Definition 2.2.12. Let A and B be subsets of a set X. The set $B - A$, called the **difference** of B and A, is the set of all elements in B which are not in A.
Thus,

$$B - A = \{x \in X \mid x \in B \text{ and } x \notin A\}.$$

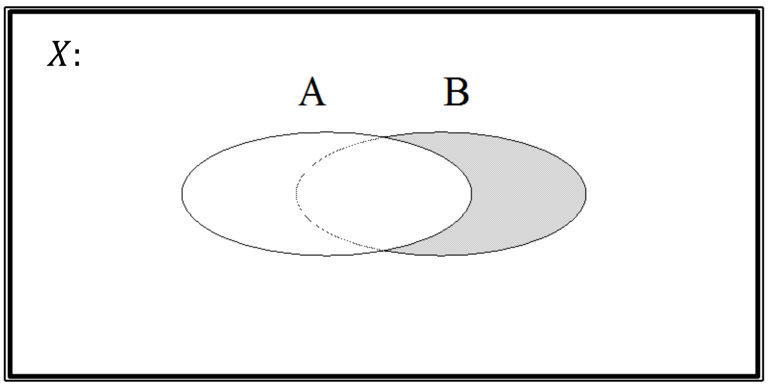
Example 2.2.13.

(i) Let $B = \{2,3,6,10,13,15\}$ and $A = \{2,10,15,21,22\}$.Then

$$B - A = \{3,6,13\}.$$

(ii) $\mathbb{Z} - \mathbb{Z}_o = \mathbb{Z}_e$.

(iii) Given that the box below represents X, the shaded area represents $B - A$.



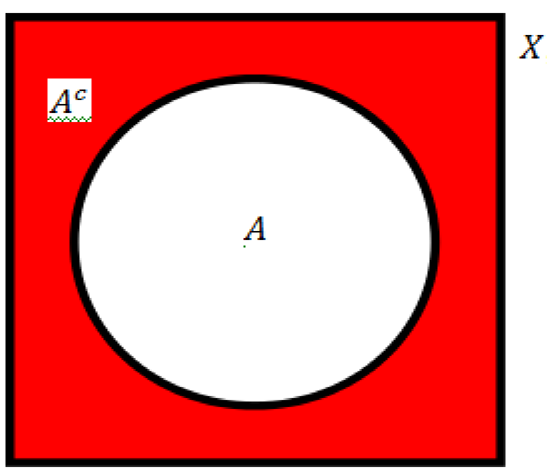
Theorem 2.2.14. Let A and B be subsets of a set X. Then

(i) $A - A = \emptyset$, $A - \emptyset = A$ and $\emptyset - A = \emptyset$

Definition 2.2.15. If $A \subseteq X$, then $X - A$ is called the **complement** of A with respect to X and denoted that by the symbol

$$X/A \text{ or } A^c.$$

Thus, $A^c = \{x \in X \mid x \notin A\}$.



Theorem 2.2.16. Let A and B be subsets of a set X. Then

(i) $A^{cc} = A$.

(ii) $X^c = \emptyset$; $\emptyset^c = X$.

(iii) $A \cup A^c = X$, $A \cap A^c = \emptyset$ (Inverse Laws)

(iv) $A \cap A^c = \emptyset$; $A \cup A^c = X$.

(v) If $A \subseteq B$, then $B^c \subseteq A^c$.

(vi) $A \cap B = \emptyset \Leftrightarrow A \subseteq B^c$.

Proof. Exercise.

Theorem 2.2.17. Let A and B be subsets of a set X. Then

(i) $(A \cup B)^c = A^c \cap B^c$, (De Morgan's Law)

$(A \cap B)^c = A^c \cup B^c$.

(ii) Let A and B be subsets of a set X. Then, $A - B = A \cap B^c$.

(iii) $A^c - B^c = B - A$.

Proof.

