Chapter Three

Properties of Binomial Coefficients

Theorem 3.41: For $0 \leq r \leq n$, the following are holds:

(1) $\binom{n}{r} = \binom{n}{n-r}$.
(2) $\binom{n}{0} = \binom{n}{n}$.
(3) $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$.
(4) $\binom{n}{0} = \binom{n+1}{0}$.
(5) $\binom{n}{n} = \binom{n+1}{n+1}$.

Proof: Exercise.

Examples 3.42:

(1) Use properties of binomial coefficients to show that:

$$\binom{6}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}$$

Solution: Here the identitiy $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ will use.

$$\binom{6}{3} = \binom{5}{3} + \binom{5}{2}$$

$$= \left[\binom{4}{3} + \binom{4}{2}\right] + \binom{5}{2} \qquad \text{Apply the identity to } \binom{5}{3}.$$

$$= \left[\binom{3}{3} + \binom{3}{2}\right] + \binom{4}{2} + \binom{5}{2} \qquad \text{Apply the identity to } \binom{4}{3}.$$

$$= \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} \qquad \text{Since } \binom{2}{2} = \binom{3}{3} = 1 \quad \blacksquare$$

(2) Show that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Solution:

We use the binomial theorem with $x = 1$ and $y = 1$. This gives

$$(1+1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}\cdot 1 + \binom{n}{2}1^{n-2}\cdot 1^2 + \cdots + \binom{n}{n}1^n$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$ ■

To find the number of subsets of a set with n elements.

(i) The number of subsets of a set with 0 elements is $\binom{n}{0}$;
(ii) The number of subsets of a set with 1 elements is $\binom{n}{1}$;
(iii) The number of subsets of a set with 2 elements is $\binom{n}{2}$;
(iv) The number of subsets of a set with $n-1$ elements is $\binom{n}{n-1}$;
(v) The number of subsets of a set with n elements is $\binom{n}{n}$.

The sum

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

is the total number of subsets of a set with *n* elements.

Theorem 3.43: A set with n elements has 2^n subsets.

Chapter Three

Exercise

Finite Mathematics: An Applied Approach 11[ed] by Michael Sullivan

1- In how many ways can 5 people be lined up?

2- All we know about Shannon, Patrick, and Ryan is that they have different birthdays. If we listed all the possible ways this could occur, how many would there be? Assume that there are 365 days in a year.

3- You own eight mathematics books and six computer science books and wish to fill seven positions on a shelf. If the first four positions are to be occupied by math books and the last three by computer science books, in how many ways can this be done?

4- List all the ordered arrangements of 5 objects $a, b, c, d,$ and e, choosing 3 at a time without repetition. What is $P(5, 3)$?

5- List all the ordered arrangements of 4 objects 1, 2, 3, and 4, choosing 3 at a time without repetition. What is $P(4, 3)$?

6- **Forming Codes:** How many two-letter codes can be formed using the letters $A, B, C,$ and D? Repeated letters are allowed.

7- Forming Codes: How many two-letter codes can be formed using the letters $A, B, C, D,$ and E? Repeated letters are allowed.

8- **Forming Numbers:** How many three-digit numbers can be formed using the digits 0 and 1? Repeated digits are allowed.

9- **Forming Numbers:** How many three-digit numbers can be formed using the digits $0, 1, 2, 3, 4, 5, 6, 7, 8,$ and 9? Repeated digits are allowed.

10- **Lining Up People:** In how many ways can 4 people be lined up?

11- **Stacking Boxes:** In how many ways can 5 different boxes be stacked?

12- **Forming Codes:** How many different three-letter codes are there if only the letters $A, B, C, D,$ and E can be used and no letter can be used more than once?

13- **Forming Codes:** How many different four-letter codes are there if only the letters $A, B, C, D, E,$ and F can be used and no letter can be used more than once?

14- **Seating Arrangements:** How many ways are there to seat 5 people in 8 chairs?

15- **Seating Arrangements:** How many ways are there to seat 4 people in a 6-passenger automobile?

