3- Write $z = (1+i)^8$ in the form $a + ib$.

Solution:

First we write $w = (1+i)$ in the polar coordinate.

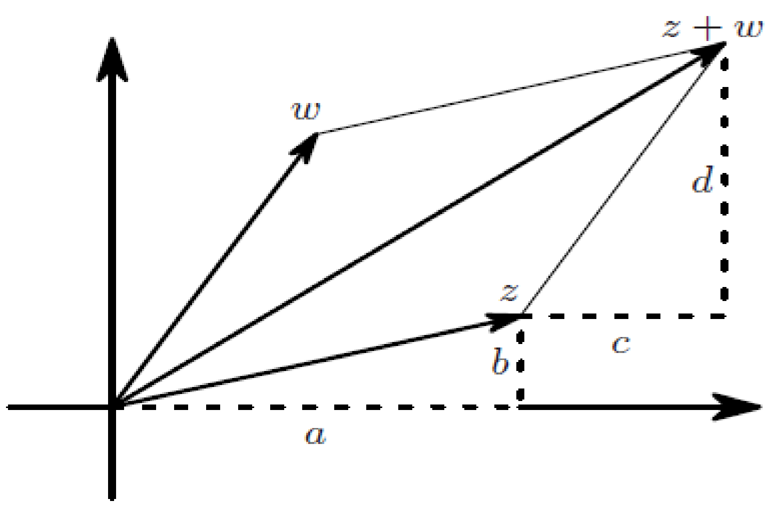
$r = |w| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$. $\theta = \arctan\left(\frac{1}{1}\right) = \arctan(1)$. Since w lies in the first quadrant, therefore, $\theta = \frac{\pi}{4} = 45^{\circ}$. So, by **De Moivre's Formula**

$$z = w^8 = r^8 cis(8\theta) = \sqrt{2}^8 cis\left(\frac{8\pi}{4}\right) = 2^4 cis(2\pi) = 16(1+0i) = 16.$$

Geometry of Arithmetic 1.7:

Since we can picture complex numbers as points in the complex plane, we can also try to visualize the arithmetic operations "addition" and "multiplication."

1- To add z and w one forms the parallelogram with the origin, z and w as vertices. The fourth vertex then is $z + w$.

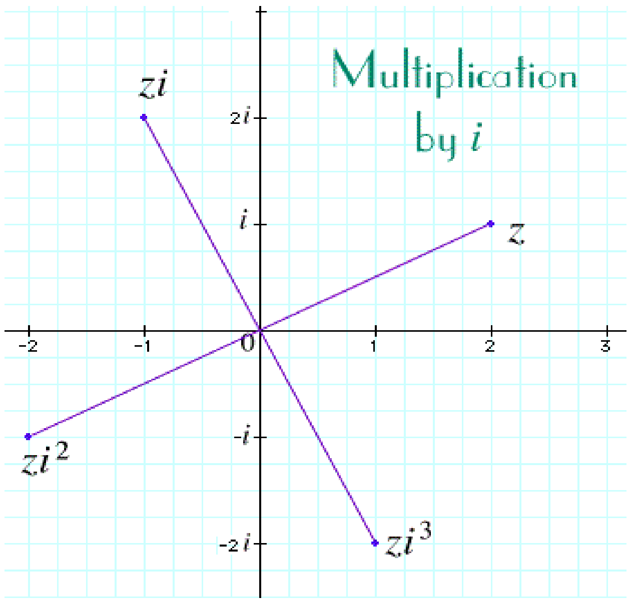


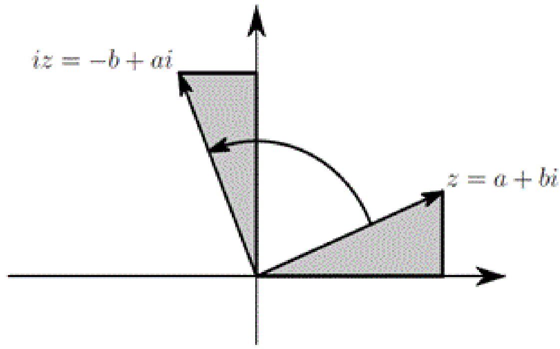
2- To understand multiplication we first look at multiplication with $\boldsymbol{i}$.