(i) Let $x \in (A \cup B)^c$

$\Leftrightarrow x \notin (A \cup B)$ Def. of complement

$\Leftrightarrow \sim(x \in A \cup B)$ Def. of $\notin$

$\Leftrightarrow \sim(x \in A \vee x \in B)$ Def. of $A \cup B$

$\Leftrightarrow \sim(x \in A) \wedge \sim(x \in B)$ De Morgan's Law

$\Leftrightarrow x \notin A \wedge x \notin B$ Def of $\notin$

$\Leftrightarrow x \in A^c \wedge x \in B^c$ Def. of complement

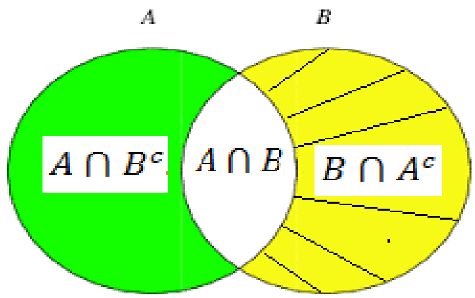
$\Leftrightarrow x \in A^c \cap B^c$ Def. of $\cap$

Hence $(A \cup B)^c = A^c \cap B^c$.

(ii) $A - B = \{x \in X \mid x \in A \text{ and } x \notin B\}$

$= \{x \in X \mid x \in A \text{ and } x \in B^c\}$ Def. of complement of B^c

$= A \cap B^c$ Def. of complement intersection



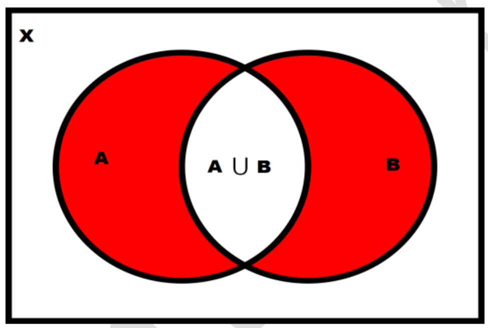
(iii) Exercise.

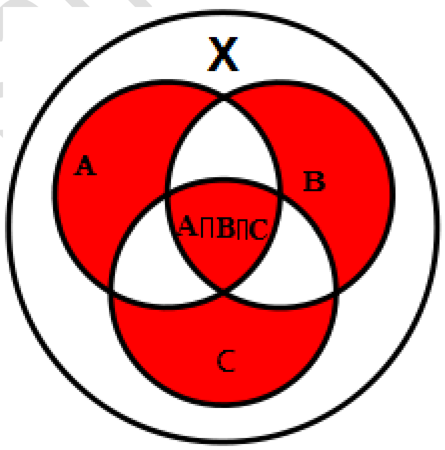
Definition 2.2.18. Let A and B be subsets of a set X. The set

$$A \vartriangle B = (A - B) \cup (B - A)$$

is called the **symmetric difference.**

Sometimes the symbol $A \oplus B$ is used for symmetric difference.



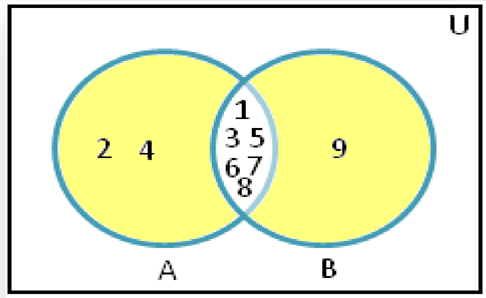


Example 2.2.19. Let $A = \{1,2,3,4,5,6,7,8\}$ and $B = \{1,3,5,6,7,8,9\}$ are subsets of $U = \{1,2,3,4,5,6,7,8,9,10\}$.

$A - B = \{2,4\}$

$B - A = \{9\}$

$A \vartriangle B = (A - B) \cup (B - A) = \{2,4,9\}$.



A △ B

Theorem 2.2.20. Let A, B and C are subsets of X. Then

(ii) $A \vartriangle \emptyset = A$.