16- **Stocks on the NYSE:** Companies whose stocks are listed on the New York Stock Exchange (NYSE) have their company name represented by either 1, 2, or 3 letters (repetition of letters is allowed). What is the maximum number of companies that can be listed on the NYSE?

17- **Stocks on the NASDAQ :** Companies whose stocks are listed on the NASDAQ stock exchange have their company name represented by either 4 or 5 letters (repetition of letters is allowed). What is the maximum number of companies that can be listed on the NASDAQ?

18- **New Printers:** An accounting firm is upgrading the network printers in each of its three branch offices and will select from among 7 different models, though each office may have different printers. How many ways can the firm purchase three printers for the offices?

19- **New Copiers:** The college of arts and sciences at a university decides to replace the copiers in each of its 6 departments. If the dean can select from 4 different models and each department can have a different type of copier, how many ways can the dean purchase the new copiers?

20- **Television Schedule:** A television network has four 30-minute time slots for comedies and has 14 shows to choose from for the fall lineup. How many different comedy lineups are possible?

21- **Arranging Books:** There are 5 different French books and 5 different Spanish books. How many ways are there to arrange them on a shelf if

(a) Books of the same language must be grouped together, French on the left, Spanish on the right?

(b) French and Spanish books must alternate in the grouping, beginning with a French book?

22- **Distributing Books:** In how many ways can 8 different books be distributed to 12 children if no child gets more than one book?

23- **Networks:** A computer must assign each of 4 outputs to one of 8 different printers. In how many ways can it do this provided no printer gets more than one output?

24- List all the combinations of 5 objects $a, b, c, d,$ and e taken 3 at a time. What is $C(5,3)$?

25- List all the combinations of 5 objects $a, b, c, d,$ and e taken 2 at a time. What is $C(5,2)$?

26- List all the combinations of 4 objects $1, 2, 3$, and 4 taken 3 at a time. What is $C(4,3)$?

27- List all the combinations of 6 objects $1, 2, 3, 4, 5$, and 6 taken 3 at a time. What is $C(6,3)$?

28- **Establishing Committees:** In how many ways can a committee of 4 students be formed from a pool of 7 students?

29- **Establishing Committees:** In how many ways can a committee of 3 professors be formed from a department having 8 professors?

30- **Tenure Selection:** A math department is allowed to tenure 4 of 17 eligible teachers. In how many ways can the selection for tenure be made?

31- Bridge Hands: How many different hands are possible in a bridge game? (A bridge hand consists of 13 cards dealt from a deck of 52 cards.)

32- **Forming a Committee:** There are 20 students in the Math Club. In how many ways can a subcommittee of 3 members be formed?

33- **Relay Teams:** How many different relay teams of 4 persons can be chosen from a group of 10 runners?

34- **Eight-Bit Strings:** How many eight-bit strings contain exactly three 1's?

35- **Eight-Bit Strings:** How many eight-bit strings contain exactly two 1's?

36- **Forming Words:** How many different 9-letter words can be formed from the letters in the word ECONOMICS?

37-

Show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$$

(The last term will be preceded by a plus or minus sign depending on whether n is even or odd.)

38- What is the coefficient of x^2y^3 in the expansion of $(x+y)^5$

39- What is the coefficient of x^8 in the expansion of $(x+3)^{10}$

40- What is the coefficient of in the expansion of

41- What is the coefficient of in the expansion of

42- How many different subsets does a set with 8 elements have?

Chapter Three

43-

Use the Binomial Theorem to find the numerical value of $(1.001)^5$ correct to five decimal places.
[**Hint:** $(1.001)^5 = (1 + 10^{-3})^5$]

44- Use the Binomial Theorem to find the numerical value of $(0.998)^6$ correct to five decimal places.

45- How many nonempty subsets does a set with 10 elements have?

46- How many subsets with an even number of elements does a

47- set with 10 elements have?

48- How many subsets with an odd number of elements does a set with 10 elements have?

