2.2. Equality of Sets

Definition 2.2.1. Two sets, A and B, are said to be **equal** if and only if A and B contain exactly the same elements and denote that by $A = B$. That is, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
The description $A \neq B$ means that A and B are not equal sets.

Example 2.2.2.
Let $\mathbb{Z}_e$ be the set of even integer numbers and $B = \{x | x \in \mathbb{Z}$ and divisible by $2\}$.
Then $\mathbb{Z}_e = B$.
Proof.
To prove $\mathbb{Z}_e \subseteq B$.
$\mathbb{Z}_e = \{2n | n \in \mathbb{Z}\}$.

	$x \in \mathbb{Z}_e \Leftrightarrow \exists\, n \in \mathbb{Z} : x = 2n$	Def. of $\mathbb{Z}_e$.
	$\Longrightarrow \frac{x}{2} = n$	Divide both side of $x = 2n$ by 2.
	$\Longrightarrow x \in B$	Def. of B.
(1)	$\Longrightarrow \mathbb{Z}_e \subseteq B$	Def. of subset.

To prove $B \subseteq \mathbb{Z}_e$.

	$x \in B \Leftrightarrow \exists\, n \in \mathbb{Z} : \frac{x}{2} = n$	Def. of $\mathbb{Z}_e$.
	$\Longrightarrow x = 2n$	Multiply $\frac{x}{2} = n$ by 2.
	$\Longrightarrow x \in \mathbb{Z}_e$	Def. of $\mathbb{Z}_e$.
(2)	$\Longrightarrow B \subseteq \mathbb{Z}_e$	Def. of subset.
	$\mathbb{Z}_e = B$	inf (1),(2) and def. of equality.

Remark 2.2.3.
(i) Two equal sets always contain the same elements. However, the rules for the sets may be written differently, as in Example 2.2.2.
(ii) Since any two empty sets are equal, therefore, there is a unique empty set.
(iii) A is said to be a **proper subset** of B is and only if
(1) $A \neq \emptyset$, (2) $A \subset B$ and (2) $A \neq B$.
(iv) the symbols $\subseteq, \subset, \subsetneq, \nsubseteq$ are used to show a relation between two sets and not between an element and a set. With one exception, if x is a member of a set A, we may write $x \in A$ or $\{x\} \subseteq A$, but **not** $x \subseteq A$.
(v) $\phi \neq \{\phi\}$.
(vi) For every set A, $\phi \subseteq A$.

Theorem 2.2.4. (Properties of Set Equality)
(i) For any set $A, A = A$. (Reflexive Property)
(ii) If $A = B$, then $B = A$. (Symmetric Property)
(iii) If $A = B$ and $B = C$, then $A = C$. (Transitive Property)

Definition 2.2.5. Let A and B be subsets of a set X. The **intersection** of A and B is the set

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\},$$

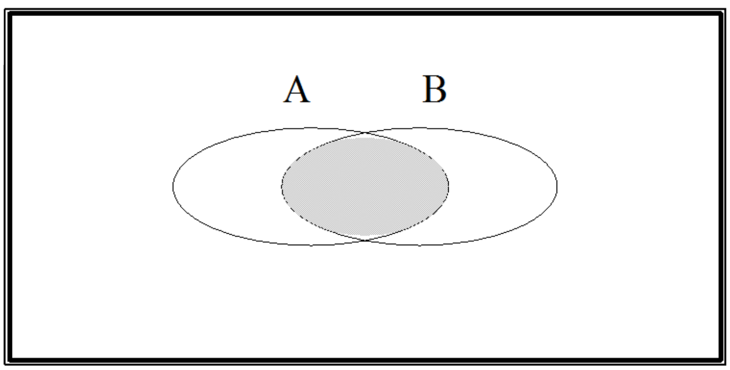
or

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

Therefore, $A \cap B$ is the set of all elements in common to both A and B.

Example 2.2.6.
(i) Given that the box below represents X, the shaded area represents $A \cap B$:

X:



(ii) Let $A = \{2,4,5\}$ and $B = \{1,4,6,8\}$. Then, $A \cap B = \{4\}$.

(iii) Let $A = \{2,4,5\}$ and $B = \{1,3\}$. Then $A \cap B = \emptyset$.

Definition 2.2.7. If two sets, A and B are two sets such that $A \cap B = \emptyset$ we say that A and B are **disjoint.**

Definition 2.2.8. Let A and B be two subsets of a set X. The **union** of A and B is the set

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\},$$

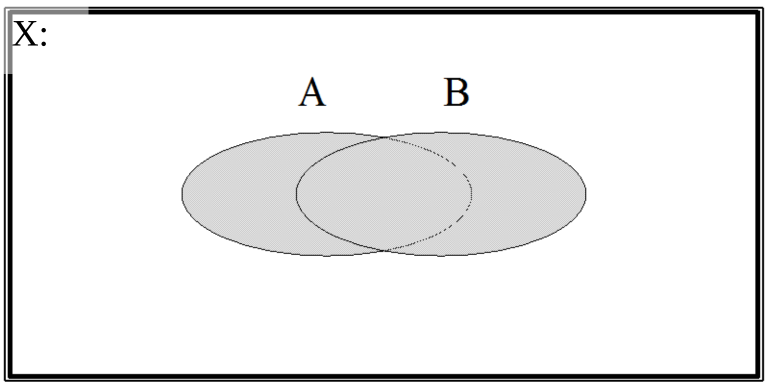
or

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

Therefore, $A \cup B$ = the set of all elements belonging to A or B.

