1.9. Logical Reasoning

Definition 1.9.1. (Arguments)
An **argument** is a series of statements starting from hypothesis(premises) and ending with the conclusion.

From the definition, an argument might be valid or invalid.

Definition 1.19.2. (Valid Arguments)
An argument is said to be **valid** if the hypothesis implies the conclusion; that is, if s is a statement implies from the statements $s_1, s_2, \dots, s_n$, then write as

$$s_1, s_2, \dots, s_n \mapsto s.$$

Example 1.9.3.
(i) Let s_1: Some mathematicians are engineering
s_2 : Ali is mathematician
s : Ali is engineering

Solution.
The argument $s_1, s_2 \mapsto s$ is not valid, since not all mathematicians are engineering.

(ii) Let s_1: There is no lazy student
s_2 : Ali is artist
s_3 : All artist are lazy

Find a conclusion s for the above premises making the argument $s_1, s_2, s_3 \mapsto s$ is valid.

Solution.
Ali is

Remark 1.9.4.
(i) An argument

$$s_1, s_2, \dots, s_n \mapsto s$$

is valid if and only if

$$(s_1 \wedge s_2 \wedge \dots \wedge s_n) \to s$$

is tautology; that is,

$$(s_1 \wedge s_2 \wedge \dots \wedge s_n) \Longrightarrow s.$$

Also, any valid argument called **the proof**.
(ii) An argument does not depend on the truth of the premises or the conclusion but it just interested only in the question "**Is the conclusion implied by the conjunction of the premises?**"

1.10. Mathematical Proof

In this section some common procedures of proofs in mathematics are given with examples.

1.10.1 To Prove Statement of Type (p → q).
(1) Rule of conditional proof.
Let p is true statement and $s_1, s_2, \dots, s_n$ all previous axioms and theorems. To prove p → q it is enough to prove

$$s_1, s_2, \dots, s_n, p \mapsto q$$

is valid argument.

Example 1.10.2. Prove that, a is an even number → a^2 is an even number.
Proof.
Suppose a is an even number.
(1) $a = 2k$, k is an integer (definition of even number).
(2) $a^2 = 4k^2$, square both sides of (1)
(3) $a^2 = 2(2k^2)$,
(4) a^2 is even number, since $2k^2$ is an integer and definition of even number.

Note that the above prove the tautology

$$(s_1 \wedge s_2 \wedge p) \rightarrow q$$

where
p: a is an even number
s_1: $a = 2k$,
s_2: $a^2 = 4k^2$,
q: a^2 is even number.

(2) Contrapositive
To prove p → q we can proof that (∼ q → ∼p) since (p → q) ≡ (∼ q → ∼p).

Example 1.10.3. Prove that, $a^2$is an even number → a is an even number.
Proof.
Let p: $a^2$is an even number,
q: a is an even number.
Then
∼p: $a^2$is an odd number,
∼q: a is an even number.
Therefore, The contrpositive statement is

a is an odd number $\rightarrow$ a^2 is an odd number.
(1) $a = 2(k+1)$, k is an integer (definition of odd number).
(2) $a^2 = 4k^2 + 4k + 1$, square both sides of (1)
(3) $a^2 = 2(2k^2 + 2k) + 1$,
(4) a^2 is odd number, since $2k^2 + 2k$ is an integer and definition of odd number.

1.10.4 To Prove Statement of Type (p ↔ q).
(i) Since (p → q) ∧ (q → p) ≡ (p ↔ q), so we can proved first p → q and then proved q → p.
(ii) Moved from p into q through series of logical equivalent statements s_i as follows:

$$p \leftrightarrow s_1$$
$$s_1 \leftrightarrow s_2$$
$$\vdots$$
$$s_n \leftrightarrow q$$

This is exactly the tautology

$$((p \leftrightarrow s_1) \wedge (s_1 \leftrightarrow s_2) \wedge \ldots \wedge (s_n \leftrightarrow q)) \rightarrow (p \leftrightarrow q).$$

1.10.5 To Prove Statement of Type ∀$\boldsymbol{x}$ $\boldsymbol{P(x)}$ or ∃$\boldsymbol{x}$ $\boldsymbol{P(x)}$.
(i) To prove a sentence of type ∀ x $P(x)$, we suppose x is an arbitrary element and then prove that $P(x)$ is true.
(ii) To prove a sentence of type ∃ x $P(x)$, we have to prove there exist at least one element x such that $P(x)$ is true.

1.10.6 To Prove Statement of Type p ∨ r → q.
Depending on the tautology

$$[(p \rightarrow q) \wedge (r \rightarrow q)] \rightarrow [(p \vee r) \rightarrow q]$$

We must prove that p→q and r→q.

Example 1.10.7. Prove that

$$(a = 0 \vee b = 0) \rightarrow ab = 0$$

where a, b are real numbers.
Proof.
Firstly, we prove that $a = 0 \rightarrow ab = 0$.
Suppose that $a = 0$, then $ab = 0.b = 0$.
Secondly, we prove that $b = 0 \rightarrow ab = 0$.
Suppose that $b = 0$, then $ab = a.0 = 0$.
Therefore, the statement $(a = 0 \vee b = 0) \rightarrow ab = 0$ is tautology.

1.10.8 . Proof by Contradiction.
The contradiction is always false statement whatever the truth values of its components. Proof by contradiction is type of indirect proof.
The way of proof logical proposition **p** by contradiction start by supposing that ∼p and then try to find sentence of type

$$R \wedge\sim R$$

where R is any sentence contain **p** or any pervious theorem or any axioms or any logical propositions.
This way supports by the tautology

$$\sim [\sim p \wedge (R \wedge\sim R)] \rightarrow p.$$

By this way we can also prove sentences of type $\forall x\, P(x)$ or $\exists x\, P(x)$ or (p → q) or (p ⟹q).

Example 1.10.9. Prove that $x \neq 0 \Longrightarrow x^{-1} \neq 0$, x is real number.
Proof.
Let p: $x \neq 0$,
q: $x^{-1} \neq 0$.
We must prove p ⟹q.
Suppose ∼(p ⟹q) is true.
(1) ∼ (p → q) is tautology, by def. of logical implication.
(2) p ∧ ∼q is tautology, since ∼ (p → q)≡p ∧ ∼q
(3) $x \neq 0 \wedge x^{-1} = 0$.
(4) $x \cdot x^{-1} = 1 \neq 0$.
(5) $x \cdot x^{-1} = x \cdot 0 = 0$.
(6) $1 = 0$, from (4) and (5). This is contradiction, since $1 \neq 0 \wedge 1 = 0$.
Thus, the statement ∼(p ⟹q) is not true. Therefore, p ⟹q.