49- Show that

$$\binom{8}{5} = \binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}$$

50- Show that

$$\binom{10}{7} = \binom{6}{6} + \binom{7}{6} + \binom{8}{6} + \binom{9}{6}$$

51- Show that

$$\binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7} = 2^6$$

52- Replace $\binom{11}{6} + \binom{11}{5}$ by a single binomial coefficient.

53- Replace $\binom{8}{8} + \binom{9}{8} + \binom{10}{8}$ by a single binomial coefficient.

54- $$\binom{5}{0}\left(\frac{1}{4}\right)^5 + \binom{5}{1}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) + \binom{5}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 + \binom{5}{3}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 + \binom{5}{4}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4 + \binom{5}{5}\left(\frac{3}{4}\right)^5 = ?$$

Chapter Three

College Algebra

Raymond A. Barnett, Michael R. Ziegler, Karl E. Byleen and Dave obecki

55-A particular new car model is available with five choices of color, three choices of transmission, four types of interior, and two types of engine. How many different variations of this model car are possible?

56- A deli serves sandwiches with the following options: three kinds of bread, five kinds of meat, and lettuce or sprouts. How many different sandwiches are possible, assuming one item is used out of each category?

57- In a horse race, how many different finishes among the first three places are possible for a 10-horse race? Exclude ties.

58- In a long-distance foot race, how many different finishes among the first five places are possible for a 50-person race? Exclude ties.

59- How many ways can a subcommittee of three people be selected from a committee of seven people? How many ways can a president, vice-president, and secretary be chosen from a committee of seven people?

60- Suppose nine cards are numbered with the nine digits from 1 to 9. A three-card hand is dealt, one card at a time. How many hands are possible where:
(A) Order is taken into consideration? (B) Order is not taken into consideration?

61-There are 10 teams in a league. If each team is to play every other team exactly once, how many games must be scheduled?

62- Given seven points, no three of which are on a straight line, how many lines can be drawn joining two points at a time?

63- How many four-letter code words are possible from the first six letters of the alphabet, with no letter repeated? Allowing letters to repeat?

64- How many five-letter code words are possible from the first seven letters of the alphabet, with no letter repeated? Allowing letters to repeat?

65-A combination lock has five wheels, each labeled with the 10 digits from 0 to 9. How many opening combinations of five numbers are possible, assuming no digit is repeated? Assuming digits can be repeated?

66- How many ways can two people be seated in a row of five chairs? Three people? Four people? Five people?

67-A basketball team has five distinct positions. Out of eight players, how many starting teams are possible if:

(A) The distinct positions are taken into consideration?

(B) The distinct positions are not taken into consideration?

(C) The distinct positions are not taken into consideration, but either Mike or Ken, but not both, must start?

Chapter Three

Exercise: (Mathematical induction)

College Algebra

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1- Prove that for all positive integers *n*: $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^n}=\frac{3^n-1}{3^n}$.

2- Prove that $(xy)^n = x^n y^n$ for all positive integers *n*.

3- Prove that $(x/y)^n = x^n/y^n$ for all positive integers *n*.

4- Prove that $8^n - 1$ is divisible by 7 for all positive integers *n*.

5- Use mathematical induction to prove that each P_n holds for all positive integers *n*.

i- $P_n: a^n a^5 = a^{n+5}$.

ii- $P_n: (a^5)^n = a^{n5}$.

iii- Prove that $9^n - 1$ is divisible by 4 for all positive integers *n*.

iv- Prove that $4^n - 1$ is divisible by 3 for all positive integers *n*.

6- use mathematical induction to prove each proposition for all positive integers *n*, unless restricted otherwise.

i- $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

ii- $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}=1-(\frac{1}{2})^n$.

iii- $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n)$.

iv- $1 + 8 + 16 + \cdots + 8(n-1) = (2n-1)^2;\ n > 1$

v- $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

vi- $1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

vii- $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

viii- $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$.

ix- $\frac{a^n}{a^3} = a^{n-3}; n \geq 3$.

x- $\frac{a^5}{a^n} = \frac{1}{a^{n-5}}; n \geq 5$.

7- Draw Pascal's Triangle from $n = 0$ to $n = 10$.