If $z = a + bi$, then

$$iz = i(a + bi) = ia + bi2 = ai - b = -b + ai.$$

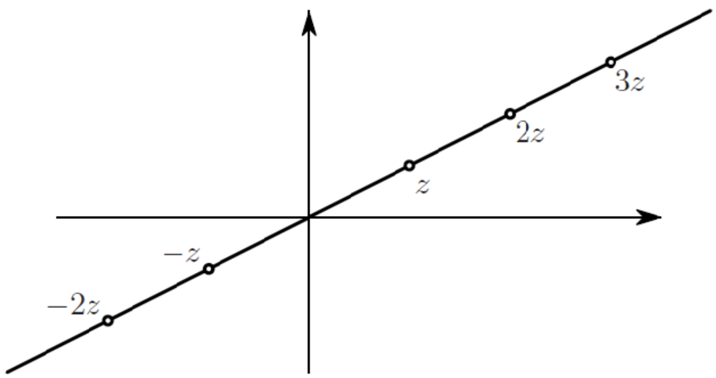
Thus, to form iz from the complex number z one rotates z counterclockwise by 90 degrees.





If a is any real number, then multiplication of $w = c + di$ by a gives

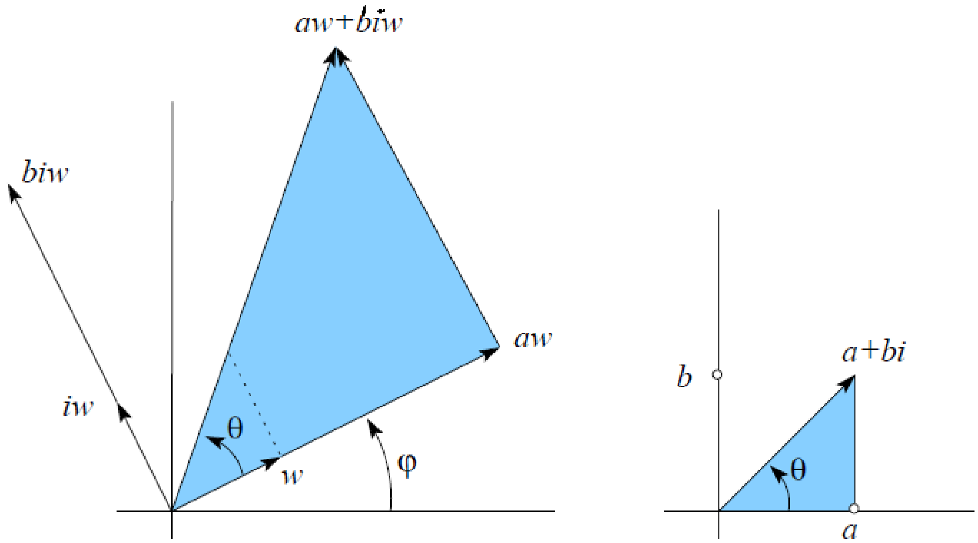
$$aw = ac + adi,$$



So, aw points in the same direction, but is a times as far away from the origin. If $a < 0$ then aw points in the opposite direction.

Next, to multiply $z = a + bi$ and $w = c + di$ we write the product as

$$zw = (a + bi)w = aw + biw.$$



Complex Roots of a Number 1.8:

For any given complex number $w \neq 0$ there is a method of finding all complex solutions of the equation

$$z^n = w \text{ -------------- } (1)$$

if n = 2, 3, 4, ... is a given integer.

To find these solutions you write w in polar form, that is, you find $r > 0$ and θ such that $w = re^{i\theta}$. Then

$$z = r^{1/n}e^{i\theta/n} = \sqrt[n]{w} = w^{1/n}$$

is a solution to (1). But it isn't the only solution, because the angle θ for which $w = re^{i\theta}$ isn't unique, it is only determined up to a multiple of 2π. Thus, if we have found one angle θ for which $w = re^{i\theta}$, then we can also write

$$w = re^{i(\theta+2k\pi)},\ k = 0, \mp 1, \mp 2, \dots$$

The n^{th} roots of w are then

$$z_k = r^{1/n}e^{i(\frac{\theta}{n}+2\pi\frac{k}{n})} = r^{1/n}[\cos\left(\frac{\theta}{n}+2\pi\frac{k}{n}\right) + i\sin(\frac{\theta}{n}+2\pi\frac{k}{n})].$$

Here k can be any integer, so it looks as if there are infinitely many solutions. However, if you increase k by n, then the exponent above increases by $2\pi i$, and hence z_k does not change. In a formula:

$$z_n = z_0,\ z_{n+1} = z_1,\ z_{n+2} = z_2, \dots, z_{n+k} = z_k.$$

So if you take $k = 0, 1, 2, \dots, n-1$ then you have had all the solutions.

Example 1.9:

1- Find all sixth roots of $w = 1$.

Solution: We have to solve $z^6 = 1$. First we write 1 in polar form.