Example 2.2.9.

(i) Given that the box below represents X, the shaded area represents $A \cup B$:



(ii) Let $A = \{2,4,5\}$ and $B = \{1,4,6,8\}$. Then, $A \cup B = \{1,2,4,5,6,8\}$.
(iii) $\mathbb{Z}_e \cup \mathbb{Z}_o = \mathbb{Z}$.

Remark 2.2.10.
It is easy to extend the concepts of intersection and union of two sets to the intersection and union of a finite number of sets. For instance, if $X_1, X_2, \ldots, X_n$ are sets, then

$$X_1 \cap X_2 \cap \ldots \cap X_n = \{x | x \in X_i \text{ for all } i = 1, \ldots, n\}$$

and

$$X_1 \cup X_2 \cup \ldots \cup X_n = \{x \mid x \in X_i \text{ for some } i = 1,2, \ldots, n\}.$$

Similarly, if we have a collection of sets $\{X_i : i = 1,2, \ldots\}$ indexed by the set of positive integers, we can form their intersection and union. In this case, the intersection of the X_i is

$$\bigcap_{i=1}^{\infty} X_i = \{x \in X_i \text{ for all } i = 1,2, \ldots\}$$

and the union of the X_i is

$$\bigcup_{i=1}^{\infty} X_i = \{x \in X_i \text{ for some } i = 1,2, \dots\}.$$

Theorem 2.2.11. Let A, B, and C be arbitrary subsets of a set X. Then

(i) $A \cap B = B \cap A$ (Commutative Law for Intersection)

$A \cup B = B \cup A$ (Commutative Law for Union)

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative Law for Intersection)
$A \cup (B \cup C) = (A \cup B) \cup C$ (Associative Law for Union)
(iii) $A \cap B \subseteq A$
(iv) $A \cap X = A$; $A \cup \emptyset = A$
(v) $A \subseteq A \cup B$
(vi) $A \cup X = X$; $A \cap \emptyset = \emptyset$
(vii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law of Union with respect to Intersection).
(viii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law of Intersection with respect to Union),
(ix) $A \cup A = A, A \cap A = A$ (Idempotent Laws)
(x) $A \cup \emptyset = A, A \cap X = A$ (Identity Laws)
(xi) $A \cup X = X$, $A \cap \emptyset = \emptyset$ (Domination Laws)
(xii) $A \cup (A \cap B) = A$ (Absorption Laws)
$A \cap (A \cup B) = A$.

Proof.
(i) $A \cap B = B \cap A$. This proof can be done in two ways.

The *first proof*
Uses the fact that the two sets will be equal only if
$(A \cap B) \subseteq (B \cap A)$ and $(B \cap A) \subseteq (A \cap B)$.

(1) Let x be an element of $A \cap B$
Therefore, $x \in A \wedge x \in B$ — Definition of $A \cap B$
Thus, $x \in B \wedge x \in A$ — Commutative property of $\wedge$
Hence, $x \in B \cap A$ — Definition of $B \cap A$
Therefore, $A \cap B \subseteq B \cap A$

(2) Let x be an element of $B \cap A$

Therefore, $x \in B \wedge x \in A$ Definition of $B \cap A$
Thus, $x \in A \wedge x \in B$ Commutative property of $\wedge$
Hence, $x \in A \cap B$ Definition of $A \cap B$
Therefore $B \cap A \subseteq A \cap B$
Thus, $A \cap B = B \cap A$

The *second proof*

$A \cap B = \{x \mid x \in A \cap B\}$
$= \{x \mid x \in A \wedge x \in B\}$ Definition of $A \cap B$
$= \{x \mid x \in B \wedge x \in A\}$ Commutative property of $\wedge$
$= \{x \mid x \in B \cap A\}$ Definition of $B \cap A$
$= B \cap A$

(iii) $(A \cap B) \subseteq A$

It must be shown that each element of $A \cap B$ is an element of A.

Let $x \in A \cap B$
Thus, $x \in A \wedge x \in B$ definition of $A \cap B$
Hence, $x \in A$

Therefore, $(A \cap B) \subseteq A$

(iv) $A \cap X = A$

(1) $A \cap X \subset A$ inf (iii) above

(2) Let $x \in A$
Thus, $x \in X$ $A \subseteq X$ is given
Hence, $x \in A \wedge x \in X$ Definition of $\wedge$
Therefore, $x \in A \cap X$ Definition of $\cap$
Thus, $A \subseteq A \cap X$ Definition of $\subseteq$

Thus, $A \cap X = A$ inf(1),(2)