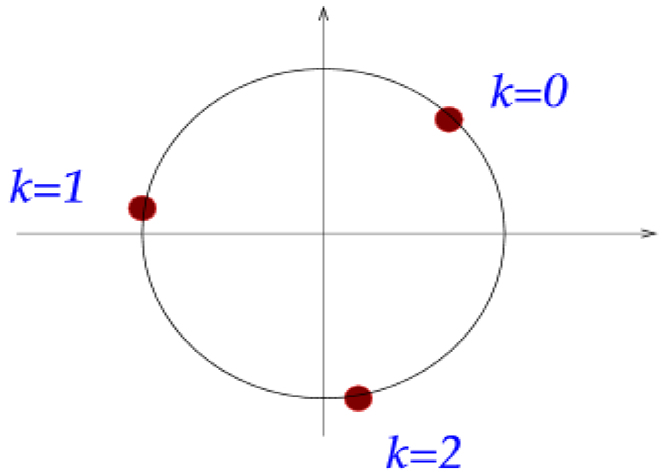
$$k = 0: \qquad z_0 = \sqrt{2}^{1/3} e^{i\left(\frac{3\pi/4}{3}\right)} = 2^{1/6}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right].$$

$$k = 1: \qquad z_1 = \sqrt{2}^{1/3} e^{i\left(\frac{\pi}{4} + 2\pi\frac{1}{3}\right)} = 2^{1/6}\left[\cos\left(\frac{\pi}{4} + 2\pi\frac{1}{3}\right) + i\sin\left(\frac{\pi}{4} + 2\pi\frac{1}{3}\right)\right.$$

$$= 2^{1/6}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right].$$

$$k = 2: \qquad z_2 = \sqrt{2}^{1/3} e^{i\left(\frac{\pi}{4} + 2\pi\frac{2}{3}\right)} = 2^{1/6}\left[\cos\left(\frac{\pi}{4} + 2\pi\frac{2}{3}\right) + i\sin\left(\frac{\pi}{4} + 2\pi\frac{2}{3}\right)\right.$$

$$= 2^{1/6}\left[\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right].$$




3- Find the solution of the equation $z^5 + 32 = 0$.

**Solution:**

First we write $w = -32$ in polar form.

$r = |w| = \sqrt{(-32)^2 + (0)^2} = 32.$

$\theta = \arctan\left(\frac{0}{-32}\right) = \arctan(0)$. Since $w$ lies in the second quadrant, therefore, $\theta = \pi$.

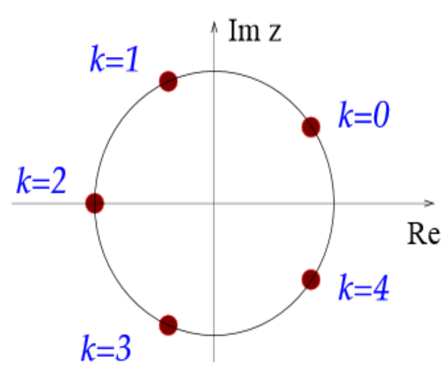
$$z_k = r^{1/n} e^{i(\frac{\theta}{n} + 2\pi\frac{k}{n})} = r^{1/n}[\cos\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right) + i\sin(\frac{\theta}{n} + 2\pi\frac{k}{n})].$$

$n$=5, $k$ =0,1,2,3,4.

$$(-32)^{1/5} = 32^{1/5}\left[\cos\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right)\right] \quad , \quad k = 0, 1, 2, 3, 4$$

that is,

$$k = 0: \quad 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$$
$$k = 1: \quad 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$$
$$k = 2: \quad -2$$
$$k = 3: \quad 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$$
$$k = 4: \quad 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$$



## <u>Exercise1.10:</u>

1- Write in polar form $re^{i\theta}$, the following:

(1) $i$ (2) $-2$ (3) $\sqrt{3}+3i$ (4) $-3i$ (5) $1-i\sqrt{3}$ (6) $1+i\sqrt{3}$ (7) $4i$ .

2- Put the following complex numbers in the form $x+iy$.

(1) $e^{3i\pi}$ (2) $3e^{i\frac{\pi}{4}}$ (3) $\pi e^{-i\frac{\pi}{3}}$ (4) $e^{-5i\frac{\pi}{4}}$ (5) $e^{i100\pi}$.

3- Put $(-1+i)^{100}$ in the form $x+iy$.

4- Let $z_1=\sqrt{3}+3i$, $z_2=1+i\sqrt{2}$, $z_3=4i$. Find the following in the form $x+iy$.

$$\frac{z_3^2(z_1^2+z_2^2)^2}{z_1\ z_2}$$

5- Prove that $\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3=1$.

6- Prove that $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos(\frac{n\pi}{4})$.

7- Prove by polar form that $i(1-i\sqrt{3})(\sqrt{3}+i)=2+i2\sqrt{3}$.

8- Find the modulus of the following:

(1) $\left|\frac{-2+3i}{3-2i}\right|$

(2) $\left|\frac{1-4i}{4+3i}\right|$

9- Find and draw all real complex solutions of the following:

(1) $z^2+7z+10=0$. (2) $z^3+8=0$.

(3) $z^5-16z=0$. (4) $z^5-32=0$.

(5) $z^4+2z^2-3=0$. (6) $3z^6=z^3+2$.

(7) $z^3-125=0$. (8) $z^2+6z+10=0$.

10- Compute the following:

(1) $\left|(1-i\sqrt{3})^2-(4-i\sqrt{3})\right|$.

(2) $\left|(3-i)^2(5-i\sqrt{2})\right|$.