(iii) $A \vartriangle B = \emptyset \Leftrightarrow A = B$.

(iv) $A \vartriangle B = B \vartriangle A$.

(v) $A \vartriangle A = \emptyset$.

Proof. Exercise.

In the following theorem the properties of union, intersection, complementation, symmetric difference and power set are given.

Theorem 2.2.21.

(i) $A - (B \cap C) = (A - B) \cup (A - C)$ De Morgan's Low on set difference
$A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A - (A \cap B) = (A - B) = (A \cup B) - B$.

(iii) $(A \cap B) - C = (A - C) \cap (B - C)$.

(iv) $(A - B) \cap (C - D) = (C - B) \cap (A - D)$

(v) If $A \subseteq B$, then $P(A) \subseteq P(B)$

(vi) $P(A) \cap P(B) = P(A) \cap P(B)$.

(vii) $P(A) \cup P(B) \subseteq P(A \cup B)$.

(viii) $A = B \Leftrightarrow P(A) = P(B)$.

(ix) $A \cap B = \emptyset \Leftrightarrow P(A) \cap P(B) = \emptyset$.

(x) $A \vartriangle B = (A \cup B) - (A \cap B)$.

(xi) $A \vartriangle (B \vartriangle C) = (A \vartriangle B) \vartriangle C$. Associative Law

(xii) $A \vartriangle C = B \vartriangle C \implies A = B$.

(xiii) If $A \subseteq B$ and $C = B - A$, then $A = B - C$.

Proof.

(i) $A-(B\cap C)=A\cap(B\cap C)^c$ Theorem 2.2.17(ii)
$=A\cap(B^c\cup C^c)$ De Morgan's Law
$=(A\ \cap B^c)\cup(A\ \cap C^c)$ Dist Law
$=(A-B)\cup(A-C)$. Theorem 2.2.17(ii)

(vii) Let $H\in P(A)\cap P(B)$
$\Longrightarrow H\in P(A)\ \wedge\ H\in P(B)$ Def. $\cap$
$\Longrightarrow H\subseteq A\ \wedge\ H\subseteq B$ Def. of power set
$\Longrightarrow H\subseteq(A\ \cap B)$ Def. $\cap$
$\Longrightarrow H\in P(A\cap B)$ Def. of power set

(xi) $x\in A\vartriangle B$ $\Leftrightarrow$ $x\in(A-B)\cup(B-A)$ Def. of $\vartriangle$

$\Leftrightarrow$ $x\in(A-B)\vee(B-A)$ Theorem $\cup$

$\Leftrightarrow$ $x\in A\ \wedge\ x\notin B\quad\vee\quad x\in B\ \wedge\ x\notin A$ Def. of difference

$\Leftrightarrow$ $x\in A\ \vee\ x\in B\quad\wedge\quad x\notin B\ \vee\ x\in B$ Dist. Law

$\wedge$

$x\in A\ \vee\notin A\quad\wedge\quad x\notin B\ \vee\ x\notin A$

$\Leftrightarrow$ $x\in A\ \vee\ x\in B\quad\wedge\quad T$ Tautology

$\wedge$

$T\quad\wedge\quad x\notin B\ \vee\ x\notin A$

$\Leftrightarrow$ $x\in A\ \vee\ x\in B$ Identity Law of $\wedge$

$\wedge$

$x\in B^c\ \vee\ x\in A^c$

$\Leftrightarrow$ $x\in(A\ \vee B)$

$$\wedge$$
$$x \in (B^c \ \vee A^c)$$

$\Leftrightarrow$	$x \in (A \cup B)$ $\cap$ $x \in (B^c \cup A^c) = (A \cap B)^c$	Def. of ∪ and De Morgan's Law
$\Leftrightarrow$	$x \in (A \cup B) \cap (A \cap B)^c$	
$\Leftrightarrow$	$x \in (A \cup B) - (A \cap B)$	Theorem 2.2.15(ii)